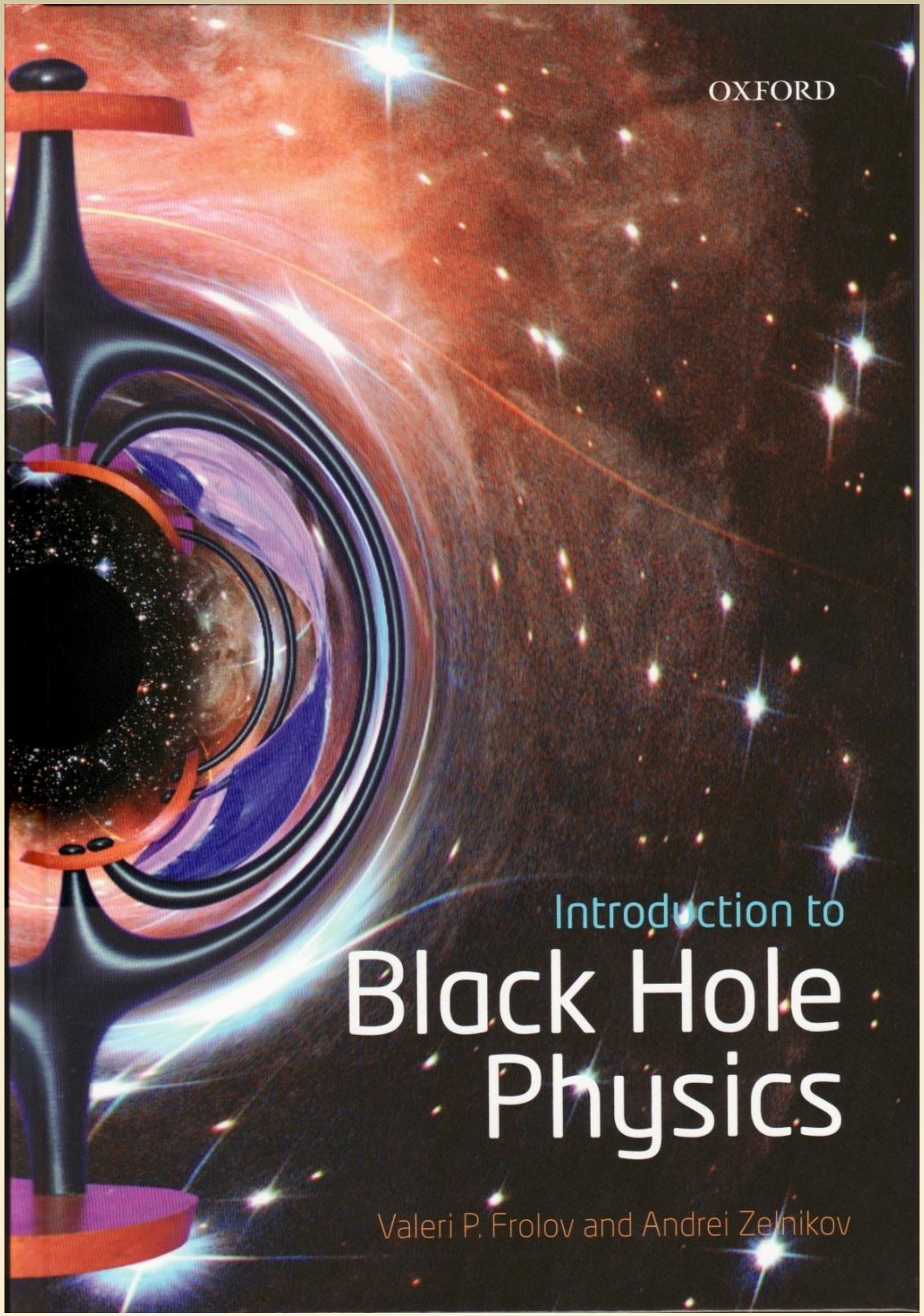

http://www.amazon.ca/Introduction-Black-Physics-Valeri-Frolov/dp/0199692297

Spinoptics in a Stationary Spacetime

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(Based on V.F. & A.Shoom, Phys.Rev. D84, 044026 (2011);
and V.F. & A.Shoom gr-qc/1205.4479 (2012))

