Discreteness, Nonlocality and Causality in NC Field Theory

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Abstract

In noncommutative (NC) field theory space can be rendered discrete in a natural way. Noncommutativity is then traded for nonlocality. A discussion of local and nonlocal oscillations and wave propagation is presented, including the exact solution of the relevant difference equations. The fields remain finite even at the location of the sources; the commutative limit can be taken without problems. The issue of causality is discussed in the continuous representation.

Equation of motion

Consider one (2+1)-dimensional scalar field Φ, depending on space coordinates forming a Heisenberg algebra (time is commutative and remains continuous):

$$\Phi(t, \hat{x}_1, \hat{x}_2), \qquad [\hat{x}_1, \hat{x}_2] = i\theta. \tag{1}$$

The scalar field Φ is a time-dependent operator acting on the Hilbert space $\mathcal{H}$ on which the algebra is represented.

Since $[\hat{x}_1, \phi(\hat{x}_1, \hat{x}_2)] = i\theta \frac{\partial \phi}{\partial \hat{x}_2}$, $[\hat{x}_2, \phi(\hat{x}_1, \hat{x}_2)] = -i\theta \frac{\partial \phi}{\partial \hat{x}_1}$, the field action, written in operatorial form, is

$$S = \int \mathrm{d}t \, \mathrm{Tr}_{\mathcal{H}} \left(\frac{1}{2} \dot{\Phi}^2 + \frac{1}{2\theta^2} [x_i, \Phi]^2 + V(\Phi) \right), \qquad i = 1, 2. \tag{2}$$

Take $V(\Phi) = 0$. The equation of motion for the field Φ is

$$\ddot{\Phi} + \frac{1}{\theta^2} [x_i, [x_i, \Phi]] = 0. \tag{3}$$

In Cartesian coordinates, have plane waves

$$\Phi \sim e^{i(k_1 x_1 + k_2 x_2) - i\omega t}, \quad k_1^2 + k_2^2 = \omega^2 \tag{4}$$

formally identical to the commutative one; in fact (4) has bilocal character [CSA Phys. Rev. D67 (2003) 045020].

Radial symmetry

If require polar coordinates (a source emiting radiation, a circular membrane oscillating), then the oscillator basis $\{|n\rangle\}$

$$N\,|n\rangle = n\,|n\rangle\,, \quad N = \bar{a}a, \quad a = \frac{1}{\sqrt{2\theta}}(x_1 + ix_2), \tag{5}$$

is the natural one and the equation of motion becomes

$$\ddot{\Phi} + \frac{2}{\theta}[a, [\bar{a}, \Phi]] = 0. \tag{6}$$

$N = \frac{1}{2}(\frac{x_1^2+x_2^2}{\theta} - 1)$ is basically the radius square operator, in units of θ.
If $\Phi = \Phi(N)$ - **radial symmetry** - Φ is diagonal in the $|n\rangle$ basis and $\langle n|\,\Phi(t)\,|n\rangle \equiv \Phi_n(t)$ obeys

$$\ddot{\Phi}_n - \frac{2}{\theta}(n\Delta^2\Phi_{n-1} + \Delta\Phi_n) = 0, \quad n = 0, 1, 2, \ldots \tag{7}$$

The discrete derivative operator Δ is defined by

$$\Delta\Phi_n = \Phi_{n+1} - \Phi_n. \tag{8}$$

If assume $\Phi_n(t) \sim e^{i\omega t}$, obtain the difference equation

$$n\Delta^2\Phi_{n-1} + \Delta\Phi_n + \lambda\Phi_n = 0, \quad n = 0, 1, 2, \ldots \tag{9}$$

with $2\lambda/\theta = \omega^2 - m^2$ (mass term reinserted).

