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On Certain Aspects of the Theory of Polynomial Bundle Lax Operators Related to Symmetric Spaces

Tihomir Valchev

Institute for Nuclear Research
and Nuclear Energy,
Bulgarian Academy of Sciences,
Sofia, Bulgaria

Joint work with: A. Mikhailov, V. Gerdjikov and G. Grahovski

1. Introduction

Heisenberg ferromagnet equation (spin of a $1D$ ferromagnetic)

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \qquad \mathbf{S}^2 = 1$$

or in a matrix notation

$$\mathrm{i}S_t = \frac{1}{2}[S, S_{xx}], \qquad S^2 = 1\!\!1, \qquad S = \sum_{k=1}^{3} \mathbf{S}^k \sigma_k.$$

Its Lax representation reads

$$\begin{aligned} L(\lambda) &:= \mathrm{i}\partial_x - \lambda S(x,t), \\ A(\lambda) &:= \mathrm{i}\partial_t + A_0(x,t) + \lambda A_1(x,t) + \lambda^2 A_2(x,t). \end{aligned}$$

All matrices above belong to $\mathfrak{su}(2)$.

Purpose of the talk: studying properties of 2-component system

$$\begin{aligned} \mathrm{i}u_t \quad &+ u_{xx} + (uu_x^* + vv_x^*)u_x + (uu_x^* + vv_x^*)_x u = 0, \\ \mathrm{i}v_t \quad &+ v_{xx} + (uu_x^* + vv_x^*)v_x + (uu_x^* + vv_x^*)_x v = 0, \end{aligned}$$

to generalize the Heisenberg model. The functions u and v are infinitely smooth to satisfy

$$\lim_{x\to\pm\infty} u(x,t) = 0, \qquad \lim_{x\to\pm\infty} v(x,t) = 1.$$

Moreover, u and v obey the constraint $|u|^2 + |v|^2 = 1$.

2. Preliminaries

- Lax representation

$$\text{NEE} \quad \Leftrightarrow \quad [L(\lambda), A(\lambda)] = 0,$$

where the Lax pair (polynomial bundle) is given by:

$$\begin{aligned} L(\lambda) &:= \mathrm{i}\partial_x + \lambda L_1(x,t), \\ A(\lambda) &:= \mathrm{i}\partial_t + \lambda A_1(x,t) + \lambda^2 A_2(x,t), \end{aligned}$$

where

$$L_1 = \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}, \quad A_2 = -\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & |u|^2 - 2/3 & u^* v \\ 0 & v^* u & |v|^2 - 2/3 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & a & b \\ a^* & 0 & 0 \\ b^* & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} a &= \mathrm{i}u_x + \mathrm{i}(uu_x^* + vv_x^*)u \\ b &= \mathrm{i}v_x + \mathrm{i}(uu_x^* + vv_x^*)v \end{aligned} \quad .$$

The specific structure of the matrices is a result of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction on generic Lax operators

$$\begin{aligned} L^\dagger(\lambda^*) &= -\breve{L}(\lambda), \qquad A^\dagger(\lambda^*) = -\breve{A}(\lambda), \\ \mathbf{C}L(-\lambda)\mathbf{C} &= L(\lambda), \qquad \mathbf{C}A(-\lambda)\mathbf{C} = A(\lambda), \end{aligned}$$

where $\mathbf{C} = \operatorname{diag}(1, -1, -1)$ and the operation $\breve{}$ is defined as follows

$$\breve{L}(\lambda)\psi(x,t,\lambda) := \mathrm{i}\partial_x\psi(x,t,\lambda) - \lambda\psi(x,t,\lambda)L_1^\dagger(x,t,\lambda).$$

The matrix $\mathbf{C}$ represents Cartan's involutive automorphism involved in the definition of $SU(3)/S(U(1)\times U(2))$, that is it induces a $\mathbb{Z}_2$-grading in the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$

$$\mathfrak{sl}(3) = \mathfrak{sl}^0(3) \oplus \mathfrak{sl}^1(3), \qquad \mathfrak{sl}^\sigma(3) = \{X \in \mathfrak{g};\ \mathbf{C}X\mathbf{C} = (-1)^\sigma X\}.$$

Thus any function $X(x,t,\lambda)$ with values in $\mathfrak{sl}(3)$ is presented as

$$X^0(x,t,\lambda) + X^1(x,t,\lambda), \qquad X^{0,1}(x,t,\lambda) \in \mathfrak{sl}^{0,1}(3).$$