XIV International Conference
Geometry, Integrability and Quantization
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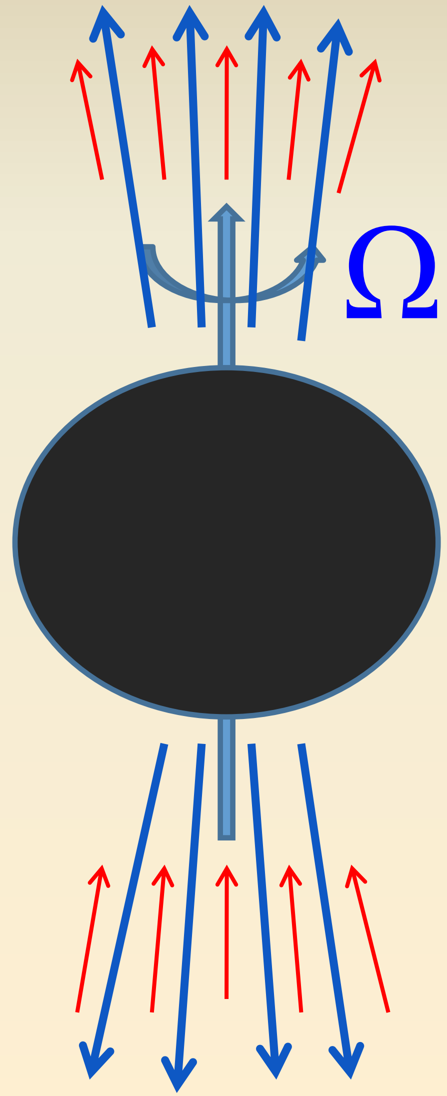
Main goal is to study how the spin of a photon affects its motion in the gravitational field.

(WKB approach for a massless field with spin)

An example of gravitational spin-spin interaction: Asymmetry of Hawking radiation for polarized light



Gravito-electromagnetism

Weak field limit:

$$ds^2 = -c^2(1-2\frac{\Phi}{c^2})dt^2 - \frac{4}{c}(\vec{A}\cdot d\vec{x})dt + (1+2\frac{\Phi}{c^2})d\vec{x}^2,$$

$$\Phi \propto \frac{GM}{r}, \quad \vec{A} \propto \frac{G}{c}\frac{\vec{J}\times\vec{x}}{r^3},$$

Transverse gauge condition: $\frac{1}{c}\frac{\partial\Phi}{\partial t} + \nabla\left(\frac{1}{2}\vec{A}\right) = 0$

Define: $\vec{E} = \nabla\Phi + \frac{1}{c}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\vec{A}\right), \quad \vec{B} = -\nabla\times\vec{A}$

Then one has:

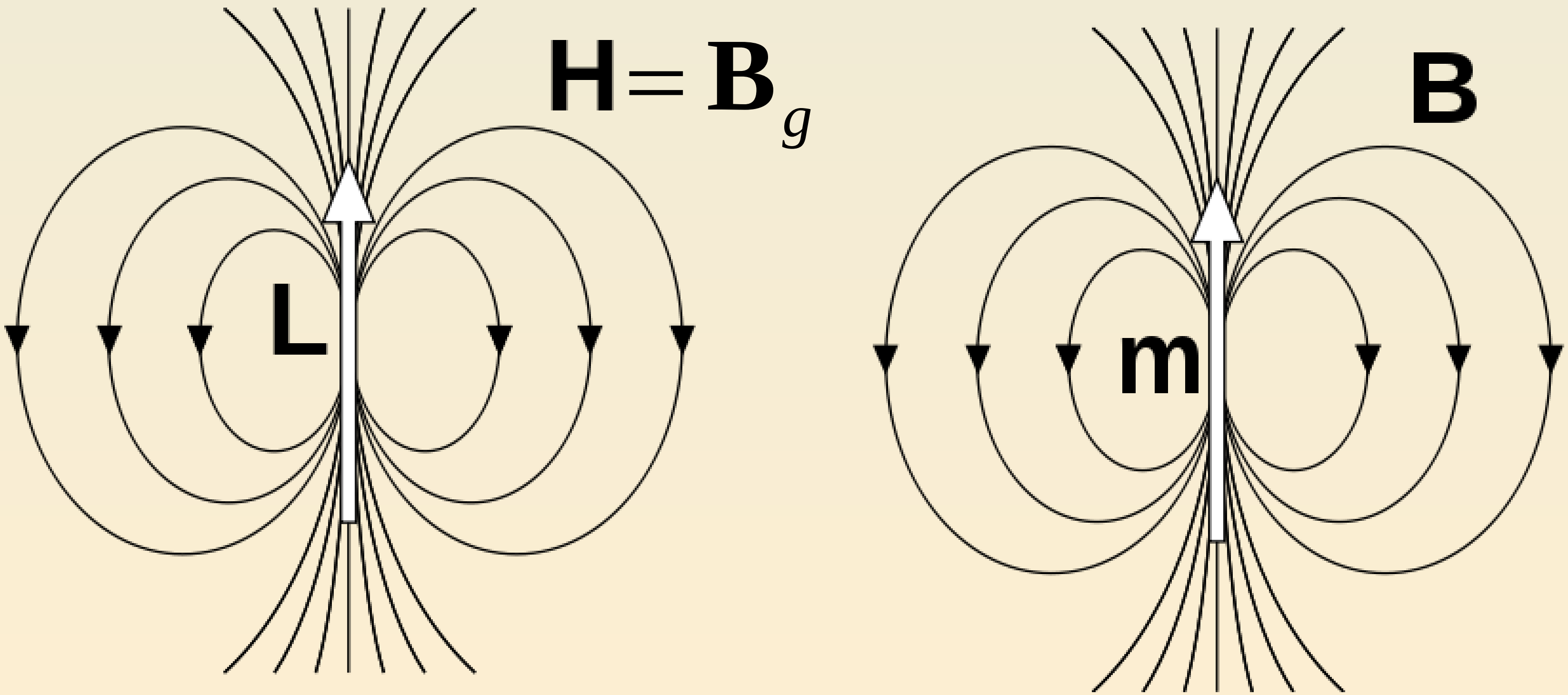
$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\vec{B}\right), \quad \nabla\cdot\left(\tfrac{1}{2}\vec{B}\right) = 0,$$

