Simons type formulas for submanifolds with parallel mean curvature in product spaces and applications

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References

- D. Fetcu and H. Rosenberg, *Surfaces with parallel mean curvature in* $\mathbb{S}^3 \times \mathbb{R}$ *and* $\mathbb{H}^3 \times \mathbb{R}$, Michigan Math. J., to appear, `arXiv:math.DG/1103.6254v1`.
- D. Fetcu, C. Oniciuc, and H. Rosenberg, *Biharmonic submanifolds with parallel mean curvature in* $\mathbb{S}^n \times \mathbb{R}$, J. Geom. Anal., to appear, `arXiv:math.DG/1109.6138v1`.
- D. Fetcu and H. Rosenberg, *On complete submanifolds with parallel mean curvature in product spaces*, Rev. Mat. Iberoam., to appear, `arXiv:math.DG/1112.3452v1`.

Using Simons inequalities to study minimal, cmc and pmc submanifolds

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Using Simons inequalities to study minimal, cmc and pmc submanifolds

- 1968 - J. Simons - a formula for the Laplacian of the second fundamental form of a submanifold in a Riemannian manifold
  - for a minimal hypersurface Σ^m in $\mathbb{S}^{m+1}$ this formula is

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + |A|^2(m - |A|^2) \geq |A|^2(m - |A|^2)$$

where ∇ and A are defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \quad \text{and} \quad \bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V$$

- for a minimal submanifold with arbitrary codimension in $\mathbb{S}^n$:

Theorem (Simons - 1968)

Let Σ^m be a closed minimal submanifold in $\mathbb{S}^n$. Then

$$\int_{\Sigma^m} \left(|A|^2 - \frac{m(n-m)}{2n-2m-1}\right)|A|^2 \geq 0.$$

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Corollary

Let Σ^m be a closed minimal submanifold in $\mathbb{S}^n$ with

$$|A|^2\leq\frac{m(n-m)}{2n-2m-1}.$$

Then, either Σ^m is totally geodesic or $|A|^2=\frac{m(n-m)}{2n-2m-1}$.

Definition

If the mean curvature vector field $H = \frac{1}{m}\operatorname{trace}\sigma$ of a submanifold Σ^m in a Riemannian manifold is parallel in the normal bundle, i.e. $\nabla^{\perp} H = 0$, then Σ^m is called a pmc submanifold. If $|H| = \text{constant}$, then Σ^m is a cmc submanifold.

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- 1969 - K. Nomizu, B. Smyth; 1973 - B. Smyth - Simons type formula for cmc hypersurfaces and, in general, pmc submanifolds in a space form
- 1971 - J. Erbacher - Simons type formula for pmc submanifolds in a space form:

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 &= |\nabla^* A|^2 + cm\{|A|^2 - m|H|^2\} \\ &+ \sum_{\alpha,\beta=m+1}^{n+1}\{(\mathrm{trace}\,A_\beta)(\mathrm{trace}(A_\alpha^2 A_\beta)) \\ &+ \mathrm{trace}[A_\alpha, A_\beta]^2 - (\mathrm{trace}(A_\alpha A_\beta))^2\},\end{aligned}$$

- 1977 - S.-Y. Cheng, S.-T. Yau - a general Simons type equation for operators S, acting on a submanifold of a Riemannian manifold and satisfying $(\nabla_X S)Y = (\nabla_Y S)X$
- 1970 - S.-S. Chern, M. do Carmo, S. Kobayashi; 1994 - H. Alencar, M. do Carmo - gap theorems for minimal hypersurfaces and cmc hypersurfaces, respectively, in $\mathbb{S}^n(c)$
- 1994 - W. Santos - a gap theorem for pmc submanifolds in $\mathbb{S}^n(c)$
- other studies on pmc submanifolds in space forms:
  - 1984, 1993, 2005, 2010, 2011 - H.-W. Xu et al.
  - 2001 - Q. M. Cheng, K. Nonaka
  - 2009 - K. Araújo, K. Tenenblat
- 2010 - M. Batista - Simons type formulas for cmc surfaces in $M^2(c) \times \mathbb{R}$

A Simons type formula for submanifolds in $\mathbb{M}^n(c) \times \mathbb{R}$

Theorem (F., Oniciuc, Rosenberg - 2011)