- Scattering problem
  - Fundamental solutions

    Introduce the auxiliary linear system

$$L(\lambda)\psi(x,t,\lambda) = \mathrm{i}\partial_x \psi(x,t,\lambda) + \lambda L_1(x,t)\psi(x,t,\lambda) = 0.$$

Since $L(\lambda)$ and $A(\lambda)$ commute ψ also satisfies the equation

$$A(\lambda)\psi(x,t,\lambda) = [\mathrm{i}\partial_t + \lambda A_1(x,t) + \lambda^2 A_2(x,t)]\psi(x,t,\lambda) = \psi(x,t,\lambda) f(\lambda)$$

for some matrix-valued function $f(\lambda)$. We choose

$$f(\lambda) = \lim_{x\to\pm\infty} g_{\mathrm{as}}^{-1}[\lambda A_1(x,t) + \lambda^2 A_2(x,t)] g_{\mathrm{as}} = -\lambda^2 I,$$

where

$$g_{\mathrm{as}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad I = \mathrm{diag}\,(1/3, -2/3, 1/3).$$

– Jost solutions and scattering matrix

$$\lim_{x \to \pm\infty} \psi_\pm(x,t,\lambda) \mathrm{e}^{-\mathrm{i}\lambda J x} g_{\mathrm{as}}^{-1} = \mathbb{1},$$

where $J = \mathrm{diag}\,(1, 0, -1)$ is the diagonal form of the asymptotic $\lim_{x\to\pm\infty} L_1(x,t)$. The transition matrix

$$T(t,\lambda) = [\psi_+(x,t,\lambda)]^{-1} \psi_-(x,t,\lambda)$$

is called scattering matrix. It can be shown that T evolves with time according to

$$\mathrm{i}\partial_t T + [f(\lambda), T] = 0 \qquad \Rightarrow \qquad T(t,\lambda) = \mathrm{e}^{\mathrm{i}f(\lambda)t} T(0,\lambda) \mathrm{e}^{-\mathrm{i}f(\lambda)t}.$$

– Construction of fundamental analytic solutions

$$\chi^{\pm}(x,\lambda) = \psi_{-}(x,\lambda)S^{\pm} = \psi_{+}(x,\lambda)T^{\mp}(\lambda)D^{\pm}(\lambda),$$

where

$$T(\lambda) = T^{\mp}(\lambda)D^{\pm}(\lambda)(S^{\pm}(\lambda))^{-1}.$$

– Riemann-Hilbert problem

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda)G(\lambda).$$

– Reduction conditions on the Jost solutions, the scattering matrix and fundamental analytic solutions

$$\begin{aligned}
\left[\psi^{\dagger}_{\pm}(x,\lambda^*)\right]^{-1} &= \psi_{\pm}(x,\lambda), \qquad \left[T^{\dagger}(\lambda^*)\right]^{-1} = T(\lambda),\\
\mathbf{C}\psi_{\pm}(x,-\lambda)\mathbf{C} &= \psi_{\pm}(x,\lambda), \qquad \mathbf{C}T(-\lambda)\mathbf{C} = T(\lambda),\\
(\chi^{+})^{\dagger}(x,\lambda^*) &= [\chi^{-}(x,\lambda)]^{-1}, \qquad \mathbf{C}\chi^{+}(x,-\lambda)\mathbf{C} = \chi^{-}(x,\lambda).
\end{aligned}$$

3. Soliton solutions

3.1. Dressing method

- Concept of the dressing method

  Let ψ_0 be a fundamental solution to

$$L_0(\lambda)\psi_0(x,\lambda) = \mathrm{i}\partial_x\psi_0(x,\lambda) + \lambda L_{1,0}(x)\psi_0(x,\lambda) = 0$$

  with some known potential $L_{1,0}$. We construct another function $\psi = g\psi_0$ and suppose it satisfies the same linear problem

$$L(\lambda)\psi(x,\lambda) = \mathrm{i}\partial_x\psi(x,\lambda) + \lambda L_1(x)\psi(x,\lambda) = 0.$$

  with a different potential to be found. This implies

$$\mathrm{i}\partial_x g + \lambda(L_1\, g - gL_{1,0}) = 0.$$

- Ansatz for the dressing factor

  We pick up a dressing factor in the form

$$g(x,\lambda) = A(x) + \frac{B(x)}{\lambda - \mu} - \frac{\mathbf{C}B(x)\mathbf{C}}{\lambda + \mu}, \qquad \mathbf{C}A\mathbf{C} = A$$

  that is compatible with the reduction conditions

$$\begin{aligned} \mathbf{C}g(x,-\lambda)\mathbf{C} &= g(x,\lambda), \\ \left[g^{\dagger}(x,\lambda^{*})\right]^{-1} &= g(x,\lambda). \end{aligned}$$