$$\nabla\cdot\vec{E} = -4\pi G\rho, \quad \nabla\times\left(\tfrac{1}{2}\vec{B}\right) = \frac{1}{c}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\vec{E}\right) - \frac{4\pi G}{c}\vec{j},$$

$$\nabla\cdot\vec{j} + \frac{\partial}{\partial t}\rho = 0$$

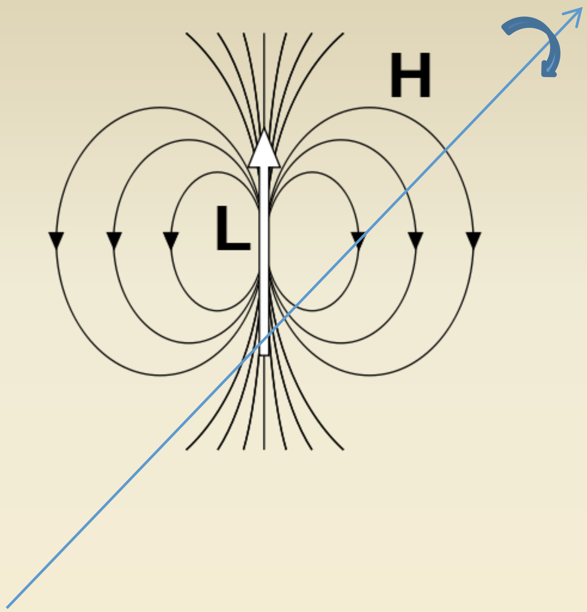
For a particle motion:

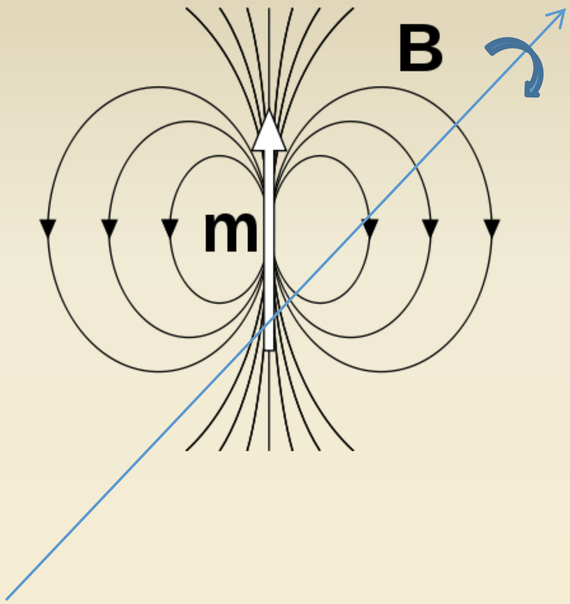
$$\frac{d\vec{p}}{dt} = \vec{F}, \quad \vec{F} = \mu\vec{E} + 2\mu\left[\frac{\vec{v}}{c} \times \vec{B}\right]$$