Let Σ^m be a submanifold of $M^n(c) \times \mathbb{R}$, with mean curvature vector field H and shape operator A. If V is a normal vector field, parallel in the normal bundle, with $\operatorname{trace} A_V = \text{constant}$, then

$$\begin{aligned}
\frac{1}{2}\Delta|A_V|^2 \quad = \quad & |\nabla A_V|^2 + c\{(m - |T|^2)|A_V|^2 - 2m|A_V T|^2 \\
& + 3(\operatorname{trace} A_V)\langle A_V T, T\rangle - m(\operatorname{trace} A_V)\langle H, N\rangle\langle V, N\rangle \\
& + m(\operatorname{trace}(A_N A_V))\langle V, N\rangle - (\operatorname{trace} A_V)^2\} \\
& + \textstyle\sum_{\alpha=m+1}^{n+1}\{(\operatorname{trace} A_\alpha)(\operatorname{trace}(A_V^2 A_\alpha)) - (\operatorname{trace}(A_V A_\alpha))^2\},
\end{aligned}$$

where $\{E_\alpha\}_{\alpha=m+1}^{n+1}$ is a local orthonormal frame field in the normal bundle, and T and N are the tangent and normal part, respectively, of the unit vector ξ tangent to $\mathbb{R}$.

Sketch of the proof.

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- Weitzenböck formula: $\frac{1}{2}\Delta|A_V|^2 = |\nabla A_V|^2 + \langle \operatorname{trace} \nabla^2 A_V, A_V \rangle$
- $C(X,Y) = (\nabla^2 A_V)(X,Y) = \nabla_X(\nabla_Y A_V) - \nabla_{\nabla_X Y} A_V$
- consider an orthonormal basis $\{e_i\}_{i=1}^m$ in $T_p\Sigma^m$, $p \in \Sigma^m$, extend e_i to vector fields E_i in a neighborhood of p such that $\{E_i\}$ is a geodesic frame field around p, and denote $X = E_k$

$$(\operatorname{trace} \nabla^2 A_V)X = \sum_{i=1}^m C(E_i, E_i)X.$$

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- Codazzi equation + Ricci formula $\Rightarrow$

$$
\begin{aligned}
C(E_i, E_i)X &= \nabla_X((\nabla_{E_i} A_V)E_i) + [R(E_i, X), A_V]E_i \\
&\quad + c\langle A_V E_i, T\rangle(\langle E_i, T\rangle X - \langle X, T\rangle E_i) \\
&\quad - c\langle V, N\rangle(\langle A_N E_i, E_i\rangle X - \langle A_N X, E_i\rangle E_i)
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- $\nabla_{E_i}A_V$ is symmetric + Codazzi eq. + $\mathrm{trace}\, A_V = \text{constant} \Rightarrow$
  $\sum_{i=1}^m (\nabla_{E_i}A_V)E_i = c(m-1)\langle V,N\rangle T$
-

$$\begin{aligned} R(X,Y)Z = &\ c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,T\rangle\langle Z,T\rangle X + \langle X,T\rangle\langle Z,T\rangle Y \\ &+ \langle X,Z\rangle\langle Y,T\rangle T - \langle Y,Z\rangle\langle X,T\rangle T\} \\ &+ \sum_{\alpha=m+1}^{n+1}\{\langle A_\alpha Y,Z\rangle A_\alpha X - \langle A_\alpha X,Z\rangle A_\alpha Y\}, \end{aligned}$$

- Codazzi equation of Σ^m:
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$$\begin{aligned} C(E_i,E_i)X &= \nabla_X((\nabla_{E_i}A_V)E_i) + [R(E_i,X),A_V]E_i \\ &\quad + c\langle A_V E_i,T\rangle(\langle E_i,T\rangle X - \langle X,T\rangle E_i) \\ &\quad - c\langle V,N\rangle(\langle A_N E_i,E_i\rangle X - \langle A_N X,E_i\rangle E_i) \end{aligned}$$