Solution of the equation of motion

The difference equation (9) has two linearly independent solutions describing travelling or stationary waves on the semi-infinite discrete space of points $n = 0, 1, 2, \ldots$

Obtain the solutions (up to a multiplicative dimensionfull constant)

$$\Phi_1(n) = \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} \; C_n^k \;, \qquad \Phi_1(0) = 1, \;\; \Phi_1(1) = 1 - \lambda \;, \tag{10}$$

$$\Phi_2(n) = \sum_{k=0}^{n-1} \frac{(-\lambda)^k}{k!} \; \sum_{j=1}^{n-k} \frac{C_{n-j}^k}{k+j} \;, \quad \Phi_2(0) = 0, \;\; \Phi_2(1) = 1 \;. \tag{11}$$

They are *finite sums*. $\Phi_2(n) = e^{-\lambda} \left(\tilde{\Phi}_2(n) - \Phi_1(n) \;\; \cdot \sum_{k=1}^{\infty} \frac{\lambda^k}{k!k} \right)$, where

$$\tilde{\Phi}_2(n) = \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} C_n^k (H_{n-k} - 2H_k) + \frac{(-\lambda)^n}{n!} \sum_{s=1}^{\infty} \frac{\lambda^s (s-1)!}{[(n+s)!/n!]^2}. \tag{12}$$

H_k is a discrete version of the logarithmic function,

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}, \;\; k = 1, 2, 3 \ldots \;\; ; \qquad H_0 = 0. \tag{13}$$

Linear independence

The two solutions are linearly independent, since

$$W(n) \equiv \Phi_1(n+1)\Phi_2(n) - \Phi_1(n)\Phi_2(n+1) = \frac{1}{n+1} \neq 0, \quad \forall n \geq 0. \tag{14}$$

The general solution is thus a linear combination of $\Phi_1(n)$ and $\Phi_2(n)$,

$$\Phi(n) = c_1\Phi_1(n) + c_2\Phi_2(n), \tag{15}$$

with the coefficients $c_{1,2}$ fixed by some physical boundary conditions.

Small distance: no classical divergences

It is worth noting that, in sharp contrast to the commutative case, in which Hankel and Neumann functions are singular at the origin, the functions $\Phi_{1,2}$ are nowhere singular (except when $\theta = 0$). This suggests that, although not finite in quantum perturbation theory, fields defined over noncommutative spaces may not display *classical* divergences. This happens simply because the sources are not localized (also, one has no access to the origin: $r/\sqrt{\theta} = \sqrt{2n+1} \geq 1$). In order to rigorously support such a claim, one has to include sources in the calculation, by solving the inhomogeneous version of equation (9).

Including sources

$$(n+1)\Phi(n+1)-(2n+1-\lambda)\Phi(n)+n\Phi(n-1)=j(n). \tag{16}$$

Consider first a nonzero source $\delta_{n_0,n}$. Adapt the method of variation of constants to the discrete case

$$\Phi_p(n)=c_1(n)\Phi_1(n)+c_2(n)\Phi_2(n). \tag{17}$$

Assuming $c_{1,2}(n)$ constant except for a jump at n_0,

$$c_i(n+1)-c_i(n)=f_1(n)\delta_{n_0,n}, \qquad i=1,2, \tag{18}$$

obtain

$$f_1(n)=\frac{\Phi_2(n)}{(n+1)W(n)}, \quad f_2(n)=-\frac{\Phi_1(n)}{(n+1)W(n)}, \quad \forall n\geq 0. \tag{19}$$

$W(n)$ is the discrete Wronskian defined in Eq. (14), which is nonzero due to the linear independence of Φ_1 and Φ_2. In the physically most interesting case $n_0=0$ the difference equation (16) becomes first-order. The above method works the same, due to the simple Ansatz (18). The solution for an arbitrary distribution of charges $j(n), \forall n$, is now obtained by linear superposition of the above type of solutions. It does not display singularities.

Large distance: commutative limit

Consider the $n \to \infty$ limit (small θ limit). Using $\lambda = \theta\omega^2/2$ and $n = \frac{r^2}{2\theta} \to \infty$, $\Phi_1(n)$ becomes, as a function of r,

$$\Phi_1(n) \overset{n\to\infty}{\to} f_1(r) = \sum_{k=0}^{\infty} \frac{(-1)^k(\omega r)^{2k}}{(k!)^2 2^{2k}} = J_0(\omega r) \overset{r\to\infty}{\sim} \sqrt{\frac{2}{\pi\omega r}} cos(\omega r - \pi/4). \tag{20}$$

$f_1(r)$ is independent of θ. Similarly, Φ_2 becomes

$$\Phi_2(n) \to f_2(r) = \sum_{k=0}^{\infty} \frac{(-1)^k(\omega r)^{2k}}{(k!)^2 2^{2k}} [2ln(\omega r) - 2H_k + \gamma - ln(2\theta\omega^2)]. \tag{21}$$

γ is the Euler-Mascheroni constant, $\gamma = \lim_{k=\infty}(H_k - lnk) \simeq 0.5772$.
$f_2(r)$ still depends on θ, via a logarithmic term; its $\theta \to 0$ limit is singular. Using the series expansion of the Bessel function of first (J_0) and second kind (Neumann function, Y_0), $f_2(r)/\pi = Y_0(\omega r) + (\gamma + ln(2\theta\omega^2))J_0(\omega r)$.