GRAVITY

Electromagnetism





Particle with spin

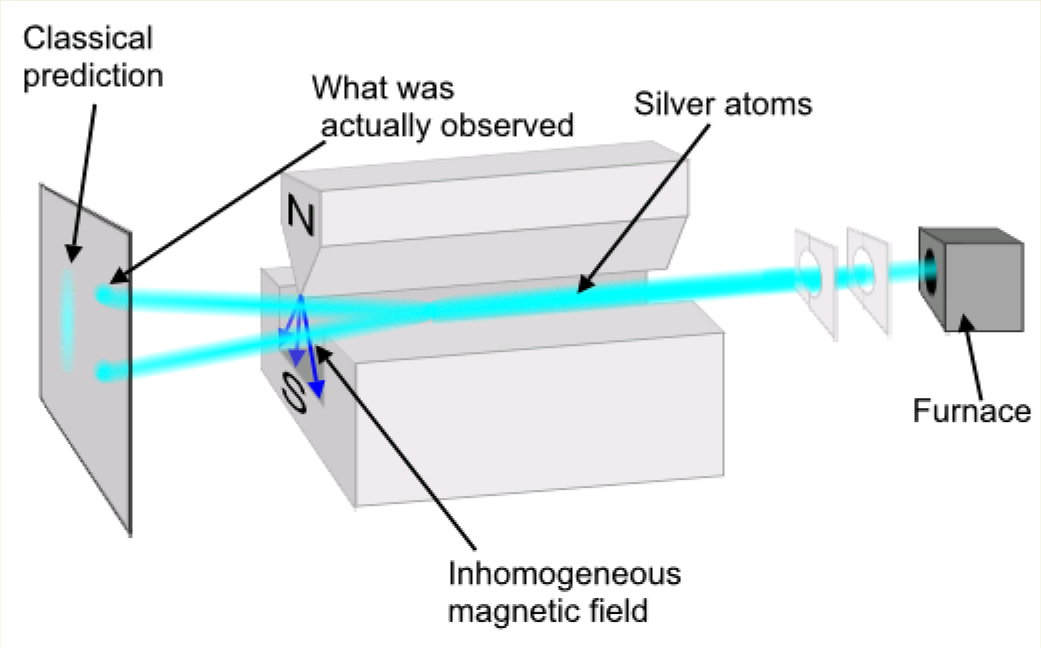
Particle with magnetic dipole moment

Maxwell equations

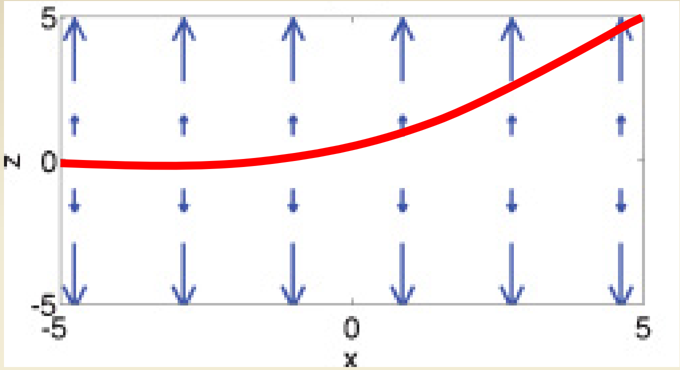
Dirac (Pauli) equation

Geometric optics (WKB) approximation

Stern-Gerlach Experiment



Hsu, Berrondo, Van Huele "Stern-Gerlach dynamics of quantum propagators" Phys. Rev. A 83, 012109 (2011)



$$H = H_z + H_x = \frac{p_x^2 + p_z^2}{2m} - \mu B_1 z \sigma_z$$

$$K(\vec{x}, \vec{x}_0;\ t\ ) = K(x, x_0;\ t\ )K(z, z_0;\ t\ )$$

$$K(x, x_0;\ t\ ) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(-\frac{m(x - x_0)^2}{2i\hbar t} \right)$$

$$K(z, z_0;\ t\ ) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(-\frac{m(z - z_0)^2}{2i\hbar t} - \frac{\mu B_1 \sigma_z (z + z_0)t}{2i\hbar} + \frac{\mu^2 B_1^2 t^3}{24 i\hbar m} \right)$$

$$H = \frac{p_z^2}{2m} - \lambda z, \quad \lambda = \mu B_1 \sigma_z,$$

$$\dot{z} = p_z / m, \quad \dot{p}_z = \lambda,$$

$$S = \int_{z_0}^{z} (p_z dz - H dt) = \frac{m(z - z_0)^2}{2t} + \frac{\mu B_1 \sigma_z (z + z_0) t}{2} + \frac{\mu^2 B_1^2 t^3}{12m},$$

$$K(z, z_0; t) \propto \exp(iS / \hbar)$$

Magnetic moment of the electron: $\mu = \frac{g}{2} \mu_B, \quad \mu_B = \frac{e\hbar}{2m_e}$

By applying formal WKB to the Pauli equation with this μ, one would get

$$K(z, z_0; t) \propto \exp(iS_0 / \hbar), \quad S_0 = \frac{m(z - z_0)^2}{2t}$$

Lessons

(i) In the exact solution for a wave packet there exists correlation between orientation of spin and spatial trajectory of electron;

(ii) At late time the up and down spin wave packets are moving along classical trajectories;

(iii) Formal WKB solution represents the motion of the `center of mass’ of two packets

$$L = x - x_0 = Vt; \quad \Delta z = \frac{\mu B_1}{m}\frac{t^2}{2};$$

Condition when terms with μ become important can be written as

$$\Delta z \sim L \Rightarrow 2m(x - x_0) \sim \mu B_1 t^2$$

or, equivalently, $L \sim mV^2 / (\mu B_1)$

To obtain a correct long time asymptotic behavior of the wave packet one needs:

(i) to `diagonalize' the field equations;
(ii) to `enhance' spin-dependent term
(iii) include it in the eikonal function

Spinoptics in gravitational field

(i) Spin induced effects
(ii) Many-component field
(iii) Helicity states
(iv) Massless field
(v) Gauge invariance

Riemann-Silberstein vector: $\vec{F}^{\pm} \equiv \vec{E} \pm i\vec{H}$

$$\vec{F}^{+}(t,\vec{r}) = \int d^3k\, \vec{e}(\vec{k})\left[a_{+}(\vec{k})e^{-i\omega t+i\vec{k}\vec{r}} + a_{-}(\vec{k})e^{+i\omega t-i\vec{k}\vec{r}} \right]$$

$a_{\pm}(\vec{k})$ are the amplitudes of right and left circularly polarized EM waves with vector $\vec{k}$

Consider purely right polarized monochromatic wave

$$\vec{F}^{+}(t,\vec{r}) = e^{-i\omega t}\boldsymbol{\mathcal{F}}^{+}(\vec{r})$$

Consider complex Maxwell field $\mathbf{F}$.

