Liouville Integrability in Field Theory

Gaetano Vilasi
Università degli Studi di Salerno, Italy
Istituto Nazionale di Fisica Nucleare, Italy

Geometry, Integrability and Quatization
Varna, June 3–8 (2011)

Gaetano Vilasi
Università degli Studi di Salerno, Italy
Istituto Nazionale di Fisica Nucleare, Italy

Outline

- Completely integrable Hamiltonian systems (Liouville 1855) are dynamical systems admitting a Hamiltonian description and possessing sufficiently many constants of motion, so that they can be integrated by quadratures.

- Some qualitative features of these systems remain true in some special classes of infinite-dimensional Hamiltonian systems expressed by nonlinear evolution equations (e.g. Korteweg-de Vries and sine-Gordon).

$$u_t + uu_x + u_{xxx} = 0, \qquad \dot{L} = [B, L]$$

$$L = \partial_{xx} + \frac{1}{6} u(x,t)$$

$$B = -4\partial_{xxx} - u(x,t)\,\partial_x - \frac{1}{2} u_x(x,t)$$

- Completely integrable Hamiltonian systems (Liouville 1855) are dynamical systems admitting a Hamiltonian description and possessing sufficiently many constants of motion, so that they can be integrated by quadratures.

- Some qualitative features of these systems remain true in some special classes of infinite-dimensional Hamiltonian systems expressed by nonlinear evolution equations (e.g. Korteweg-de Vries and sine-Gordon).

$$u_t + uu_x + u_{xxx} = 0, \qquad \dot{L} = [B, L]$$

$$L = \partial_{xx} + \frac{1}{6} u(x,t)$$

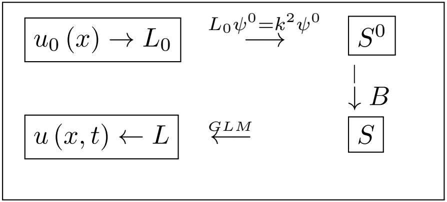
$$B = -4\partial_{xxx} - u(x,t)\,\partial_x - \frac{1}{2} u_x(x,t)$$

- For ∞-d phase manifold $\mathcal{M}$, *Lax Representation* (Lax, 68) has played an important role in formulating the *Inverse Scattering Method* (*ISM*) and of the *AKNS scheme*.

- *ISM* allows the integration of non linear dynamics, both with a finitely or infinitely many degrees of freedom, for which a Lax representation can be given (Gardner, Greene, Kruskal and Miura, 67), this being both of physical and mathematical relevance (Faddeev and Takhtajan book, 87).

- For ∞-d phase manifold $\mathcal{M}$, *Lax Representation* (Lax, 68) has played an important role in formulating the *Inverse Scattering Method* (*ISM*) and of the *AKNS scheme*.

- *ISM* allows the integration of non linear dynamics, both with a finitely or infinitely many degrees of freedom, for which a Lax representation can be given (Gardner, Greene, Kruskal and Miura, 67), this being both of physical and mathematical relevance (Faddeev and Takhtajan book, 87).



- Most of the evolution equations admitting a Lax Representation are generally Hamiltonian dynamics on infinite dimensional *(weakly) symplectic* manifolds.

- So the natural arena, for the analysis of their integrability, is represented by the phase space with its natural symplectic structure.

- In terms of this structure, the scattering data associated with the Lax operator have a natural interpretation as *action-angle* type variables (Faddeev Zakharov 71).

- Most of the evolution equations admitting a Lax Representation are generally Hamiltonian dynamics on infinite dimensional *(weakly) symplectic* manifolds.
- So the natural arena, for the analysis of their integrability, is represented by the phase space with its natural symplectic structure.
- In terms of this structure, the scattering data associated with the Lax operator have a natural interpretation as *action-angle* type variables (Faddeev Zakharov 71).

- Most of the evolution equations admitting a Lax Representation are generally Hamiltonian dynamics on infinite dimensional *(weakly) symplectic* manifolds.

- So the natural arena, for the analysis of their integrability, is represented by the phase space with its natural symplectic structure.

- In terms of this structure, the scattering data associated with the Lax operator have a natural interpretation as *action-angle* type variables (Faddeev Zakharov 71).

- Many of these systems are Hamiltonian dynamics with respect to two *compatible* symplectic structures (Magri 78, Gelfand, I. Ya. Dorfman, G. V. 80), this leading to a geometrical interpretation of the *recursion operator* (Lenard, 67).

- Then integrability of non linear field theories could be naturally explained in terms of mixed tensor fields, whose relation with Lax operators is still unclear (De Filippo, Marmo, Salerno, G. V. 82) .

- Indeed, integrability criteria can be given in terms of invariant mixed tensor field, having 2-d eigenspaces and vanishing Nijenhuis torsion. (De Filippo, Marmo, Salerno, G. V. 82, 84 and 85, Florko, Yanovski 83, Landi, Marmo, G. V. 94)

- Many of these systems are Hamiltonian dynamics with respect to two *compatible* symplectic structures (Magri 78, Gelfand, I. Ya. Dorfman, G. V. 80), this leading to a geometrical interpretation of the *recursion operator* (Lenard, 67).

- Then integrability of non linear field theories could be naturally explained in terms of mixed tensor fields, whose relation with Lax operators is still unclear (De Filippo, Marmo, Salerno, G. V. 82) .

- Indeed, integrability criteria can be given in terms of invariant mixed tensor field, having 2-d eigenspaces and vanishing Nijenhuis torsion. (De Filippo, Marmo, Salerno, G. V. 82, 84 and 85, Florko, Yanovski 83, Landi, Marmo, G. V. 94)

- Many of these systems are Hamiltonian dynamics with respect to two *compatible* symplectic structures (Magri 78, Gelfand, I. Ya. Dorfman, G. V. 80), this leading to a geometrical interpretation of the *recursion operator* (Lenard, 67).