Standing and travelling waves

The $n \to \infty$ limits of $\Phi_1(n)$ and $\Phi_2(n)$ obey the Bessel equation, in agreement with the $n = \frac{r^2}{2\theta} \to \infty$ limit of the difference operator

$$\frac{2}{\theta}(n\Delta^2\Phi_{n-1} + \Delta\Phi_n) \overset{n\to\infty}{\to} \frac{2}{\theta}(n\frac{d^2}{dn^2} + \frac{d}{dn})\Phi(n) \overset{n=\frac{r^2}{2\theta}}{=} (\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr})f(r). \tag{22}$$

Thus, at $r >> \sqrt{\theta}$, NC radial waves behave like commutative ones. Usual standing waves are described by $J_0(r)$, radially expanding ones by the first Hankel function $H_0^1(r) = J_0(r) + iY_0(r)$. The linear combination of $\Phi_1(n)$ and $\Phi_2(n)$ which at $n \to \infty$ tends to $J_0(r)$ will describe standing noncommutative waves (oscillations). This is $\Phi_1(n)$. On the other hand, the function which tends to $H_0^1(r)$ as $r \to \infty$, namely

$$\Phi_3(n) = \Phi_1(n) + \frac{i}{\pi}\left(\Phi_2(n) + [\gamma + ln(\theta\omega^2/2)]\Phi_1(n)\right), \tag{23}$$

represents a travelling radial NC wave propagating outwards towards $n = \infty$. Any solution $\Phi(n)$ of (9) can be written as a linear superposition of $\Phi_1(n)$ and either $\Phi_2(n)$ or $\Phi_3(n)$, with coefficients determined by the boundary conditions one wishes to impose.

No radial symmetry - Bilocal waves

We encountered only Bessel functions of zero order since the angular dependence of Φ is lost if it depends only on the "radius squared" N, $\Phi(N)$, and not on the angle θ. If $\Phi(\hat{x},\hat{y}) = \Phi(\hat{N}, "\hat{\theta}")$ however, $\langle n'|\,\Phi\,|n\rangle \equiv \Phi(n,n') \neq 0$ even for $n' \neq n$. Φ becomes bilocal. Define $\Phi_n^{(m)} \equiv \Phi(n',n)$, $m = n' - n > 0$; its classical equation of motion is

$$\sqrt{n+m+1}\sqrt{n+1}\Phi_{n+1}^{(m)} + \sqrt{n+m}\sqrt{n}\Phi_{n-1}^{(m)} + (\lambda - 2n - m - 1)\Phi_n^{(m)} = 0. \tag{24}$$

In the $n \to \infty$ limit, $m << n$, $\frac{n+n'}{2} \sim \frac{r^2}{2\theta}$, $\Phi_n^{(m)} \to f^{(m)}(r)$ obeying

$$\frac{d^2 f^{(m)}}{dr^2} + \frac{1}{r}\frac{df^{(m)}}{dr} + (\lambda - \frac{m^2}{r^2})f^{(m)}(r) = 0 \tag{25}$$

precisely the equation of the m^{th} order Bessel function $J^m(r)$! In fact, the solutions are consistent with that, for instance the first one

$$\Phi_n^{1(m)} = \sum_{k=0}^{n} \frac{(-1)^k \lambda^{k+\frac{m}{2}} \sqrt{(n+m)^{(k+m)} n^{(k)}}}{k!(m+k)!} \to J^m(r). \tag{26}$$

$\Phi_n^{2(m)}$ involves also the higher order Neumann function $Y^m(r)$.