(Anti-)Self-dual field $\mathbf{F}^{\pm} = \pm i * \mathbf{F}^{\pm}$.

For monochromatic wave $\mathbf{F}^{\pm} \sim e^{-i\omega t} \boldsymbol{\mathcal{F}}^{\pm}$.

In a curved ST there is a unique splitting of a complex Maxwell field into its self-dual and anti-self-dual parts. As a result the right and left polarized photons are well defined and the helicity is preserved.

Maxwell equations in a stationary ST

$$ds^2 = -h dS^2, \quad h = -\xi_{(t)}^2,$$

$$dS^2 = (dt - g_i\, dx^i)^2 + \gamma_{ij}\, dx^i\, dx^j$$

- ultrastationary metric

$$t \to \tilde{t} = t + q(x^i), \quad g_i \to \tilde{g}_i = g_i + q_{,i}$$

Since Maxwell eqns are conformally invariant it is convenient to perform calculations in the ultrastationary metric

3+1 form of Maxwell equations

$$E_i \equiv F_{i0}\,, \quad B_{ij} \equiv F_{ij}\,, \quad D^i \equiv h^2 F^{0i}\,, \quad H^{ij} \equiv h^2 F^{ij}\,.$$

$$D_i = E_i - H_{ij} g^j\,, \quad B^{ij} = H^{ij} - E^i g^j + E^j g^i\,.$$

$$B_{ij} = e_{ijk} B^k\,, \quad H^{ij} = e^{ijk} H_k\,.$$

$$C = [A \times B], C^i = e^{ijk} A_j B_k \quad \Rightarrow \quad D = E - [g \times H], B = H + [g \times E].$$

$$\mathrm{div}\vec{B} = 0, \mathrm{curl}\dot{\vec{E}} = -\vec{B}, \quad \mathrm{div}\vec{D} = 0, \mathrm{curl}\vec{H} = \vec{D}.$$

$$\mathrm{E} \equiv \frac{1}{8\pi}[(\vec{E},\vec{D}) + (\vec{B},\vec{H})], \quad \vec{V} \equiv \frac{1}{4\pi}[\vec{E} \times \vec{H}], \qquad \dot{\mathrm{E}} + \mathrm{div}\vec{V} = 0$$

Master equation for c-polarized light

Riemann-Silberstein vectors: $\vec{F}^{\pm} \equiv \vec{E} \pm i\vec{H}$, $\vec{G}^{\pm} \equiv \vec{D} \pm i\vec{B}$

$$\vec{E} = e^{-i\omega t}\boldsymbol{\mathcal{E}} + e^{i\omega t}\boldsymbol{\mathcal{E}}^*, \quad \vec{H} = e^{-i\omega t}\boldsymbol{\mathcal{H}} + e^{i\omega t}\boldsymbol{\mathcal{H}}^*$$

$$\vec{D} = e^{-i\omega t}\boldsymbol{\mathcal{D}} + e^{i\omega t}\boldsymbol{\mathcal{D}}^*, \quad \vec{B} = e^{-i\omega t}\boldsymbol{\mathcal{B}} + e^{i\omega t}\boldsymbol{\mathcal{B}}^*,$$

$$\boldsymbol{\mathcal{F}}^{\pm} = \boldsymbol{\mathcal{E}} \pm i\boldsymbol{\mathcal{H}}, \quad \boldsymbol{\mathcal{G}}^{\pm} = \boldsymbol{\mathcal{D}} \pm i\boldsymbol{\mathcal{B}}.$$

$$\mathrm{div}\boldsymbol{\mathcal{G}}^{\pm} = 0, \quad \mathrm{curl}\boldsymbol{\mathcal{F}}^{\pm} = \pm\omega\boldsymbol{\mathcal{G}}^{\pm}, \quad \boldsymbol{\mathcal{G}}^{\pm} = \boldsymbol{\mathcal{F}}^{\pm} \pm i[\vec{g} \times \boldsymbol{\mathcal{F}}^{\pm}]$$

$$\mathrm{curl}\boldsymbol{\mathcal{F}}^{\pm} = \pm\omega\boldsymbol{\mathcal{F}}^{\pm} + i\omega[\vec{g} \times \boldsymbol{\mathcal{F}}^{\pm}]$$

"Standard" geometric optics

Small dimensionless parameter: $\varepsilon=(\omega\ell)^{-1}$

ℓ is characteristic length scale of the problem

Geometric optics ansatz $\mathcal{F} = \vec{f}\, e^{i\omega S}$

There is a phase factor ambiguity

$$\vec{f} \Rightarrow e^{i\varphi(x)}\,\vec{f}, \quad S \Rightarrow S - \varphi(x)/\omega$$