- Then integrability of non linear field theories could be naturally explained in terms of mixed tensor fields, whose relation with Lax operators is still unclear (De Filippo, Marmo, Salerno, G. V. 82) .

- Indeed, integrability criteria can be given in terms of invariant mixed tensor field, having 2-d eigenspaces and vanishing Nijenhuis torsion. (De Filippo, Marmo, Salerno, G. V. 82, 84 and 85, Florko, Yanovski 83, Landi, Marmo, G. V. 94)

- The analysis of the integrability realized with the help of a such tensor field leads to the formulation of an integrability criterion (DMSV 82,84, DSV85, LMV94) which, for finite dimensional systems, is *essentially* equivalent to the classic Liouville theorem.

- The mentioned *essential equivalence* means that the equivalence holds for *non resonant* Hamiltonian systems. This is the case when the number of first integrals, defined on the entire phase space, is larger than one half of the phase space dimension. The prototype is Kepler dynamics which, however, is bihamiltonian and has a recursion operator with the right properties (Marmo and G.V. 92).

- The analysis of the integrability realized with the help of a such tensor field leads to the formulation of an integrability criterion (DMSV 82,84, DSV85, LMV94) which, for finite dimensional systems, is *essentially* equivalent to the classic Liouville theorem.

- The mentioned *essential equivalence* means that the equivalence holds for *non resonant* Hamiltonian systems. This is the case when the number of first integrals, defined on the entire phase space, is larger than one half of the phase space dimension. The prototype is Kepler dynamics which, however, is bihamiltonian and has a recursion operator with the right properties (Marmo and G.V. 92).

- The analysis of symmetries shows that generally one is faced with a non Abelian algebra corresponding, for Hamiltonian systems, to a non Abelian algebra of first integrals (Krasil'shchik, Lichagin and Vinogradov 86). The integrability of such systems, with finitely many degrees of freedom, has been analyzed in several papers (Mishenko and Fomenko 78).

- However, there exist field dynamics, related to vector and matrix nonlinear Schroedinger equation (Kulish 85, Gerdjikov 94), possessing a noncommutative set of first integrals, and for them it is useful to have a noncommutative integrability criterion formulated in terms of a recursion operator.

- The analysis of symmetries shows that generally one is faced with a non Abelian algebra corresponding, for Hamiltonian systems, to a non Abelian algebra of first integrals (Krasil'shchik, Lichagin and Vinogradov 86). The integrability of such systems, with finitely many degrees of freedom, has been analyzed in several papers (Mishenko and Fomenko 78).

- However, there exist field dynamics, related to vector and matrix nonlinear Schroedinger equation (Kulish 85, Gerdjikov 94), possessing a noncommutative set of first integrals, and for them it is useful to have a noncommutative integrability criterion formulated in terms of a recursion operator.

The best known criterion of integrability goes back to Liouville

LIOUVILLE

If on a $2n$-d symplectic manifold $\mathcal{M}$ are defined a Hamiltonian dynamics and n functionally independent first integrals $f_1, \ldots, f_n$ in involution $\{f_i, f_j\} = 0 \quad \forall i, j = 1, \ldots, n$, whose associated Hamiltonian fields X_i are complete, then the level manifolds

$$\mathcal{M}_{f(\pi)} = \{p \in \mathcal{M} : f_i(p) = \pi_i\,,\ \ i = 1, \ldots, n\}\,,$$

are invariant with respect to the dynamics and each of their connected components is diffeomorphic either to $T^m \times \Re^{n-m}$ or, if compact, to a torus T^n. Moreover, for every point $p \in \mathcal{M}$ near which m is constant, there exists a neighborhood $\mathcal{U}$ invariant under the composed flow of the vector fields X_i, and canonical coordinates $\left(P_1, \cdots, P_n, Q^1, \cdots, Q^n\right)$, where $Q^1, \cdots, Q^m$ are angles, such that the equation of the motion take the form:

$$\dot{P}_i = 0, \quad \dot{Q}^i = \nu^i(P)\,, \quad 1 \leq\ i \leq n.$$

Invariance

Because of the bi-Hamiltonian structure of (some) evolution equations, the first relevant property is the existence of an invariant the tensor field T

$$L_{\triangle} T = 0$$

This characterization of the dynamics is very suggestive because of the similitude

Dynamics		**Invariant structure**
Symplectic	ω	*a not degenerate, skewsymmetric, closed* $\binom{0}{2}$ *tensor field*
Geodesical	Γ	*a connection 2-form*
Killing	g	*a symmetric, not degenerate* $\binom{0}{2}$ *tensor field*
Hamiltonian	Λ	*a skewsymmetric* $\binom{2}{0}$ *tensor field, fulfilling Jacobi Identity*
Liouville	Ω	*a volume form*
Lax	T	*a* $\binom{1}{1}$ *tensor field with vanishing torsion*

Vanishing of Nijenhuis torsion

The second relevant property, coming by the Lenard sequence, is $\delta(\check{T}^n\alpha) \equiv 0$ if α is δ-closed and δ_T-closed, *i.e.*, if $\delta\alpha = 0$ and $\delta(\check{T}\alpha) = 0$.
Such a property is ensured by

$$\mathcal{N}_T(\alpha, X, Y) = 0$$

where

$$\mathcal{N}_T(\alpha, X, Y) \equiv \langle \alpha, \mathcal{H}_T(X, Y) \rangle$$

and

$$\mathcal{H}_T(X, Y) \equiv \left[(L_{\hat{T}X} T)^\wedge - \hat{T}(L_X T)^\wedge \right] Y$$

Invariance

Because of the bi-Hamiltonian structure of (some) evolution equations, the first relevant property is the existence of an invariant the tensor field T

$$L_{\triangle} T = 0$$

This characterization of the dynamics is very suggestive because of the similitude