  The inverse of g looks like

$$g^{-1}(x,\lambda) = A^{\dagger}(x) + \frac{B^{\dagger}(x)}{\lambda - \mu^{*}} - \frac{\mathbf{C}B^{\dagger}(x)\mathbf{C}}{\lambda + \mu^{*}}.$$

- Algebraic constraints

  From the identity $gg^{-1} = \mathbb{1}$ it follows

$$B\left(A^{\dagger} + \frac{B^{\dagger}}{\mu - \mu^{*}} - \frac{\mathbf{C}B^{\dagger}\mathbf{C}}{\mu + \mu^{*}}\right) = 0.$$

At this point we introduce the factorization $B = XF^T$. Hence we have

$$AF^* = \frac{F^T F^*}{\mu - \mu^*} X + \frac{F^T \mathbf{C} F^*}{\mu + \mu^*} \mathbf{C} X.$$

Solving the equation above: $F, A \Rightarrow X$. But $F, A = ?$.

- Analysis of the differential constraint on g

  At $\lambda = 0$ it leads to

$$\partial_x A - \frac{1}{\mu} \partial_x (B + \mathbf{C} B \mathbf{C}) = 0 \quad \Rightarrow \quad A = \frac{1}{\mu}(B + \mathbf{C} B \mathbf{C}) + A_0.$$

  It suffices to pick up $A_0 = \mathbb{1}$. On the other hand comparing the residues leads us to the coclusion that

$$\mathrm{i}\partial_x F^T - \mu F^T L_{1,0} = 0 \qquad \Rightarrow \qquad F^T(x) = F_0^T \psi_0^{-1}(x, \lambda = \mu).$$

  After substituting A one obtains

$$F^* = (a + bJ_1)\, X.$$

where

$$a := \frac{\mu^* F^T F^*}{\mu(\mu - \mu^*)}, \qquad b := -\frac{\mu^* F^T \mathbf{C} F^*}{\mu(\mu + \mu^*)} .$$

By inverting the linear system above we get X expressed by F:

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} (a+b)^{-1} F_1^* \\ (a-b)^{-1} F_2^* \\ (a-b)^{-1} F_3^* \end{pmatrix} .$$

In order to obtain a relation between L_1 and $L_{1,0}$ one takes the limit $\lambda \to \infty$ in

$$\frac{\mathrm{i}}{\lambda} \partial_x g + L_1 g - g L_{1,0} = 0$$

to deduce that

$$L_1 = A L_{1,0} A^\dagger .$$

This relation allows us to generate another solution starting from a known one by following the sequence

$$L_{1,0} \to \psi_0 \to F \to X, A \to L_1 .$$

- Recovering the time dependence

  In order to recover the time dependence one needs to compare the linear problems

$$
\begin{aligned}
A(\lambda)\psi &= [\mathrm{i}\partial_t + \lambda A_1 + \lambda^2 A_2]\psi = \psi f, \\
A_0(\lambda)\psi &= [\mathrm{i}\partial_t + \lambda A_{1,0} + \lambda^2 A_{2,0}]\psi_0 = \psi_0 f.
\end{aligned}
$$

  Therefore the dressing factor satisfies

$$
\mathrm{i}\partial_t g + [\lambda A_1 + \lambda^2 A_2]g - g[\lambda A_{1,0} + \lambda^2 A_{2,0}] = 0.
$$

  A more analysis shows that F_0 depends on time exponentially

$$
F_0^T \quad \mapsto \quad F_0^T \mathrm{e}^{-\mathrm{i}f(\mu)t} = \left(\mathrm{e}^{\frac{\mathrm{i}\mu^2 t}{3}} F_{0,1},\ \mathrm{e}^{-\frac{2\mathrm{i}\mu^2 t}{3}} F_{0,2},\ \mathrm{e}^{\frac{\mathrm{i}\mu^2 t}{3}} F_{0,3} \right).
$$