Exact equation: $L\vec{f} = \sigma\omega^{-1}\text{curl}\,\vec{f}$

$$L\vec{f} \equiv \vec{f} - i\sigma[\vec{n} \times \vec{f}],$$

$$\vec{n} \equiv \vec{p} - \vec{g}, \quad \vec{p} \equiv \nabla S,$$

Standard Geometric Optics

$$f = f_0 + \omega^{-1} f_1 + \omega^{-2} f_2 + \ldots$$

$$L f_0 +$$
$$\omega^{-1}[L f_1 - \sigma \mathrm{curl}\, f_0] + \ldots +$$
$$\omega^{-2}[L f_2 - \sigma \mathrm{curl}\, f_1] + \ldots = 0$$

L is a degenerate operator. Condition of existence of solutions of eqn $L f_0 = 0$ implies the eikonal equation $(\nabla S - \vec{g})^2 = 1$

Effective Hamiltonian is:

$$H(x^i, p_i) \equiv \frac{1}{2}(\vec{p} - \vec{g})^2 = \frac{1}{2}\gamma^{ij}(p_i - g_i)(p_j - g_j)$$

Characteristic scale $L_F : \Delta\phi = L_F \, |\nabla \vec{g}| \propto 2\pi$

$L_F \propto 4\pi / |\nabla \vec{g}|$

4 - D point of view :

(i) Light ray is a 4D null geodesic

(ii) Vector of linear polarization is 4D parallel transported

Modified Geometric Optics

To fix an ambiguity in the choice of the phase,

we require that vectors of the basis $(\vec{n}, \vec{m}, \vec{m}^*)$ along rays are Fermi transported;

$$\mathbf{F}_n\,\vec{a} = \nabla_n \vec{a} - (\vec{n}, \vec{a})\,\vec{w} + (\vec{w}, \vec{a})\,\vec{n}, \quad \vec{w} = \nabla_n \vec{n}$$

As a result the lowest order polarization dependent correction is included in the phase.

$$\mathscr{F}^{\sigma} \approx f_0^{\sigma}\, m^{\sigma} e^{i\omega \tilde{S}(\vec{x})}, \quad \tilde{S}(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} \left[1 + \left(\vec{\tilde{g}}, \frac{d\vec{x}}{d\ell} \right) \right] d\ell,$$

$$\vec{\tilde{g}} = \vec{g} + \frac{\sigma}{2\omega} \operatorname{curl} \vec{g}$$

Modified geometric optics

$$\vec{\tilde{n}} \equiv \vec{p} - \vec{\tilde{g}}, \quad \vec{p} \equiv \nabla \tilde{S}, \quad \vec{\tilde{g}} \equiv \vec{g} + \frac{\sigma}{2\omega} \operatorname{curl} \vec{g},$$

$$\tilde{L} f \equiv f - i\sigma[\vec{\tilde{n}} \times f] = 0,$$

$$\tilde{L} f = \frac{\sigma}{\omega} \operatorname{curl} f + \frac{i}{2\omega}[\operatorname{curl} \vec{g} \times f]$$

$$\det \tilde{L} = 0 \quad \Rightarrow \quad (\vec{\tilde{n}}, \vec{\tilde{n}}) = 1 \quad \Rightarrow \quad (\nabla \tilde{S} - \vec{\tilde{g}}) = 1,$$

$$\tilde{H}(x^i, p_i) = \frac{1}{2}(\vec{p} - \vec{\tilde{g}})^2 \equiv \frac{1}{2}\gamma^{ij}(p_i - \tilde{g}_i)(p_j - \tilde{g}_j),$$

$$\frac{D^2\vec{x}}{d\ell^2} = \left[\frac{d\vec{x}}{d\ell} \times \vec{f}_\varepsilon\right], \qquad \vec{f}_\varepsilon = \mathrm{curl}\,\vec{g} + \varepsilon\,\mathrm{curl}\,\mathrm{curl}\,\vec{g},$$

$$\frac{d\tau}{d\ell} = 1 + (\vec{g}, \frac{d\vec{x}}{d\ell}), \qquad \varepsilon = \pm(2\omega M)^{-1}$$

4D form of the effective equations (in the ultrastationary metric)

$$\frac{D^2 x^\mu}{d\lambda^2} = \varepsilon F^\mu{}_\nu \frac{Dx^\mu}{d\lambda}$$