Dynamics		**Invariant structure**
Symplectic	ω	*a not degenerate, skewsymmetric, closed* $\binom{0}{2}$ *tensor field*
Geodesical	Γ	*a connection 2-form*
Killing	g	*a symmetric, not degenerate* $\binom{0}{2}$ *tensor field*
Hamiltonian	Λ	*a skewsymmetric* $\binom{2}{0}$ *tensor field, fulfilling Jacobi Identity*
Liouville	Ω	*a volume form*
Lax	T	*a* $\binom{1}{1}$ *tensor field with vanishing torsion*

Vanishing of Nijenhuis torsion

The second relevant property, coming by the Lenard sequence, is $\delta(\check{T}^n \alpha) \equiv 0$ if α is δ-closed and δ_T-closed, *i.e.*, if $\delta\alpha = 0$ and $\delta(\check{T}\alpha) = 0$.
Such a property is ensured by

$$\mathcal{N}_T(\alpha, X, Y) = 0$$

where

$$\mathcal{N}_T(\alpha, X, Y) \equiv \langle \alpha, \mathcal{H}_T(X, Y)\rangle$$

and

$$\mathcal{H}_T(X, Y) \equiv \left[(L_{\hat{T}X} T)^\wedge - \hat{T}(L_X T)^\wedge \right] Y$$

Bidimensionality of eigenspaces of T

Since T is a $(1,1)$-tensor field, we can put a corresponding eigenvalue problem for the associated endomorphism $\check{T}$ on $\Lambda(\mathcal{M})$:

$$\check{T}G_\lambda = \lambda G_\lambda .$$

It is not difficult to see that for each λ there exist two (generalized) eigenvectors, namely G_λ^1 , G_λ^2 such that

$$\check{T}G_\lambda^1 = \lambda G_\lambda^1, \qquad \check{T}G_\lambda^2 = \lambda G_\lambda^2 + G_\lambda^1,$$

this corresponding to Jordan's normal form for a finite matrix.
Explicitly, we have:

$$G_\lambda^1 = e^{2\ell j}[f_2(ik_j, x)]^2, \qquad G_\lambda^2 = e^{2\ell j}\frac{\partial}{\partial k_j}[f_2(ik_j, x)]^2$$

where $f(k,x)$ are the Jost solutions of the Lax operator L

$$L^2 f = -k^2 f, \qquad k^2 = -\lambda.$$

Outline

A more general setting for the commutative integrability is

LMV

Let $\mathcal{M}$ be a a smooth $2n$-d manifold $\mathcal{M}$, where n vector fields $X_1, \ldots, X_n \in \mathcal{X}(\mathcal{M})$ and n functions $f_1, \ldots, f_n \in \mathcal{F}(\mathcal{M})$ exist:

$$[X_i, X_j] = 0 \ ,$$
$$L_{X_i} f^j = 0 \ . \quad i, j \in \{1, \ldots, n\} \ .$$

Then, a dynamical system Δ on $\mathcal{M}$ which is of the form

$$\Delta = \sum_{i=1}^{n} \nu^i X_i \ , \quad \nu^i = \nu^i(f^1, \ldots f^n) \ ,$$

is completely integrable in any open dense submanifold where

$$X_1 \wedge \cdots \wedge X_n \neq 0 \ ,$$
$$df^1 \wedge \cdots \wedge df^n \neq 0 \ .$$

If the fields X_i are complete, by using the n-functions $f^1, \ldots, f^n$, a family of symplectic structures can defined with respect to which the dynamics is Hamiltonian.R

An alternative integrability criterium, suggested by the analysis of integrable models in field theory, can be formulated (DMSV82, DMSV84, DSV85) using invariant tensor fields and it reads:

DMSV

Let Δ be a dynamical vector field on a differential manifold $\mathcal{M}$ which admits a mixed tensor field T which

- *is invariant*
$$L_\Delta T = 0,$$
- *has a vanishing Nijenhuis torsion*
$$\mathcal{N}_T = 0,$$
- *is diagonalizable with doubly degenerate eigenvalues λ_j whose differentials $d\lambda_j$ are independent at each point*

Then, the vector field Δ is separable, completely integrable and Hamiltonian.

Gaetano Vilasi
Università degli Studi di Salerno, Italy
Istituto Nazionale di Fisica Nucleare, Italy

- The Hamiltonian character of the dynamics Δ is not assumed *a priori* but it follows from the properties of the tensor field T, so that all dynamics, satisfying the given hypotheses, result to be Liouville integrable.

- Integrability of dissipative dynamics can be put in the same setting by assuming different spectral hypothesis for the tensor field T.

- The last formulation has the advantage of being more appropriate to deal with dynamics with infinitely many degrees of freedom (completely integrable field theories). We also observe that the Lax Representation, the powerful integration tool for such systems, may not be useful in more than one space dimension since the inverse problem in Quantum Mechanics has been solved only for 1-dimensional systems.

- The Hamiltonian character of the dynamics Δ is not assumed *a priori* but it follows from the properties of the tensor field T, so that all dynamics, satisfying the given hypotheses, result to be Liouville integrable.

- Integrability of dissipative dynamics can be put in the same setting by assuming different spectral hypothesis for the tensor field T.

- The last formulation has the advantage of being more appropriate to deal with dynamics with infinitely many degrees of freedom (completely integrable field theories). We also observe that the Lax Representation, the powerful integration tool for such systems, may not be useful in more than one space dimension since the inverse problem in Quantum Mechanics has been solved only for 1-dimensional systems.

- The Hamiltonian character of the dynamics Δ is not assumed *a priori* but it follows from the properties of the tensor field T, so that all dynamics, satisfying the given hypotheses, result to be Liouville integrable.

- Integrability of dissipative dynamics can be put in the same setting by assuming different spectral hypothesis for the tensor field T.

- The last formulation has the advantage of being more appropriate to deal with dynamics with infinitely many degrees of freedom (completely integrable field theories). We also observe that the Lax Representation, the powerful integration tool for such systems, may not be useful in more than one space dimension since the inverse problem in Quantum Mechanics has been solved only for 1-dimensional systems.