3.2. Elementary solitons

Let us choose as a seed solution

$$L_{1,0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then for ψ_0 one obtains

$$\psi_0(x,t,\lambda) = \begin{pmatrix} \cos\lambda x & 0 & \mathrm{i}\sin\lambda x \\ 0 & 1 & 0 \\ \mathrm{i}\sin\lambda x & 0 & \cos\lambda x \end{pmatrix}.$$

Inserting ψ_0 into the expressions for F leads to

$$F(x,t) = \begin{pmatrix} F_{0,1}\cos\mu x - \mathrm{i}F_{0,3}\sin\mu x \\ F_{0,2} \\ F_{0,3}\cos\mu x - \mathrm{i}F_{0,1}\sin\mu x \end{pmatrix}.$$

Particular cases (elementary solitons):

1. Consider $F_{0,2} = 0$. In this case the solution is stationary, namely

$$\begin{aligned} u(x) &= 0, \\ v(x) &= \exp\left[4\mathrm{i}\arctan\frac{\gamma\cos(2\omega x+\delta_0)}{\omega\cosh(2\gamma x+\xi_0)}\right], \end{aligned}$$

where $\omega := \operatorname{Re}\mu > 0$, $\gamma := \operatorname{Im}\mu > 0$ and

$$\sinh\xi_0 = \frac{2|F_{0,1}F_{0,3}|\cos(\arg F_{0,1} - \arg F_{0,3})}{\sqrt{\left(|F_{0,1}|^2+|F_{0,3}|^2\right)^2 - 4|F_{0,1}F_{0,3}|^2\cos^2(\arg F_{0,1} - \arg F_{0,3})}},$$

$$\sin\delta_0 = \frac{2|F_{0,1}F_{0,3}|\sin(\arg F_{0,1} - \arg F_{0,3})}{\sqrt{\left(|F_{0,1}|^2+|F_{0,3}|^2\right)^2 - 4|F_{0,1}F_{0,3}|^2\cos^2(\arg F_{0,1} - \arg F_{0,3})}}.$$

2. $F_{0,1} = F_{0,3}$. Now F is given by

$$F(x,t) = \begin{pmatrix} F_{0,1}\mathrm{e}^{-\mathrm{i}\mu x} \\ F_{0,2} \\ F_{0,1}\mathrm{e}^{-\mathrm{i}\mu x} \end{pmatrix}$$

and after recovering the time dependence the soliton solution reads

$$\begin{aligned} u(x,t) &= -\frac{4\mathrm{i}\omega\gamma\left[2\omega\mathrm{e}^{\gamma(x-2\omega t+\vartheta_0)} + (\omega-\mathrm{i}\gamma)\mathrm{e}^{-\gamma(x-2\omega t+\vartheta_0)}\right]}{(\omega-\mathrm{i}\gamma)\left[2\omega\mathrm{e}^{\gamma(x-2\omega t+\vartheta_0)} + (\omega+\mathrm{i}\gamma)\mathrm{e}^{-\gamma(x-2\omega t+\vartheta_0)}\right]^2}, \\ &\times \exp[\mathrm{i}\omega x + \mathrm{i}(\gamma^2-\omega^2)t + \mathrm{i}\phi_0], \\ v(x,t) &= 1 - \frac{8\omega\gamma^2}{(\omega-\mathrm{i}\gamma)\left[2\omega\mathrm{e}^{\gamma(x-2\omega t+\vartheta_0)} + (\omega+\mathrm{i}\gamma)\mathrm{e}^{-\gamma(x-2\omega t+\vartheta_0)}\right]^2}, \end{aligned}$$

where

$$\vartheta_0 = \frac{1}{\gamma}\ln\frac{|F_{0,1}|}{|F_{0,2}|}, \qquad \phi_0 = \arg F_{0,2} - \arg F_{0,1}.$$

4. Generalized Fourier Transform

- Wronskian relations and 'squared solutions'

$$(\chi^{\pm})^{-1} L_1 \chi^{\pm}\Big|_{x=-\infty}^{\infty} = \int_{-\infty}^{\infty} \mathrm{d}\, x\, \hat{\chi}^{\pm} L_{1,x} \chi^{\pm}$$

Therefore one can write

$$\left\langle (\chi^{\pm})^{-1} L_1 \chi^{\pm}, E_\alpha \right\rangle \Big|_{-\infty}^{\infty} = \int_{-\infty}^{\infty} \mathrm{d}\, x\, \left\langle L_{1,x}, e_\alpha^{\pm} \right\rangle .$$

The quantities $e_\alpha^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda) E_\alpha [\chi^{\pm}(x,\lambda)]^{-1}$ introduced above are called 'squared solutions' and

$$\langle X, Y \rangle := \mathrm{tr}\,(XY)$$

is the Cartan-Killing form.