Non-local solutions

More explicitely, the difference equation

$$\sqrt{n+m+1}\sqrt{n+1}\Phi_{n+1}^{(m)} + \sqrt{n+m}\sqrt{n}\Phi_{n-1}^{(m)} + (\lambda - 2n - m - 1)\Phi_n^{(m)} = 0$$

is solved by

$$\Phi_n^{1(m)} = \sqrt{\frac{(n+m)!}{n!}\lambda^m} \sum_{k=0}^{n} \frac{(-\lambda)^k}{(k+m)!} C_n^k \, ,$$

$$\Phi_n^{2(m)} = \sqrt{\frac{(n+m)!}{n!}\lambda^m} \sum_{L=0}^{n-1} (-\lambda)^L \left\{ \sum_{s=1}^{n-L} \frac{(-)^{s-1}(m+s-1)!}{(m+s+L)!} C_{n-L+L}^{s+L} \right\} \, .$$

Finding the second solution through the series expansion method was quite involved. However, no 'smarter' method (generating function, reduction of order, etc.) worked satisfactorily, until now.

For $m \neq 0$ one can rewrite the second solution as

$$\Phi_n^{2(m)} = \sqrt{\frac{(n+m)!}{n!}\lambda^m} \sum_{L=0}^{n-1} (-\lambda)^L \left\{ \sum_{j=0}^{n-L-1} C_{L-1+j}^{L-1} \frac{\left(1 - \frac{(m+L)!(n-j)!}{L!(n-j+m)!}\right)}{m(m+1)\cdots(m+L)} \right\} .$$

For $m = 0$ the second solution becomes

$$\Phi_n^{2(0)} = \sum_{L=0}^{n-1} (-\lambda)^L \left\{ \sum_{s=1}^{n-L} \frac{(-)^{s-1} C_{n-L+L}^{s\ \ +L}}{s(s+1)\cdots(s+L)} = \sum_{j=1}^{n-L} \frac{C_{L+n-L-j}^{L}}{L+j} \right\} \stackrel{!}{\equiv} \Phi_2(n) .$$

At this point, one has all that is needed for

- including sources (easy)
- performing the commutative limit (requires some care)
- solving decay through radiation of field configurations possessing angular momentum (in project)

Continuous representation

Consider again $\Phi(t,\hat{x},\hat{y})$; $[\hat{x},\hat{y}] = i\theta\hat{I}$; $\hat{x}$, $\hat{y}$: $\mathcal{H} \to \mathcal{H}$. Choose the basis $\{|x>\}$ of eigenstates of $\hat{x}$: $\hat{x}|x>= x|x>$, $\hat{y}|x>= -i\theta\frac{\partial}{\partial x}|x>$.

To quantize Φ, promote normal modes expansion coefficients a and a^*. to annihilation/creation operators a, $a^\dagger$ on a standard Fock space $\mathcal{F}$.

To introduce NC space, apply Weyl (not Weyl-Moyal!) quantization to the exponentials $e^{i(k_x x + k_y y)}$ (the normal modes). The result is

$$\Phi = \int\int \frac{dk_x dk_y}{2\pi\sqrt{2\omega_{\vec{k}}}} \left[\hat{a}_{k_x k_y} e^{i(\omega_{\vec{k}} t - k_x\hat{x} - k_y\hat{y})} + \hat{a}^\dagger_{k_x k_y} e^{-i(\omega_{\vec{k}} t - k_x\hat{x} - k_y\hat{y})}\right]. \quad (27)$$

Φ acts on a direct product of two Hilbert spaces, $\Phi : \mathcal{F}\otimes\mathcal{H} \to \mathcal{F}\otimes\mathcal{H}$. Saturate the action of Φ on $\mathcal{H}$ by working with expectation values $< x'|\Phi|x >$: $\mathcal{F} \to \mathcal{F}$. Bilocality appears explicitely due to

$$< x'|e^{i(k_x\hat{x}+k_y\hat{y})}|x >= e^{ik_x(x+k_y\theta/2)}\delta(x'-x-k_y\theta) = e^{ik_x\frac{x+x'}{2}}\delta(x'-x-k_y\theta). \quad (28)$$

The span along the x axis is $(x'-x) = \theta k_y$; the energy is

$$\omega_{\vec{k}} = \sqrt{k_x^2 + \frac{\Delta x^2}{\theta^2} + m^2}. \quad (29)$$

Notice the intrinsic IR/UV-dual character of the dipoles: both big momentum (UV) and big extension (IR) give big energy.