The Burger Hierarchy

An instance of a dynamics which admits an invariant mixed tensor field T which satisfies Nijenhuis condition, but which is not diagonalizable without complexification and whose eigenvalues are trivially constant, is given by Burger's equation. It can be linearized through the Hopf-Cole transformation $u = v_x/v$, where v satisfies the heat equation $v_t = v_{xx}$. Burger's equation is a member of a whole hierarchy of nonlinear evolution equations which linearize to equations of the type $v_t = D^n v \quad n = 1, 2, \ldots,$ D denoting x-derivative. The even elements of sequence obviously define dissipative dynamics, while the odd ones are integrable Hamiltonian evolution equations with respect to the following symplectic form:

$$\omega = \int_{-\infty}^{+\infty} \delta_1 v(x)(D^{-1}\delta_2 v)(x)dx, \qquad (D^{-1}f)(x) = \int_{-\infty}^{+\infty} f(y)dy,$$

with Hamiltonian functionals given by

$$H_p = \frac{1}{2}\int_{-\infty}^{+\infty} (D^p v)^2 dx.$$

In order that previous equations make sense, some assumptions on the functional space $\mathcal{M}$ must be made, for example that $\mathcal{M}$ consists of fast-decreasing infinitely differentiable functions. Then clearly

$$T[v] = D$$

is a Nijenhuis Δ-invariant tensor operator for heat equation hierarchy. In the present geometrical approach, Hopf-Cole transformation plays the role of a coordinate transformation and thus a Nijenhuis Δ invariant tensor operator for Burgers's hierarchy is readily obtained from $\hat{T}[v]$ by,

$$T[u] = \left(\frac{\delta v}{\delta u}\right)^{-1} T[v] \left(\frac{\delta v}{\delta u}\right)$$

which easily yields

$$T[u] = D + DuD^{-1}.$$

Burgers hierarchy is then obtained by repeated applications, on the translation group generator $\Delta_0 = u_x$, of the tensor operator $T[u]$

$$\Delta_k = T^k \Delta_0$$

The first fields of the hierarchy are

$$\begin{aligned} \Delta_0 &= u_x \\ \Delta_1 &= 2uu_x + u_{xx} \\ \Delta_2 &= (3u^3 + 3uu_x + u_{xx})_x \quad . \end{aligned}$$

This hierarchy is just the transcription in the new coordinate frame of the linear one and, apart some technical points on the phase manifold $\mathcal{M}$, one can translate what has been said for heat hierarchy to the Burgers hierarchy.

Liouville integrability *vs* invariant mixed tensor fields

How to construct invariant mixed tensor fields, with the appropriate properties (also called a *recursion tensor field*), for a given Liouville's integrable Hamiltonian dynamics Δ?
If H is the Hamiltonian function and $\{\cdot,\cdot\}$ denotes the Poisson bracket, we have

$$\Delta f = \{H, f\} \ .$$

Let us introduce in some neighborhood of a Liouville's torus T^n action-angle coordinates $\left(J_1, ..., J_n, \varphi^1, ..., \varphi^n\right)$, in which we have:

$$\omega = \sum_h dJ_h \wedge d\varphi^h, \qquad \Delta = \frac{\partial H}{\partial J_h}\frac{\partial}{\partial \varphi^h} \ .$$

Let us distinguish two cases:

- The Hamiltonian H is separable

$$H = \sum_k H_k(J_k) \quad .$$

In this case a class of recursion tensor fields can be easily defined

$$T = \sum_h \lambda_h(J_h)(dJ_h \otimes \frac{\partial}{\partial J_h} + d\varphi^h \otimes \frac{\partial}{\partial \varphi^h}) \quad ,$$

with the λ's arbitrary and functionally independent. Indeed, the tensor field T is invariant and has vanishing Nijenhuis torsion and doubly degenerate eigenvalues.

- The Hamiltonian has a non vanishing Hessian:

$$\det\left(\frac{\partial^2 H}{\partial J_h \partial J_k}\right) \neq 0$$

In this case, in the chosen neighborhood, setting

$$\nu^h(J) = \frac{\partial H}{\partial J_h},$$

new coordinates (ν/φ) can be introduced, so that the dynamics can be described, with respect to the new symplectic structure

$$\omega_1 = \sum_h d\nu^h \wedge d\varphi^h = \sum_{hk} \frac{\partial^2 H}{\partial J_h \partial J_k} dJ_k \wedge d\varphi^h,$$

by a separable Hamiltonian function:

$$H_1 = \frac{1}{2} \sum_h (\nu^h)^2 \quad .$$

As before, a class of recursion tensor fields is then given by

$$T = \sum_h \lambda_h(\nu^h)(d\nu^h \otimes \frac{\partial}{\partial \nu^h} + d\varphi^h \otimes \frac{\partial}{\partial \varphi^h}) \quad .$$

Outline

If the number of independent first integrals is larger than half the dimension of the symplectic manifold, they cannot be in involution anymore and one will have to deal with non commuting sets of first integrals. For a finite number of degrees of freedom a non commutative generalization of Liouville theorem is the following:

MISHENKO E FOMENKO I

A Hamiltonian vector field on a symplectic manifold $(\mathcal{M}, \omega)$ having a noncommutative Lie algebra $\mathcal{A}$ of first integrals satisfying the condition

$$\dim \mathcal{A} + \mathrm{rank}\mathcal{A} = \dim \mathcal{M},$$

is completely integrable, i.e. the joint level surfaces of the first integrals are invariant, and in a neighborhood of each invariant surface one can define canonical coordinates $(\lambda/\chi/p/q)$, the χ's being the coordinates on the invariant surfaces, such that Hamilton's equations take the form

$$\dot{\lambda}_i = 0,\ \dot{\chi}^i = \nu_i,\ \dot{p}_\alpha = 0,\ \dot{q}^\alpha = 0,\quad 1 \leq\ i \leq r, \qquad r+1 \leq \alpha \leq n,$$

with $r = \mathrm{rank}\mathcal{A}$. If these invariant surfaces are compact and connected one can prove, as in the commutative case, that they are tori, and the χ's can be chosen to be angle variables. The canonical coordinates are called, in this case, "generalized action-angle variables".