- Splitting of $e_\alpha^{\pm}(x,\lambda)$ due to $\mathbb{Z}_2$-grading of the Lie algebra

$$e_\alpha(x,\lambda) = \mathcal{H}_\alpha(x,\lambda) + \mathcal{K}_\alpha(x,\lambda), \quad \mathcal{H}_\alpha(x,\lambda) \in \mathfrak{sl}^0(3), \quad \mathcal{K}_\alpha(x,\lambda) \in \mathfrak{sl}^1(3).$$

In addition, $\mathcal{H}_\alpha(x,\lambda)$ and $\mathcal{K}_\alpha(x,\lambda)$ split into

$$\begin{aligned} \mathcal{H}_\alpha(x,\lambda) &= H_\alpha(x,\lambda) + h_\alpha(x,\lambda) L_2(x), \qquad && \langle H_\alpha, L_2\rangle = 0, \\ \mathcal{K}_\alpha(x,\lambda) &= K_\alpha(x,\lambda) + k_\alpha(x,\lambda) L_1(x), \qquad && \langle K_\alpha, L_1\rangle = 0, \end{aligned}$$

where $L_2 := L_1^2 - 2/3 \in \mathfrak{sl}^0(3)$.

It can be proven that the 'squared solutions' form a complete system in the space of smooth functions with values in $\mathfrak{sl}(3)/\ker \mathrm{ad}_{L_1}$. Hence they play a role quite similar to the exponential functions in the usual Fourier analysis — by expanding all quantities involved in NEE one obtains a linearized version of the NEE.

- Recursion operators

  Consider the equation

$$\mathrm{i}\partial_x e_\alpha + \lambda [L_1, e_\alpha] = 0.$$

Due to the grading condition $\mathcal{H}_\alpha$ and $\mathcal{K}_\alpha$ are interrelated through

$$\begin{aligned} \mathrm{i}\partial_x \mathcal{H}_\alpha + \lambda [L_1, \mathcal{K}_\alpha] &= 0, \\ \mathrm{i}\partial_x \mathcal{K}_\alpha + \lambda [L_1, \mathcal{H}_\alpha] &= 0. \end{aligned}$$

After we extract the terms proportional to L_1 we obtain

$$\begin{aligned}
\langle L_2, \partial_x H_\alpha \rangle + \partial_x h_\alpha = 0 \qquad &\Rightarrow \qquad h_\alpha = h_{\alpha,0} - \tfrac{3}{2}\partial_x^{-1} \langle L_2, \partial_x H_\alpha \rangle \,, \\
\langle L_1, \partial_x K_\alpha \rangle + \partial_x k_\alpha = 0 \qquad &\Rightarrow \qquad k_\alpha = k_{\alpha,0} - \tfrac{1}{2}\partial_x^{-1} \langle L_1, \partial_x K_\alpha \rangle \,.
\end{aligned}$$

On the other hand the orthogonal part reads:

$$\begin{aligned}
\mathrm{i}\pi\partial_x H_\alpha + \mathrm{i} h_\alpha L_{2,x} \quad &= -\lambda[L_1, K_\alpha], \\
\mathrm{i}\pi\partial_x K_\alpha + \mathrm{i}\, k_\alpha L_{1,x} \quad &= -\lambda[L_1, H_\alpha].
\end{aligned}$$

After substituting h_α and k_α in the equations above we get

$$\begin{aligned}
\Lambda_1 K_\alpha \quad &= \lambda H_\alpha - k_{\alpha,0}\Lambda_2 L_2, \\
\Lambda_2 H_\alpha \quad &= \lambda K_\alpha - h_{\alpha,0}\Lambda_1 L_1,
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 \quad &= \quad -\mathrm{iad}\,_{L_1}^{-1} \left(\pi\partial_x(\cdot) - L_{1,x}\frac{1}{2}\partial_x^{-1}\langle \partial_x(\cdot), L_1 \rangle \right), \\
\Lambda_2 \quad &= \quad -\mathrm{iad}\,_{L_1}^{-1} \left(\pi\partial_x(\cdot\,) - \frac{3}{2} L_{2,x}\partial_x^{-1}\langle \partial_x(\cdot\,), L_2 \rangle \right).
\end{aligned}$$