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- Ricci eq. $\langle R^\perp(X,Y)V,U\rangle = \langle [A_V,A_U]X,Y\rangle + \langle \bar{R}(X,Y)V,U\rangle \Rightarrow$

$$[A_V,A_U] = 0, \forall U \in N\Sigma^m$$

pmc surfaces in $M^3(c) \times \mathbb{R}$

- Let Σ^2 be a non-minimal pmc surface in $M^3(c) \times \mathbb{R}$
- Consider the orthonormal frame field $\{E_3 = \frac{H}{|H|}, E_4\}$ in the normal bundle $\Rightarrow E_4 =$ parallel
- $\phi_3 = A_3 - |H| \mathrm{I}$ and $\phi_4 = A_4$
- $\phi(X,Y) = \sigma(X,Y) - \langle X,Y\rangle H = \langle \phi_3 X, Y\rangle E_3 + \langle \phi_4 X, Y\rangle E_4$
- $|\phi|^2 = |\phi_3|^2 + |\phi_4|^2 = |\sigma|^2 - 2|H|^2$

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Proposition (F., Rosenberg - 2011)

If Σ^2 is an immersed pmc surface in $M^n(c) \times \mathbb{R}$, then

$$\frac{1}{2}\Delta|T|^2 = |A_N|^2 - \frac{1}{2}|T|^2|\phi|^2 - 2\langle \phi(T,T), H \rangle + c|T|^2(1-|T|^2) - |T|^2|H|^2.$$

Theorem (F., Rosenberg - 2011)

Let Σ^2 be an immersed pmc 2-sphere in $M^n(c) \times \mathbb{R}$, such that

1. $|T|^2 = 0$ *or* $|T|^2 \geq \frac{2}{3}$ *and* $|\sigma|^2 \leq c(2 - 3|T|^2)$*, if* $c < 0$*;*
2. $|T|^2 \leq \frac{2}{3}$ *and* $|\sigma|^2 \leq c(2 - 3|T|^2)$*, if* $c > 0$*.*

Then, Σ^2 *is either a minimal surface in a totally umbilical hypersurface of* $M^n(c)$ *or a standard sphere in* $M^3(c)$*.*

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Then, Σ^2 is either a minimal surface in a totally umbilical hypersurface of $M^n(c)$ or a standard sphere in $M^3(c)$.

Proof.

- $Q(X,Y)=2\langle\sigma(X,Y),H\rangle-c\langle X,\xi\rangle\langle Y,\xi\rangle\Rightarrow$
  $Q^{(2,0)}=$ holomorphic

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- assume $|T|\neq 0$ on an open dense set, and consider $\{e_1=T/|T|,e_2\}$
- Σ^2 is a sphere $\Rightarrow Q^{(2,0)}=0\Rightarrow\langle\phi(T,T),H\rangle=\frac{1}{4}c|T|^2\Rightarrow$
- $\frac{1}{2}\Delta|T|^2=|A_N|^2+\frac{1}{2}|T|^2(-|\sigma|^2+c(2-3|T|^2))\geq 0$

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- $\frac{1}{2}\Delta|T|^2=|A_N|^2+\frac{1}{2}|T|^2(-|\sigma|^2+c(2-3|T|^2))\geq 0$
- $K\geq 0\Rightarrow\Sigma^2$ is a parabolic space $\Rightarrow$
  $|T|=\text{constant},\quad A_N=0,\quad\nabla_X T=0\Rightarrow K=0$ (contradiction)
  $\Rightarrow T=0$ (the result then follows from [Yau - 1974])

Proposition (F., Rosenberg - 2011)

If Σ^2 is a non-minimal pmc surface in $M^3(c)\times\mathbb{R}$, then

$$\begin{aligned}\tfrac{1}{2}\Delta|\phi|^2 = \;& |\nabla\phi_3|^2+|\nabla\phi_4|^2-|\phi|^4+\{c(2-3|T|^2)+2|H|^2\}|\phi|^2\\ & -2c\langle\phi(T,T),H\rangle+2c|A_N|^2-4c\langle H,N\rangle^2.\end{aligned}$$

Theorem

Let Σ^2 be a complete non-minimal pmc surface in $M^3(c) \times \mathbb{R}$, $c > 0$. Assume

i) $|\phi|^2 \leq 2|H|^2 + 2c - \frac{5c}{2}|T|^2$, *and*

ii)
   a) $|T| = 0$, *or*
   b) $|T|^2 > \frac{2}{3}$ *and* $|H|^2 \geq \frac{c|T|^2(1-|T|^2)}{3|T|^2-2}$.