MISHENKO E FOMENKO II

If $\mathcal{M}$ is compact, then, under the hypotheses of the previous theorem, one can find $n = \frac{1}{2} \dim \mathcal{M}$ first integrals which are in involution.

Even in this case, however, the noncommutative theorem, showing the full symmetry of the system, remains of interest.
In the commutative case the level surfaces of the first integrals f_i define an invariant Lagrangian foliation $\mathcal{F}_1$ of $\mathcal{M}$. The Hamiltonian vector fields X_i associated to the functions f_i are then a basis of commuting tangent vector fields for the leaves and can be used to define local coordinates χ^i on the leaves. These fields also commute with the Hamiltonian vector field Δ which, consequently, can be expressed as $\Delta = \nu^i(f) X_i$. In a neighborhood of a point $p \in \mathcal{M}$, the set (χ/f) define canonical coordinates and Hamilton's equations of motion take the simple following form:

$$\dot{\chi}^i = \nu^i, \quad \dot{f}_i = 0 .$$

In the noncommutative case the first integrals $f_a, \quad 1 \leq a \leq 2n - r$, still define an invariant foliation, but the leaves now have dimension $r \leq n$ and the Hamiltonian vector fields X_a, associated with the first integrals f_a, are not all tangent to the leaves. However, the condition $\dim \mathcal{A} + \mathrm{rank}\mathcal{A} = \dim \mathcal{M}$ ensures, for each leaf l, the existence of a subalgebra $\mathcal{A}_l$ which commutes with $\mathcal{A}$ on l.

To obtain a set of canonical coordinate, in a neighborhood of a point of l and eventually of the whole of l, one needs to exploit further properties of this isotropic foliation. At each point p of l consider the subspace $T_p l \subseteq T_p \mathcal{M}$ and the resulting distribution of symplectically orthogonal subspaces $(T_p l)^{\perp}$. Since $\omega\left(\overline{X_i}, X_a\right)\Big|_l = 0$, this distribution is generated, for all leaves, by the vector fields X_a, and, furthermore, since X_a satisfy the hypotheses of the Frobenius theorem, we obtain a second coisotropic foliation $\mathcal{F}_2$ whose leaves are themselves foliated by those of the first foliation $\mathcal{F}_1$. The regularity of this foliation follows from the independence of the functions f_a. One can now prove the existence of canonical coordinates $\left(\lambda_i, \chi^i, p_\alpha, q^\alpha\right)$, such that the symplectic structure and the dynamical vector field take the following form

$$\omega = d\lambda_i \wedge d\chi^i + dp_\alpha \wedge dq^\alpha, \quad \Delta = \nu^i(\lambda) X_i$$

so that the equations of motion become

$$\dot{\lambda}_i = 0, \; \dot{\chi}^i = \nu^i, \; \dot{p}_\alpha = 0, \; \dot{q}^\alpha = 0.$$

The functions λ_i describe locally $\mathcal{F}_2$, and their associated Hamiltonian vector fields X_i define coordinates χ^i on $\mathcal{F}_1$. The fields X_i are independent and, since $\omega\left(X_i, X_a\right) = d\lambda_i\left(X_a\right) = 0$, they are tangent to the leaves of $\mathcal{F}_1$, and thus commute among themselves and with Δ. To understand better this canonical coordinates, one can actually observe that the momentum map $J : \mathcal{M} \to \mathcal{A}^*$ defined by $J : x \to \xi_x \in \mathcal{A}^*$ where $\xi_x(f) \equiv f(x)$, $f \in \mathcal{A}$, defines a fibration of a neighborhood $\mathcal{U}$ of a leaf of $\mathcal{F}_2$ with fiber $l_x = J^{-1}\left(\xi_x\right)$, namely a leaf of $\mathcal{F}_1$. The neighborhood $\mathcal{U}$ can then be represented as $l_x \times S \times \mathcal{O}$, where $\mathcal{O}$ is the coadjoint orbit through ξ_x of the Lie group corresponding to $\mathcal{A}$ and S is a linear manifold transverse to $\mathcal{O}$. The symplectic structure ω restricted to $\mathcal{O}$ coincides with the Lie-Kirillov-Kostant-Souriau symplectic form; $\left(p_\alpha, q^\alpha\right)$ are canonical coordinates on $\mathcal{O}$ and λ_i coordinates on S. It has been actually proved (Fasso and Ratiu, 98) that all what is needed for the existence of such local canonical coordinates is the double foliation, namely that $\mathcal{M}$ has an isotropic foliation such that the distribution of subspaces, symplectically orthogonal to the tangent spaces to its leaves, is integrable.

Noncommutative integrability and invariant tensor field

SV

Let Δ be a dynamical vector field on a $2n$-differential manifold $\mathcal{M}$ which admits a $(1,1)$ mixed tensor field T which

- *is invariant*

$$L_\Delta T = 0,$$

- *is diagonalizable with* only simple and *doubly degenerate eigenvalues whose differentials are independent at each point* $p \in \mathcal{M}$.
- *has the property*

$$\mathcal{N}_T(\alpha, X, Y) = 0,$$

  $\forall X : X(p) \in S(p)$, $\forall Y \in \mathcal{D}(\mathcal{M})$ *and for all* 1*-forms* α, $S(p)$ *denoting the sum of eigenspaces associated to the doubly eigenvalues* of $T(p)$.

Then, the vector field Δ is separable, completely integrable and Hamiltonian.