Let us apply Λ_2 to the first relation and Λ_1 to the second one. The result reads:

$$\begin{aligned}
\Lambda_2^{\pm}\Lambda_1^{\pm}K_{\alpha}^{\pm} &= \lambda^2 K_{\alpha}^{\pm} - \lambda h_{\alpha,0}\Lambda_1^{\pm}L_1 - k_{\alpha,0}\Lambda_2^{\pm}\Lambda_2^{\pm}L_2, \\
\Lambda_1^{\pm}\Lambda_2^{\pm}H_{\alpha}^{\pm} &= \lambda^2 H_{\alpha}^{\pm} - \lambda k_{\alpha,0}\Lambda_2^{\pm}L_2 - h_{\alpha,0}\Lambda_1^{\pm}\Lambda_2^{\pm}L_2.
\end{aligned}$$

The constants $h_{\alpha,0}$ and $k_{\alpha,0}$ are determined by the asymptotic of the relevant 'squared solution' for $x \to \infty$ (or for $x \to -\infty$). More detailed analysis shows that the following equalities holds:

$$\begin{aligned}
\Lambda_2^{+}\Lambda_1^{+}K_{\mp\alpha}^{\pm}(x,\lambda) &= \lambda^2 K_{\mp\alpha}^{\pm}(x,\lambda), \quad \Lambda_2^{-}\Lambda_1^{-}K_{\pm\alpha}^{\pm}(x,\lambda) = \lambda^2 K_{\pm\alpha}^{\pm}(x,\lambda) \\
\Lambda_1^{+}\Lambda_2^{+}H_{\mp\alpha}^{\pm}(x,\lambda) &= \lambda^2 H_{\mp\alpha}^{\pm}(x,\lambda), \quad \Lambda_1^{-}\Lambda_2^{-}H_{\pm\alpha}^{\pm}(x,\lambda) = \lambda^2 H_{\pm\alpha}^{\pm}(x,\lambda)
\end{aligned}$$

The operator $\Lambda^{\pm}$ introduced as

$$\begin{aligned}
\Lambda^{\pm}X &:= \Lambda_1^{\pm}\Lambda_2^{\pm}X, \quad X \in \mathfrak{sl}^0(3), \\
\Lambda^{\pm}Y &:= \Lambda_2^{\pm}\Lambda_1^{\pm}Y, \quad Y \in \mathfrak{sl}^1(3).
\end{aligned}$$

is the called recursion operator.

5. Integrable hierarchy

- Description of the integrable hierarchy in terms of the recursion operator

  Any member of the integrable hierarchy under consideration has a Lax pair in the form

$$
\begin{aligned}
L(\lambda) &= \mathrm{i}\partial_x + \lambda L_1(x,t), \\
A(\lambda) &= \mathrm{i}\partial_t + \sum_{k=1}^{N} \lambda^k A_k(x,t).
\end{aligned}
$$

  As before the operators L and A are subject to the reductions

$$
\begin{aligned}
\mathbf{C}A_{2q-1}\mathbf{C} &= -A_{2q-1} \qquad \Rightarrow \qquad A_{2q-1} \in \mathfrak{sl}^1(3), \\
\mathbf{C}A_{2q}\mathbf{C} &= A_{2q}, \qquad \Rightarrow \qquad A_{2q} \in \mathfrak{sl}^0(3).
\end{aligned}
$$

  The original Lax pair corresponds to the simplest nontrivial case ($N = 2$) of this general flow pair.

  The compatibility condition $[L(\lambda), A(\lambda)] = 0$ gives rise to the following set of recurrence relations:

$$
\begin{aligned}
&[L_1, A_N] = 0,\\
&\mathrm{i}\partial_x A_N + [L_1, A_{N-1}] = 0,\\
&\dots\\
&\mathrm{i}\partial_x A_k + [L_1, A_{k-1}] = 0, \qquad k = 2, \dots, N-1,\\
&\dots\\
&\partial_x A_1 - \partial_t L_1 = 0.
\end{aligned}
$$

It follows from the first relation that the highest order term is a polynomial of L_1 and hence we have two possibilities for A_N:

$$
\begin{aligned}
&\text{a)} \qquad A_N = f_{2p} L_2, \qquad \text{for} \quad N = 2p,\\
&\text{b)} \qquad A_N = f_{2p+1} L_1, \qquad \text{for} \quad N = 2p+1.
\end{aligned}
$$

It suffices to restrict oursleves with the case when $N = 2p$.