Symmetries

Reintroduce the commutative z direction and use the notation

$$< x'|\phi|x > \equiv \phi(x',x) \equiv \phi(\bar{x},\Delta x), \qquad \bar{x} \equiv \frac{x+x'}{2}, \qquad \Delta x \equiv (x'-x)$$

The free equation of motion for $\phi(t,\hat{x},\hat{y},z)$ follows from the action

$$S = Tr_H \int dt \int dz \left((\dot{\phi})^2 + \frac{1}{\theta^2}[\hat{x},\phi]^2 + \frac{1}{\theta^2}[\hat{y},\phi]^2 - (\partial_z\phi)^2 + m^2\phi^2 \right),$$

and reads $(\partial_t^2 - \partial_z^2 + m^2)\phi + \frac{1}{\theta^2}[\hat{y},[\hat{y},\phi]] + \frac{1}{\theta^2}[\hat{x},[\hat{x},\phi]] = 0$. Sandwiching it between $|x>$ states, one gets rid of NC and obtains the wave equation

$$\left(\partial_t^2 - \partial_{\bar{x}}^2 - \partial_z^2 + \frac{(x'-x)^2}{\theta^2} + m^2 \right) \phi(x,x') = 0$$

for a dipole living in (2+1) commutative dimensions at $t,\bar{x},z$ *and* having extension Δx. Notice the full agreement with the dispersion relation (29). In the interacting case, the relevant Lagrangian is thus

$$2L = (\partial_t\phi)^2 - (\partial_{\bar{x}}\phi)^2 + [(\theta^{-1}\Delta x)^2 + m^2]\phi^2 - 2V(\phi)$$

and is invariant under Lorentz boosts along the $\bar{x}$-axis, and along the z-axis, independently (recall the tensorial character of $\theta = \theta_{xy} \sim xy$ and $\Delta x \sim x$). These bilocal representation symmetries are at variance with the Moyal approach claim $O(2)_{x-y} \otimes O(1,1)_{t-z}$.

Causality

Free NC fields behave like lower-dimensional commutative fields with a modified dispersion relation $\omega^2 = k_x^2 + \frac{(x-x')^2)}{\theta^2}$, hence they are causal. For *interacting* fields, assume the vanishing of the following commutator

$$[\phi(t_1, \bar{x}_1), \phi(t_2, \bar{x}_2)] = 0, \tag{30}$$

with $\bar{x}_1 = \frac{x_1 + x_1'}{2}$, $\bar{x}_2 = \frac{x_2 + x_2'}{2}$ the average positions of the two dipoles considered. We want (30) to be true for a space-like interval

$$(t_1 - t_2)^2 - (\bar{x}_1 - \bar{x}_2)^2 \leq 0. \tag{31}$$

Since one *can* apply a boost along x to render equal the two times appearing in Eq. (30), Eqs. (30, 31) are generically equivalent to

$$[\phi(t, \bar{x}), \phi(t, \bar{y})] = 0, \qquad \vec{x} \neq \vec{y}. \tag{32}$$

In consequence, Eqs. (30, 31) are tantamount, via a boost, to

$$e^{iH't}[\phi(0, \bar{x}), \phi(0, \bar{y})]e^{-iH't} = 0 \tag{33}$$

which is true at $t = 0$, (by definition) the time at which the fields behave like free fields ($H' \equiv V_{I.P.}$). Adding now the (passive) commutative coordinate z, we conclude that the correct causality criterion for NCFT is

$$[\phi(t_1, \bar{x}_1, z_1), \phi(t_2, \bar{x}_2, z_2)] = 0, \qquad (t_1 - t_2)^2 - (\bar{x}_1 - \bar{x}_2)^2 - (z_1 - z_2)^2 \leq 0.$$

Summary

- On the NC plane defined by $[x^1, x^2] = i\theta$, local and non-local waves propagate on a discrete space, given by the eigenvalues $r = \sqrt{2n+1}, n = 0, 1, 2, \ldots$ of the radius square operator. At finite distance, the amplitude of the waves is given by a finite series.
- In the large radius limit, $r >> \sqrt{\theta}$, or $n \to \infty$, the amplitudes become Bessel-type functions, consequently the waves behave like commutative ones.
- At small radius, if $\theta \neq 0$, there are no signs of singularities appearing, even at the location of the sources.
- The degree of non-locality is proportional to the angular momentum of the field configuration.
- 'Residual' Lorentz symmetry persists, involving also one of the NC coordinates. Using this symmetry, a satisfactory causality criterion can be formulated.

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