Noncommutative integrability *vs* invariant tensor fields.

For a non commutative Mishenko-Fomenko integrable system, we have $\omega = d\lambda_i \wedge d\chi^i + dp_\alpha \wedge dq^\alpha$ and the equations of the motion

$$\dot{\lambda}_i = 0,\ \dot{\chi}^i = \nu_i,\ \dot{p}_\alpha = 0,\ \dot{q}^\alpha = 0,\quad 1 \leq\ i \leq r,\qquad r+1 \leq \alpha \leq n,$$

or, calling μ the collection of the p's and q's, more simply

$$\dot{\lambda}_i = 0,\ \dot{\chi}_i = \nu_i,\ \dot{\mu}_\alpha = 0.$$

It is easily verified that the following tensor field

$$T = \sum_{j=1}^r \lambda_j \left(\frac{\partial}{\partial \lambda_i} \otimes d\lambda_i + \frac{\partial}{\partial \chi^i} \otimes d\chi^i \right) + C_\rho^\sigma\left(\mu\right) \frac{\partial}{\partial \mu_\rho} \otimes d\mu_\sigma .$$

is invariant and, for all diagonalizable matrix $C_\rho^\sigma\left(\mu\right) = \delta_\rho^\sigma \mu_\sigma$, has a vanishing torsion, provided that the Hamiltonian function can be written in the form:

$$H = K_1\left(\lambda\right) + K_2\left(\mu\right),$$

with a separable

$$K_1\left(\lambda\right) = \sum_{i=1}^r H_i\left(\lambda_i\right)$$

If K_1 is not separable but $\det\left(\frac{\partial^2 K_1}{\partial \lambda_j \partial \lambda_i}\right) \neq 0$, the construction of the invariant tensor field follows strictly follows the lines of commutative case.

Gaetano Vilasi
Università degli Studi di Salerno, Italy
Istituto Nazionale di Fisica Nucleare, Italy

This shows that also in the noncommutative case an invariant torsionless tensor field can be always found. Of course, such a tensor field always generates, by repeated application, Abelian algebras of symmetries. Regardless of the vanishing of the torsion on the whole space, the noncommutative features are linked to the non degenerate eigenvalues and, then, are still described by the term $C_{\rho}^{\sigma}(\mu)\frac{\partial}{\partial\mu_{\rho}}\otimes d\mu_{\sigma}$.

Example

A Recursion operator for Kepler dynamics in the commutative case.
The vector field Δ for the Kepler problem, in spherical-polar coordinates, for $\Re^3 - \{0\}$, is Hamiltonian with respect to the symplectic form:
$\omega = \sum_i dp_i \wedge dq^i \quad (i = r, \vartheta, \varphi)$,

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \vartheta}) + V(r), \qquad V(r) = -\frac{k}{r} \tag{1}$$

In action-angle coordinates (J, φ), H, ω and Δ become:

$$H = -mk^2 \left(J_r + J_\vartheta + J_\varphi\right)^{-2} \qquad \omega = \sum_h dJ_h \wedge d\varphi^h \tag{2}$$

$$\Delta = \frac{2mk^2}{(J_r + J_\vartheta + J_\varphi)^3}\left(\frac{\partial}{\partial\varphi^1} + \frac{\partial}{\partial\varphi^2} + \frac{\partial}{\partial\varphi^3}\right)$$

It is globally Hamiltonian also with respect to: $\omega_1 = \sum_{hk} S^h{}_k \, dJ_h \wedge d\varphi^k$

$$S = \frac{1}{2}\left\| \begin{array}{ccc} J_1 & J_2 & J_3 \\ J_2 - J_3 & J_1 + J_3 & J_3 \\ J_3 - J_2 & J_2 & J_1 + J_2 \end{array} \right\|, \quad H_1 = -\frac{mk^2}{(J_r + J_\vartheta + J_\varphi)}$$

$$T = \sum_{hk}(S^h{}_k \, dJ_h \otimes \tfrac{\partial}{\partial J_k} + \left(S^+\right)_h{}^k \, d\varphi^h \otimes \tfrac{\partial}{\partial \varphi^k}) \tag{3}$$

has double degenerate eigenvalues and vanishing Nijenhuis torsion.

Example

A Recursion operator for Kepler dynamics in the non commutative case.
The Kepler dynamics has five first integrals given by the components of the angular momentum and the components of the orthogonal Laplace-Runge-Lenz vector.
In action-angle coordinates (J/φ) such first integrals are given by

$$J_1, J_2, J_3, \varphi_1 - \varphi_2, \varphi_2 - \varphi_3.$$

By using the Delauney action-angle coordinates

$$\begin{aligned}
I_1 &= J_1 + J_2 + J_3 \equiv \lambda_1 \\
I_2 &= J_2 + J_3 \equiv \mu_3 \\
I_3 &= J_3 \equiv \mu_4 \\
\alpha_1 &= \varphi_1 \equiv \chi_1 \\
\alpha_2 &= \varphi_2 - \varphi_1 \equiv \mu_5 \\
\alpha_3 &= \varphi_3 - \varphi_2 \equiv \mu_6,
\end{aligned}$$

$$T = \lambda_1 \left(\frac{\partial}{\partial \lambda_1} \otimes d\lambda_1 + \frac{\partial}{\partial \chi_1} \otimes d\chi_1 \right) + \sum_{\alpha=3}^{6} \mu_\alpha \frac{\partial}{\partial \mu_\alpha} \otimes d\mu_\alpha .$$

WHAT ABOUT LAX REPRESENTATION?

Example

Einstein metrics invariant for a Lie algebra of Killing vector fields generating a 2 dimensional distribution $\mathcal{D}$ can be classified according to properties of the orthogonal distribution $\mathcal{D}^{\perp}$.