Then either

1. $|\phi|^2 = 0$ *and* Σ^2 *is a round sphere in* $M^3(c)$, *or*
2. $|\phi|^2 = 2|H|^2 + 2c$ *and* Σ^2 *is a torus* $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{\frac{1}{c} - r^2})$, $r^2 \neq \frac{1}{2c}$, *in* $M^3(c)$.

Sketch of the proof.

- $$\begin{aligned}\frac{1}{2}\Delta(|\phi|^2 - c|T|^2) &= |\nabla\phi_3|^2 + |\nabla\phi_4|^2 \\ &\quad + \{-|\phi|^2 + \frac{c}{2}(4 - 5|T|^2) + 2|H|^2\}|\phi|^2 \\ &\quad + c|A_N|^2 - 4c\langle H, N\rangle^2 + c|T|^2|H|^2 \\ &\quad - c^2|T|^2(1 - |T|^2)\end{aligned}$$

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- $|A_N|^2 \geq 2\langle H, N\rangle^2$ and $\langle H, N\rangle^2 \leq (1 - |T|^2)|H|^2$
- $\frac{1}{2}\Delta(|\phi|^2 - c|T|^2) \geq \{-|\phi|^2 + 2c + 2|H|^2\}|\phi|^2 \geq 0$, if $T = 0$
- $$\begin{aligned} \tfrac{1}{2}\Delta(|\phi|^2 - c|T|^2) &\geq \{-|\phi|^2 + \tfrac{c}{2}(4 - 5|T|^2) + 2|H|^2\}|\phi|^2 \\ &\quad + c(3|T|^2 - 2)|H|^2 - c^2|T|^2(1 - |T|^2) \\ &\geq 0, \end{aligned}$$

  otherwise

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  otherwise
- $2K = 2c(1 - |T|^2) + 2|H|^2 - |\phi|^2 \geq \frac{1}{2}c|T|^2 \geq 0$ and $|\phi|^2 - c|T|^2$ is bounded and subharmonic $\Rightarrow$

- $|\phi|^2 - c|T|^2 = \text{constant}$ and $\phi = 0$ or $|\phi|^2 = 2|H|^2 + 2c - \frac{5c}{2}|T|^2$ and
  $|A_N|^2 = 2\langle H, N\rangle$, $\langle H, N\rangle^2 = (1 - |T|^2)|H|^2$

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- $A_N = \langle H, N\rangle \mathrm{I}$ and $N \parallel H \Rightarrow A_H = |H|^2 \mathrm{I} \Rightarrow \Sigma^2$ is pseudo-umbilical $\Rightarrow \Sigma^2$ lies in $M^3(c)$

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  $A_N = \langle H, N\rangle \mathrm{I}$ and $N = 0$ or $N \parallel H$
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- $A_N = \langle H, N\rangle \mathrm{I}$ and $N \parallel H \Rightarrow A_H = |H|^2 \mathrm{I} \Rightarrow \Sigma^2$ is pseudo-umbilical $\Rightarrow \Sigma^2$ lies in $M^3(c)$
- in conclusion Σ^2 lies in $M^3(c)$ and the result follows from [Alencar, do Carmo - 1994; Santos - 1994], using $\nabla\phi = 0$.

Another Simons type formula

Proposition (F., Rosenberg - 2011)

Let Σ^m be a pmc submanifold of $M^n(c) \times \mathbb{R}$, with mean curvature vector field H, shape operator A, and second fundamental form σ. Then we have

$$\begin{aligned}
\frac{1}{2}\Delta|\sigma|^2 &= |\nabla^{\perp}\sigma|^2 + c\{(m-|T|^2)|\sigma|^2 - 2m\textstyle\sum_{\alpha=m+1}^{n+1}|A_\alpha T|^2 \\
&\quad +3m\langle\sigma(T,T),H\rangle + m|A_N|^2 - m^2\langle H,N\rangle^2 - m^2|H|^2\} \\
&\quad +\textstyle\sum_{\alpha,\beta=m+1}^{n+1}\{(\operatorname{trace}A_\beta)(\operatorname{trace}(A_\alpha^2 A_\beta)) + \operatorname{trace}[A_\alpha,A_\beta]^2 \\
&\quad -(\operatorname{trace}(A_\alpha A_\beta))^2\},
\end{aligned}$$

where $\{E_\alpha\}_{\alpha=m+1}^{n+1}$ is a local orthonormal frame field in the normal bundle.