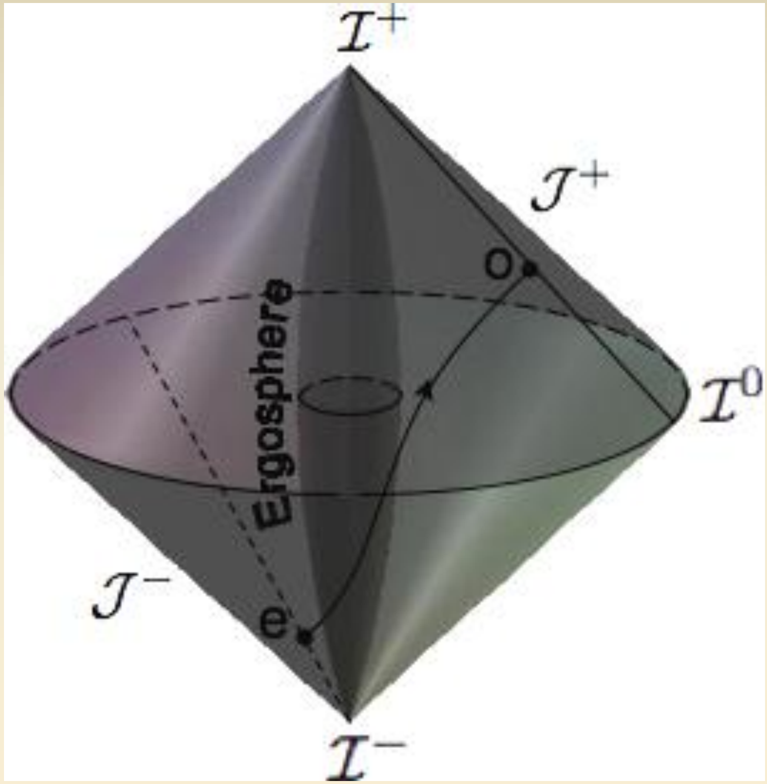
Null curves are solutions of these equations. For ε=0, null geodesics.

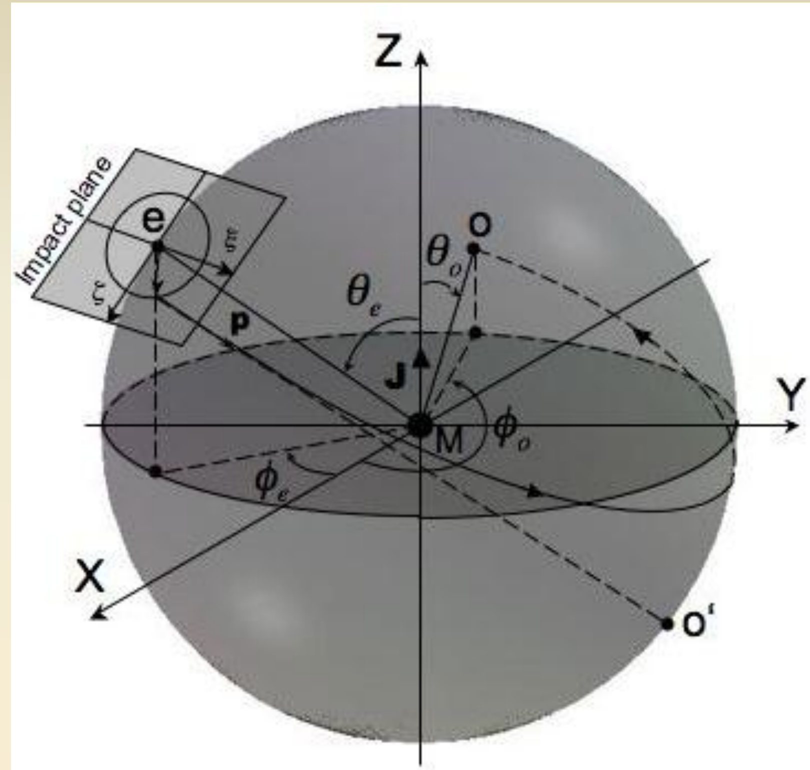
Polarized Photon Scattering in Kerr ST

(i) How does the photon bending angle depend on its polarization?

(ii) How does the position of the image of a photon arriving to an observer depend on its polarization?

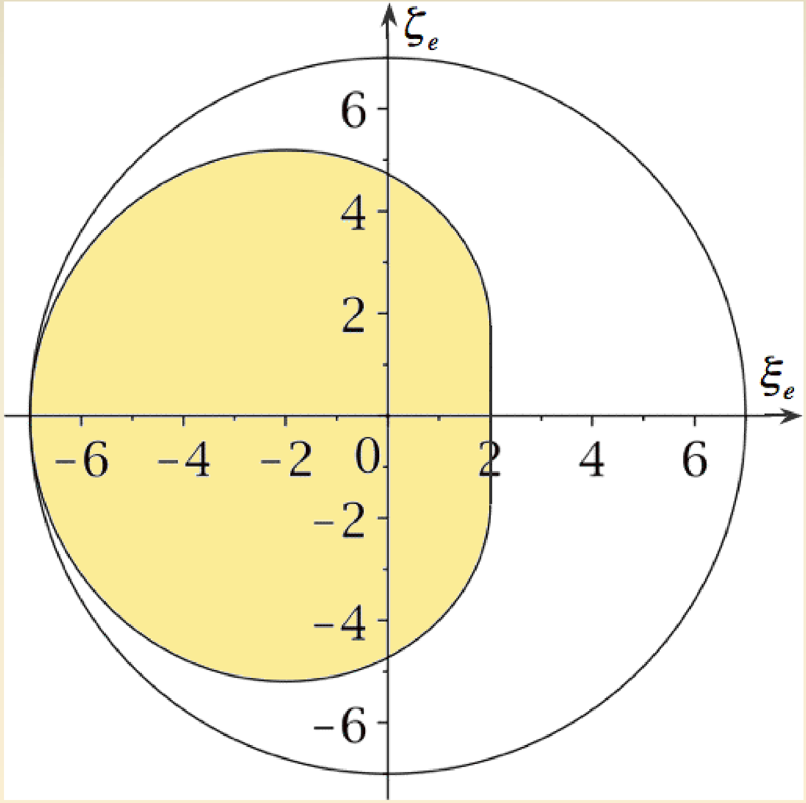
(iii) How does the arrival time of such photons depend on their polarization?