We shall split each element A_k into two mutually orthogonal parts:

$$
\begin{aligned}
A_{2q-1} &= A^{\perp}_{2q-1} + f_{2q-1} L_1,\\
A_{2q} &= A^{\perp}_{2q} + f_{2q} L_2.
\end{aligned}
$$

Substituting the splitting of A_{N-1} we have

$$\mathrm{i} f_{2p,x} L_2 + \mathrm{i} f_{2p} L_{2,x} + [L_1, A^{\perp}_{2p-1}] = 0.$$

After taking the Killing form $\langle . \, , L_2 \rangle$ to separate the L_1-commuting part and its orthogonal complement we deduce that

$$f_N = c_N = \text{const}, \qquad A^{\perp}_{2p-1} = -\mathrm{i} c_{2p} \, \mathrm{ad}\,_{L_1}^{-1} L_{2,x}.$$

Similarly, after extracting the L_1-commuting part from the generic recurrence relations we determine for the coefficient f_{2q-1} (resp. f_{2q})

$$\begin{aligned} f_{2q-1} &= c_{2q-1} - \frac{1}{2} \partial_x^{-1} \left\langle \left(A^{\perp}_{2q-1} \right)_x , L_1 \right\rangle , \\ f_{2q} &= c_{2q} - \frac{3}{2} \partial_x^{-1} \left\langle \left(A^{\perp}_{2q} \right)_x , L_2 \right\rangle , \end{aligned}$$

where c_{2q-1} (resp. c_{2q}) is a constant of integration. On the other hand from the orthogonal parts of generic recurrence relations one can express $A^{\perp}_{2q-1}$ (resp. $A^{\perp}_{2q}$), namely :

$$\begin{aligned} A^{\perp}_{2q} &= \Lambda_1\left(A^{\perp}_{2q+1}\right) - \mathrm{i}c_{2q+1}\mathrm{ad}\,_{L_1}^{-1}L_{1,x}, \\ A^{\perp}_{2q-1} &= \Lambda_2\left(A^{\perp}_{2q}\right) - \mathrm{i}c_{2q}\,\mathrm{ad}\,_{L_1}^{-1}L_{2,x}. \end{aligned}$$

The last recurrence relation yields to

$$f_1 = c_1 - \frac{1}{2}\partial_x^{-1}\langle (A_1^{\perp})_x, L_1\rangle,$$
$$\mathrm{iad}\,_{L_1}^{-1}\partial_t L_1 + \Lambda_1 A_1^{\perp} - \mathrm{i}c_1\mathrm{ad}\,_{L_1}^{-1}L_{1,x} = 0.$$

Finally for an arbitrary member of the integrable hierarchy we obtain

$$\text{a)}\qquad \partial_t L_1 = \sum_{q=1}^{p} c_{2q}(\Lambda_1\Lambda_2)^{q-1}\Lambda_1\mathrm{ad}\,_{L_1}^{-1}L_{2,x} + \sum_{q=0}^{p-1} c_{2q+1}(\Lambda_1\Lambda_2)^{q}\mathrm{ad}\,_{L_1}^{-1}L_{1,x},$$

$$\text{b)}\qquad \partial_t L_1 = \sum_{q=1}^{p} c_{2q}(\Lambda_1\Lambda_2)^{q-1}\Lambda_1\mathrm{ad}\,_{L_1}^{-1}L_{2,x} + \sum_{q=0}^{p} c_{2q+1}(\Lambda_1\Lambda_2)^{q}\mathrm{ad}\,_{L_1}^{-1}L_{1,x}.$$

The coefficients c_k are involved in the dispersion law of the corresponding NEE:

$$f(\lambda) = \lim_{x\to\pm\infty} g_{\rm as}^{-1} \sum_{k=1}^{N} \lambda^k A_k(x,t) g_{\rm as}.$$