Complete pmc submanifolds in product spaces

Case I. pmc submanifolds with dimension higher than 2

Theorem (F., Rosenberg - 2011)

Let Σ^m be a complete non-minimal pmc submanifold in $M^n(c)\times\mathbb{R}$, $n>m\geq 3$, $c>0$, with mean curvature vector field H and second fundamental form σ. If the angle between H and ξ is constant and

$$|\sigma|^2+\frac{2c(2m+1)}{m}|T|^2\leq 2c+\frac{m^2}{m-1}|H|^2,$$

then Σ^m is a totally umbilical cmc hypersurface in $M^{m+1}(c)$.

Theorem (F., Rosenberg - 2011)

Let Σ^m be a complete non-minimal pmc submanifold in $M^n(c) \times \mathbb{R}$, $n > m \geq 3$, $c < 0$, with mean curvature vector field H and second fundamental form σ. If H is orthogonal to ξ and

$$|\sigma|^2 + \frac{2c(m+1)}{m}|T|^2 \leq 4c + \frac{m^2}{m-1}|H|^2,$$

then Σ^m is a totally umbilical cmc hypersurface in $M^{m+1}(c)$.

Case II. pmc surfaces

Theorem (F., Rosenberg - 2011)

Let Σ^2 be a complete non-minimal pmc surface in $M^n(c)\times\mathbb{R}$, $n>2$, $c>0$, such that the angle between H and ξ is constant and

$$|\sigma|^2+3c|T|^2\leq 4|H|^2+2c.$$

Then, either

1. *Σ^2 is pseudo-umbilical and lies in $M^n(c)$; or*
2. *Σ^2 is a torus $\mathbb{S}^1(r)\times\mathbb{S}^1\left(\sqrt{\frac{1}{c}-r^2}\right)$ in $M^3(c)$, with $r^2\neq\frac{1}{2c}$.*

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Let Σ^2 be a complete non-minimal pmc surface in $M^n(c) \times \mathbb{R}$, $n > 2$, $c < 0$, such that H is orthogonal to ξ and

$$|\sigma|^2 + 5c|T|^2 \leq 4|H|^2 + 4c.$$

Then Σ^2 is pseudo-umbilical and lies in $M^n(c)$.

A gap theorem for biharmonic pmc submanifolds in $\mathbb{S}^n \times \mathbb{R}$

Definition

A *harmonic map* $\psi : (M, g) \to (\bar{M}, h)$ *between two Riemannian manifolds is a critical point of the* *energy functional*

$$E(\psi) = \frac{1}{2} \int_M |d\psi|^2 \, v_g.$$

The Euler-Lagrange equation for the energy functional:

$$\tau(\psi) = \operatorname{trace} \nabla d\psi = 0$$

and τ is called the tension field.

Definition

A biharmonic map is a critical point of the bienergy functional

$$E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 \, v_g.$$

If ψ is a biharmonic non-harmonic map, then it is called a proper-biharmonic map.

Theorem (Jiang - 1986)

A map $\psi : (M,g) \to (\bar{M},h)$ is biharmonic if and only if

$$\tau_2(\psi) = \Delta\tau(\psi) - \operatorname{trace}\bar{R}(d\psi,\tau(\psi))d\psi = 0$$

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A submanifold of a Riemannian manifold is called a biharmonic submanifold if the inclusion map is biharmonic.

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A submanifold of a Riemannian manifold is called a biharmonic submanifold if the inclusion map is biharmonic.