Let g be a metric on a manifold and $\mathcal{H}$ be a Killing algebra of g with 2-d orbits. Then dim $\mathcal{H} = 2$, or 3. If dim $\mathcal{H} = 3$, $\mathcal{H}$ is isomorphic either to $\mathcal{A}_2$, or to $\mathcal{G}_2$. Restrictions of the considered metric to the Killing leaves (orbits) are nondegenerate iff the distribution orthogonal to them is transversal to them. so, the geometrical structure to be studied first is a nonintegrable bidimensional distribution on a $4-fold$ manifold M together with a bidimensional algebra of vector fields of its symmetries whose orbits (leaves) are transversal to the distribution. These structures have been completely described and this allows us to construct a privileged local chart in which the equation $Ric(g) = 0$ is more easily studied. All the possible situations, corresponding to a 2−dimensional Lie algebra of isometries, are described by the following table

	$\mathcal{D}^{\perp}$, $r = 0$	$\mathcal{D}^{\perp}$, $r = 1$	$\mathcal{D}^{\perp}$, $r = 2$
$\mathcal{G}_2$	***integrable***	***integrable***	***integrable***
$\mathcal{G}_2$	***semi − integrable***	***semi − integrable***	*semi − integrable*
$\mathcal{G}_2$	***non − integrable***	***non − integrable***	*non − integrable*
$\mathcal{A}_2$	***integrable***	***integrable***	*integrable*
$\mathcal{A}_2$	*semi − integrable*	*semi − integrable*	***semi − integrable***
$\mathcal{A}_2$	*non − integrable*	*non − integrable*	*non − integrable*

,

where a non integrable 2-dimensional distribution which is part of a 3-dimensional integrable distribution has been called *semi-integrable* and in which the cases indicated with bold letters have been essentially solved.

The Ricci tensor field components in the Abelian case

- **Components** $R_{ij} = S_{ij} + T_{ij}$:

$$S_{ii} = \frac{1}{4} tr\left[\mathbf{H}^{-1}\partial_i\left(\mathbf{H}\right)\mathbf{H}^{-1}\partial_i\left(\mathbf{H}\right)\right] + \partial_i^2\left(\ln\alpha\right) - \partial_i\left(\ln\alpha\right)\partial_i\left(\ln f\right)$$

$$S_{12} = \frac{1}{4} tr\left[\mathbf{H}^{-1}\partial_1\left(\mathbf{H}\right)\mathbf{H}^{-1}\partial_2\left(\mathbf{H}\right)\right] + \partial_1\partial_2\left(\ln\alpha f\right)$$

$$T_{ii} = 0$$

$$T_{12} = -\frac{\bar{s}^2 h_{11}}{2f}\phi_{,p}^2 = -\frac{\bar{s}^2 h_{11}}{4f}\left(\psi_{,1}^2 + \psi_{,2}^2 - 2\psi_{,1}\psi_{,2}\right)$$

- **Components** $R_{ab} = S_{ab} + T_{ab}$:

$$\left(S_{ab}\right) = \frac{\mathbf{H}}{2f\alpha}\left[\partial_2\left(\alpha\mathbf{H}^{-1}\partial_1\left(\mathbf{H}\right)\right) + \partial_1\left(\alpha\mathbf{H}^{-1}\partial_2\left(\mathbf{H}\right)\right)\right]$$

$$T_{ab} = \frac{\bar{s}^2 h_{1a-2}h_{1b-2}}{2f^2}\phi_{,p}^2 = \frac{\bar{s}^2 h_{1a-2}h_{1b-2}}{4f^2}\left(\psi_{,1}^2 + \psi_{,2}^2 - 2\psi_{,1}\psi_{,2}\right)$$

- **Components** $R_{aj} = S_{ai} + T_{ai}$:

$$S_{ai} = 0$$

$$T_{ai} = -\frac{\bar{s}}{2\sqrt{2}f}h_{1a-2}\left(\psi_{,1} - \psi_{,2}\right)\partial_i\left(\ln\left|\frac{f}{\alpha\bar{s}\left(\psi_{,1} - \psi_{,2}\right)h_{3a}}\right|\right)$$

The Abelian integrable case

In the Abelian integrable case main equations

$$\partial_2 \left(\alpha \mathbf{H}^{-1} \partial_1 (\mathbf{H})\right) + \partial_1 \left(\alpha \mathbf{H}^{-1} \partial_2 (\mathbf{H})\right) = 0$$

have a *generalized Lax form* and have been integrated by Belinski and Zakharov by using the Inverse Scattering Method.

But, what about the Recursion Operator?

Burgers hierarchy splits into two sub hierarchies

- *Dissipative hierarchy*

$$T\Delta_0, T^3\Delta_0, \ldots, T^{2n+1}\Delta_0, \ldots$$

- *Hamiltonian hierarchy*

$$\Delta_0, T^2\Delta_0, \ldots, T^{2n}\Delta_0, \ldots$$

which are, respectively, a sequence of dissipative and Hamiltonian vector fields. The foregoing statement can be understood by examining the spectral properties of T, whose *block diagonal form* is

$$T = \int dk \left(e_{(k)} \otimes \vartheta'^k - e'_k \otimes \vartheta^k \right) k \quad ,$$

The vector fields

$$e_{(k)}[u] = \int_{-\infty}^{\infty} dx(-u\cos kx - k\sin kx) exp\left[- \int_{-\infty}^{x} u \ \ dy \right] \frac{\delta}{\delta u(x)}$$

$$e'_{(k)}[u] = \int_{-\infty}^{\infty} dx(-u\sin kx + k\cos kx) exp\left[- \int_{-\infty}^{x} u \ dy \ \right] \frac{\delta}{\delta u(x)} \quad ,$$

are a basis of a generic invariant subspace:

$$Te_{(k)} = -ke'_{(k)}, \qquad Te'_{(k)} = ke_{(k)},$$

$$\left\langle \vartheta'^{(k)}, e'_{(k)} \right\rangle = \left\langle \vartheta^{(k)}, e_{(k)} \right\rangle = \delta(h-k), \quad \left\langle \vartheta'^{(k)}, e_{(h)} \right\rangle = \left\langle \vartheta^{(k)}, e'_{(h)} \right\rangle = 0.$$