The dispersion law determines the evolution of scattering matrix

$$T(t,\lambda) = \mathrm{e}^{\mathrm{i}f(\lambda)t} T(0,\lambda) \mathrm{e}^{-\mathrm{i}f(\lambda)t}.$$

It is not hard to check that the equalities below are valid

$$\text{a)} \qquad f(\lambda) = \sum_{q=0}^{p-1} c_{2q+1}\lambda^{2q+1} J + \sum_{q=1}^{p} c_{2q}\lambda^{2q} I,$$

$$\text{b)} \qquad f(\lambda) = \sum_{q=0}^{p} c_{2q+1}\lambda^{2q+1} J + \sum_{q=1}^{p} c_{2q}\lambda^{2q} I,$$

where $J = \operatorname{diag}(1,0,-1)$, $I = \operatorname{diag}(1/3,-2/3,1/3)$. The initial NEE can be derived from the above formulae in the simplest case $N = 2$ after inserting $c_2 = -1$ and $c_1 = 0$.

- Integrable hierarchy in terms of the scattering data

  The following theorem holds true:

Theorem 1 *Let the Lax operator L be such that its potential satisfies the conditions:*

1. $L_1(x) - \lim_{x\to\pm\infty} L_1$ *is complex valued function of Schwartz type;*

2. $L_1(x)$ *is such that the principal minors of the scattering matrix have a finite number of simple zeroes which do not coincide;*

3. *The principal minors have no zeroes on the real axis of the complex λ-plane.*

Then the NEE are equivalent to each of the following set of linear evolution equations:

$$
\begin{aligned}
\mathrm{i}\partial_t S^{\pm} + [f(\lambda), S^{\pm}(\lambda)] &= 0, \qquad \partial_t D^{+}(\lambda) = 0, \\
\mathrm{i}\partial_t s_k^{\pm} + \left[f(\lambda_k^{\pm}), s_k^{\pm}\right] &= 0, \qquad \frac{\mathrm{d}\,\lambda_k^{\pm}}{\mathrm{d}\,t} = 0.
\end{aligned}
$$

where $S^{\pm}(\lambda) = \exp s^{\pm}(\lambda)$ *and*

$$s^{+}(\lambda) \quad = \quad \begin{pmatrix} 0 & s^{+}_{\alpha_1} & s^{+}_{\alpha_3} \\ 0 & 0 & s^{+}_{\alpha_2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad s^{+}_{k} = \begin{pmatrix} 0 & \operatorname*{Res}_{\lambda=\lambda^{+}_{k}} s^{+}_{\alpha_1} & \operatorname*{Res}_{\lambda=\lambda^{+}_{k}} s^{+}_{\alpha_3} \\ 0 & 0 & \operatorname*{Res}_{\lambda=\lambda^{+}_{k}} s^{+}_{\alpha_2} \\ 0 & 0 & 0 \end{pmatrix},$$

$$s^{-}(\lambda) \quad = \quad \begin{pmatrix} 0 & 0 & 0 \\ s^{-}_{\alpha_1} & 0 & 0 \\ s^{-}_{\alpha_3} & s^{-}_{\alpha_2} & 0 \end{pmatrix}, \qquad s^{-}_{k} = \begin{pmatrix} 0 & 0 & 0 \\ \operatorname*{Res}_{\lambda=\lambda^{-}_{k}} s^{-}_{\alpha_1} & 0 & 0 \\ \operatorname*{Res}_{\lambda=\lambda^{-}_{k}} s^{-}_{\alpha_3} & \operatorname*{Res}_{\lambda=\lambda^{-}_{k}} s^{-}_{\alpha_2} & 0 \end{pmatrix}$$

The variables $\{s^{\pm}_{\alpha}(\lambda), \quad \lambda \in \mathbb{R}, \qquad s^{\pm}_{\alpha;k}, \quad \forall\, \alpha \in \Delta_{+}\}_{k=1}^{n}$ define a minimal set of scattering data.

Conclusions

- A 2-component generalization of HF equation has been studied. The direct scattering problem for the corresponding L operator has been developed in terms of Jost solutions, scattering matrix and fundamental analytic solutions.

- The soliton solutions have been constructed analytically. For that purpose we have used the dressing technique.

- The basic notions of the generalized Fourier intepretation of the inverse scattering method has been introduced. These are Wronskian relations, 'squared solutions' and recursion operators. By using them we have described the integrable hierarchy of NEE, associated to L in terms of recursion operators and scattering data.