Proposition (F., Oniciuc, Rosenberg - 2011)

If Σ^m *is a compact biharmonic submanifold in* $\mathbb{S}^n(c) \times \mathbb{R}$*, then* Σ^m *lies in* $\mathbb{S}^n(c)$*.*

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Theorem (Oniciuc - 2003)

A proper-biharmonic cmc submanifold Σ^m *in* $\mathbb{S}^n(c)$*, with mean curvature equal to* $\sqrt{c}$*, is minimal in a small hypersphere* $\mathbb{S}^{n-1}(2c) \subset \mathbb{S}^n(c)$*.*

Theorem (Balmuş, Oniciuc - 2010)

If Σ^m *is a proper-biharmonic pmc submanifold in* $\mathbb{S}^n(c)$*, with mean curvature vector field* H *and* $m > 2$*, then* $|H| \in \left(0, \frac{m-2}{m}\sqrt{c}\right] \cup \{\sqrt{c}\}$*. Moreover,* $|H| = \frac{m-2}{m}\sqrt{c}$ *if and only if* Σ^m *is* (*an open part of*) *a standard product*

$$\Sigma_1^{m-1} \times \mathbb{S}^1(2c) \subset \mathbb{S}^n(c),$$

where Σ_1^{m-1} *is a minimal submanifold in* $\mathbb{S}^{n-2}(2c)$*.*

Theorem (Balmuş, Montaldo, Oniciuc - 2011)

A submanifold Σ^m in a Riemannian manifold $\bar{M}$ is biharmonic iff

$$\begin{cases} -\Delta^{\perp} H + \operatorname{trace} \sigma(\cdot, A_H \cdot) + \operatorname{trace}(\bar{R}(\cdot, H)\cdot)^{\perp} = 0 \\ \frac{m}{2} \operatorname{grad} |H|^2 + 2 \operatorname{trace} A_{\nabla^{\perp}_{\cdot} H}(\cdot) + 2 \operatorname{trace}(\bar{R}(\cdot, H)\cdot)^{\top} = 0, \end{cases}$$

where $\Delta^{\perp}$ is the Laplacian in the normal bundle and $\bar{R}$ is the curvature tensor of $\bar{M}$.

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Corollary

A pmc submanifold Σ^m in $M^n(c) \times \mathbb{R}$, with $m \geq 2$, is biharmonic iff

$$\begin{cases} H \perp \xi, \quad |A_H|^2 = c(m - |T|^2)|H|^2 \\ \operatorname{trace}(A_H A_U) = 0 \quad \text{for any normal vector} \quad U \perp H. \end{cases}$$

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A pmc submanifold Σ^m in $M^n(c)\times\mathbb{R}$, with $m \geq 2$, is biharmonic iff

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Remark

There are no proper-biharmonic pmc submanifolds in $M^n(c)\times\mathbb{R}$ with $c \leq 0$.

Definition

A submanifold Σ^m *of* $M^n(c) \times \mathbb{R}$ *is called a* ***vertical cylinder*** *over* Σ^{m-1} *if* $\Sigma^m = \pi^{-1}(\Sigma^{m-1})$*, where* $\pi : M^n(c) \times \mathbb{R} \to M^n(c)$ *is the projection map and* Σ^{m-1} *is a submanifold of* $M^n(c)$*.*

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Proposition (F., Oniciuc, Rosenberg - 2011)

Let Σ^m, $m\geq 2$, *be a proper-biharmonic pmc submanifold in* $\mathbb{S}^n(c)\times\mathbb{R}$. *Then* σ *satisfies* $|\sigma|^2\geq c(m-1)$, *and the equality holds if and only if* Σ^m *is a vertical cylinder* $\pi^{-1}(\Sigma^{m-1})$ *in* $\mathbb{S}^m(c)\times\mathbb{R}$, *where* Σ^{m-1} *is a proper biharmonic cmc hypersurface in* $\mathbb{S}^m(c)$.

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Proposition (F., Oniciuc, Rosenberg - 2011)

Let Σ^m*,* $m\geq 2$*, be a proper-biharmonic pmc submanifold in* $\mathbb{S}^n(c)\times\mathbb{R}$*. Then* $|H|^2\leq c$*, and the equality holds if and only if* Σ^m *is minimal in a small hypersphere* $\mathbb{S}^{n-1}(2c)\subset\mathbb{S}^n(c)$*.*

Theorem (F., Oniciuc, Rosenberg - 2011)

Let Σ^m be a complete proper-biharmonic pmc submanifold in $\mathbb{S}^n \times \mathbb{R}$, with $m \geq 2$, such that its mean curvature satisfies

$$|H|^2 > C(m) = \frac{(m-1)(m^2+4) + (m-2)\sqrt{(m-1)(m-2)(m^2+m+2)}}{2m^3}$$

and the norm of its second fundamental form σ is bounded. Then $m < n$, $|H| = 1$ and Σ^m is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-1}(2) \subset \mathbb{S}^n$.