The conditions

$$[e_{(k)}, e_{(h)}] = [e'_{(k)}, e'_{(h)}] = [e'_{(k)}, e_{(h)}] = 0$$

imply the holonomicity of the frame, *i.e.*, the existence of coordinates $(q^{(k)}, p^{(k)})$ such that:

$$e_{(k)} = \frac{\delta}{\delta q^{(k)}}, \quad e'_{(k)} = \frac{\delta}{\delta p^{(k)}}.$$

In the bidimensional integral manifold of $\{e_{(k)}, e'_{(k)}\}$, the operator T can be projected to:

$$\delta\varphi^{(k)} \otimes \frac{\delta}{\delta J^{(k)}} - \delta J^{(k)} \frac{\delta}{\delta \varphi^{(k)}} \qquad \textit{no sum over } k \quad ,$$

where

$$J^{(k)} = \frac{1}{2}\left(q^{(k)2} + p^{(k)2}\right) \quad ; \quad \varphi^{(k)} = \arctan \frac{q^{(k)}}{p^{(k)}}$$

are action-angle type variables. Then, $\hat{T}$ transforms a dissipative integrable field of the type

$$X_D^{(k)} = \Delta(J^{(k)}) \frac{\partial}{\partial J^{(k)}}$$

in a Hamiltonian one

$$X_H^{(k)} = \Delta(J^{(k)}) \frac{\partial}{\partial \varphi^{(k)}}$$

and *vice versa*

This alternating character of T is responsible for the splitting of hierarchy into two subhierarchies. Furthermore, we observe that

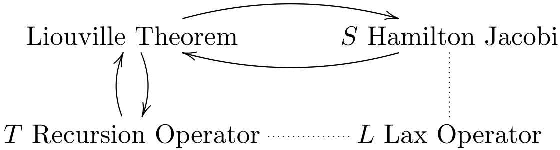
- $\hat{T}$ has a 2d invariant spaces, but is not diagonalizable without complexification.
- $\hat{T}^2$, which characterizes the Hamiltonian subhierarchy, is diagonalizable with doubly degenerate constant eigenvalues.

Thus, for none of the subhierarchies one can use the integrability criterion to establish their integrability. However, we observe that the projections of dissipative dynamics on the bidimensional invariant spaces simply are one degree of freedom dynamics, while for the Hamiltonian ones, the existence of a functional $J^{(k)}[u]$, which is not trivially conserved on each bidimensional space, ensures its integrability. It is worthwhile remarking that this same functional $J^{(k)}[u]$ obviously plays the role of a Ljapunov functional for the projection of the dissipative dynamics on the bidimensional invariant submanifold, thus ensuring the asymptotic stability of the solution $J^{(k)}[u] = 0$.

The Hamiltonian subhierarchy

We discuss in more details the Hamiltonian character of subhierarchy . In order to do that, some care is needed for the appropriate choice of the functional space $\mathcal{M}$ on which dynamics is defined. The most natural one would be to take $\mathcal{M}$ as the functional space whose elements u go to a constant as $x \to \pm\infty$, as it is the space on which there lies the typical solitary wave of Burgers' hierarchy. However, with such a choice it would not be possible to introduce a Hamiltonian structure on $\mathcal{M}$.

This can be understood easily by going back to the linear hierarchy for which $\mathcal{M}$ becomes, *via* the transformation , the space of functions which as $x \to \pm\infty$ behave like $\exp[kx]$ and the Hamiltonian becomes meaningless. One is then tempted to restrict $\mathcal{M}$ in such a way, that both symplectic structures and Hamiltonian one be well defined. This can be accomplished by considering only function $v(x)$ tending to some nonvanishing fixed constants as $x \to \pm\infty$ or, equivalently, functions $u(x)$ vanishing as $x \to \pm\infty$, whose integral has fixed value. More precisely, as for what refers to tangent spaces, the derivative of Hopf-Cole map is a bijection $\delta v \to \delta u$ between $\mathcal{S}(\Re)$, *i.e.,* the space of all *fast decreasing* test functions, and the space of functions which are derivatives of elements of $\mathcal{S}(\Re)$, this ensuring the existence of a symplectic structure with respect to which the subhierarchy is Hamiltonian.


Liouville Theorem
S Hamilton Jacobi
T Recursion Operator
L Lax Operator

Gaetano Vilasi
Università degli Studi di Salerno, Italy
Istituto Nazionale di Fisica Nucleare, Italy

1. AKNS74 M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems.* Stud.Appl.Math., **53** (1974) 249
2. AM78 R. Abraham, J. E. Marsden, *Foundations of Mechanics* (Benjamin/Cummings, Reading, MA, 1978).
3. Ar76 V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Mir, Moscow 1976).
4. AKN88 V. I. Arnold, V.V.Kozlov, A.I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics* , Dynamical Systems III (Springer-Verlag, Berlin 1988).
5. Bo96 O.I. Bogoyavlenskij, *Theory of Tensor invariants of Integrable Hamiltonian Systems. I*, Comm. Math. Phys., **180** (1996)529-586; *Theory of Tensor invariants of Integrable Hamiltonian Systems. II*, Comm. Math. Phys., **184** (1996)301-365.
6. Bo98 O. I. Bogoyavlenskij, *Extended Integrability and biHamiltonian systems*, Comm. Math. Phys., **196**, n.1 (1998)19–51.
7. DMSV82 S. De Filippo, G. Marmo, M. Salerno and G. Vilasi, *On the phase manifold geometry of integrable non-linear field theories*, Preprint IFUSA, Salerno (1982).
8. DMV83 S. De Filippo, G. Marmo and G. Vilasi, *A*