Sketch of the proof.

- $\langle H, \xi \rangle = 0 \quad \Rightarrow \quad 0 = \langle \bar{\nabla}_X H, \xi \rangle = -\langle A_H T, X \rangle \quad \Rightarrow \quad A_H T = 0$

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- $$\begin{aligned} \frac{1}{2}\Delta |\phi_H|^2 &= |\nabla \phi_H|^2 + m(\operatorname{trace} \phi_H^3) + 3m|H|^2|\phi_H|^2 \\ &\quad - m^2|H|^4(1 - |H|^2) \end{aligned}$$

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$$\frac{1}{2}\Delta|\phi_H|^2 \geq \frac{P(|T|^2)}{\sqrt{m-1}|H|((m-2)\sqrt{1-|H|^2}+2\sqrt{m-1}|H|)}|\phi_H|^2$$

- $$\geq \frac{P(1)}{\sqrt{m-1}|H|((m-2)\sqrt{1-|H|^2}+2\sqrt{m-1}|H|)}|\phi_H|^2$$

$$\geq 0$$

$$P(t) = m(m-1)t^2 - (3m^2-4)|H|^2 t + m|H|^2(m^2|H|^2 - (m-2)^2)$$

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- **Theorem (Omori-Yau Maximum Principle)**

  If Σ^m is a complete Riemannian manifold with Ricci curvature bounded from below, then for any smooth function $u \in C^2(\Sigma^m)$ with $\sup_{\Sigma^m} u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma^m$ satisfying

  $$\lim_{k\to\infty} u(p_k) = \sup_{\Sigma^m} u, \quad |\nabla u|(p_k) < \frac{1}{k} \quad \text{and} \quad \Delta u(p_k) < \frac{1}{k}.$$

- $\begin{cases} \phi_H = 0 \ (\Sigma^m = \text{pseudo-umbilical}) \\ A_H T = 0 \end{cases} \quad \Rightarrow T = 0 \ (\Sigma^m \text{ lies in } \mathbb{S}^n)$
- $|H|^2 > C(m) > (\frac{m-1}{m})^2 > (\frac{m-2}{m})^2$

- $\begin{cases} \phi_H = 0 \ (\Sigma^m = \text{pseudo-umbilical}) \\ A_H T = 0 \end{cases} \Rightarrow T = 0$ (Σ^m lies in $\mathbb{S}^n$)
- $|H|^2 > C(m) > (\frac{m-1}{m})^2 > (\frac{m-2}{m})^2$
- $|H| = 1$ and Σ^m is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-1}(2) \subset \mathbb{S}^n$

Biharmonic pmc surfaces in $\mathbb{S}^n(c) \times \mathbb{R}$

Lemma (F., Oniciuc, Rosenberg - 2011)

A pmc surface Σ^2 in $\mathbb{S}^n(c) \times \mathbb{R}$ is proper-biharmonic iff either

1. *Σ^2 is pseudo-umbilical and, therefore, it is a minimal surface of a small hypersphere $\mathbb{S}^{n-1}(2c) \subset \mathbb{S}^n(c)$; or*
2. *the mean curvature vector field H is orthogonal to ξ, $|A_H|^2 = c(2 - |T|^2)|H|^2$, and $A_U = 0$ for any normal vector field U orthogonal to H.*

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Corollary

If Σ^2 is a proper-biharmonic pmc surface in $\mathbb{S}^n(c) \times \mathbb{R}$ then the tangent part T of ξ has constant length.

Proof.

- the map $p \in \Sigma^2 \to (A_H - \mu\,\mathrm{I})(p)$, where μ is a constant, is analytic, and, therefore, either
  - Σ^2 is a pseudo-umbilical surface (at every point), or
  - $H(p)$ is an umbilical direction on a closed set without interior points
- $\Sigma^2 \neq$ pseudo-umbilical + $[A_H, A_U] = 0 \Rightarrow$
  at $p \in \Sigma^2$ $\exists\{e_1, e_2\}$ - orthonormal basis that diagonalizes A_H and A_U, $\forall U \perp H$
- $H \perp U \Rightarrow \operatorname{trace} A_U = 2\langle H, U\rangle = 0$
- $A_H = \begin{pmatrix} a+|H|^2 & 0 \\ 0 & -a+|H|^2 \end{pmatrix}$ and $A_U = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$

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- $A_H = \begin{pmatrix} a + |H|^2 & 0 \\ 0 & -a + |H|^2 \end{pmatrix}$ and $A_U = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$
- $\begin{cases} 0 = \operatorname{trace}(A_H A_U) = 2ab \\ a \neq 0 \end{cases} \Rightarrow b = 0$, i.e. $A_U = 0$

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- (Corollary) $H \perp N \Rightarrow \nabla_X T = A_N X = 0 \Rightarrow X(|T|^2) = 0$

Proposition (F., Rosenberg - 2010)

If Σ^2 is a pmc surface in $M^n(c) \times \mathbb{R}$, then

$$\frac{1}{2}\Delta|T|^2 = |A_N|^2 + K|T|^2 + 2T(\langle H, N\rangle),$$

where K is the Gaussian curvature of the surface.

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where K is the Gaussian curvature of the surface.

Corollary

If Σ^2 is a non-pseudo-umbilical proper-biharmonic pmc surface in $\mathbb{S}^n(c)\times\mathbb{R}$, then it is flat.

Theorem (F., Oniciuc, Rosenberg - 2011)

Let Σ^2 *be a proper-biharmonic pmc surface in* $\mathbb{S}^n(c)\times\mathbb{R}$. *Then either*

1. Σ^2 *is a minimal surface of a small hypersphere* $\mathbb{S}^{n-1}(2c)\subset\mathbb{S}^n(c)$; *or*
2. Σ^2 *is* (*an open part of*) *a vertical cylinder* $\pi^{-1}(\gamma)$, *where* γ *is a circle in* $\mathbb{S}^2(c)$ *with curvature equal to* $\sqrt{c}$, *i.e.* γ *is a biharmonic circle in* $\mathbb{S}^2(c)$.

Sketch of the proof.

- assume $\Sigma^2 \neq$ pseudo-umbilical $\Rightarrow |T| = \text{constant} \neq 0$, i.e. $|N| = \text{constant} \in [0,1)$

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- $A_U = 0$, $\forall U \perp H \Rightarrow \dim L = \dim \operatorname{span}\{\operatorname{Im} \sigma, N\} \leq 2 \Rightarrow$
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    - $\mathbb{S}^2(c) \times \mathbb{R}$ (if $N = 0$), or
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- $|A_H|^2 = c(2 - |T|^2)|H|^2 = \text{constant} \Rightarrow N = 0 \Rightarrow \Sigma^2 = \pi^{-1}(\gamma)$, where γ is a proper-biharmonic pmc curve with curvature $\kappa = 2|H| = \sqrt{c}$

Remark

$\nabla A_H = 0$ *for all proper-biharmonic surfaces in* $\mathbb{S}^n(c) \times \mathbb{R}$.

Theorem (F., Oniciuc, Rosenberg - 2011)

If Σ^m, *with* $m \geq 3$, *is a proper-biharmonic pmc submanifold in* $\mathbb{S}^n(c) \times \mathbb{R}$ *such that* $\nabla A_H = 0$, *then either*

1. Σ^m *is a proper-biharmonic pmc submanifold in* $\mathbb{S}^n(c)$, *with* $\nabla A_H = 0$; *or*
2. Σ^m *is* $($*an open part of*$)$ *a vertical cylinder* $\pi^{-1}(\Sigma^{m-1})$, *where* Σ^{m-1} *is a proper-biharmonic pmc submanifold in* $\mathbb{S}^n(c)$ *such that the shape operator corresponding to its mean curvature vector field in* $\mathbb{S}^n(c)$ *is parallel.*