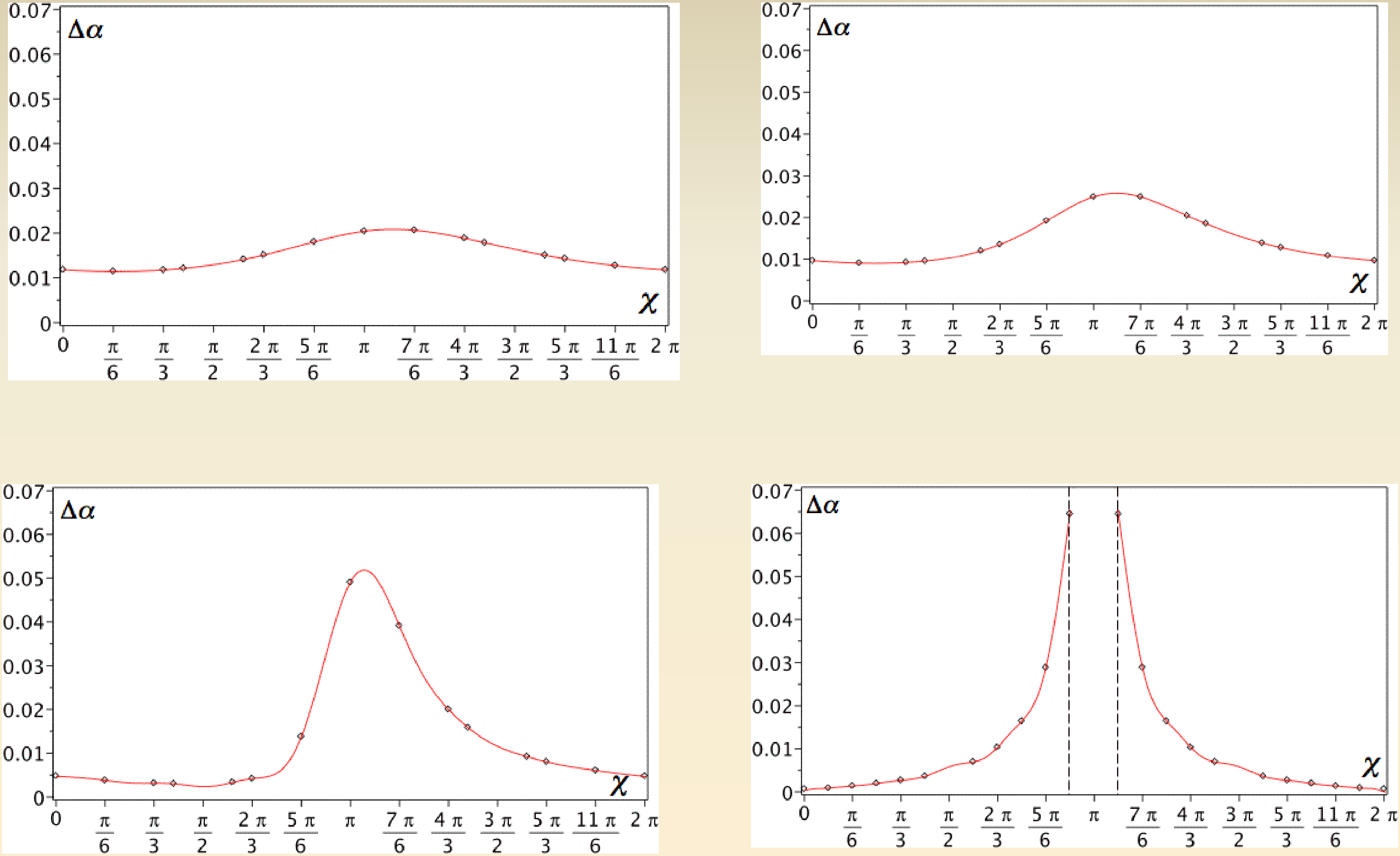


All calculations are for an extremal Kerr BH (M=a=1)

Capture Domain (equatorial plane)

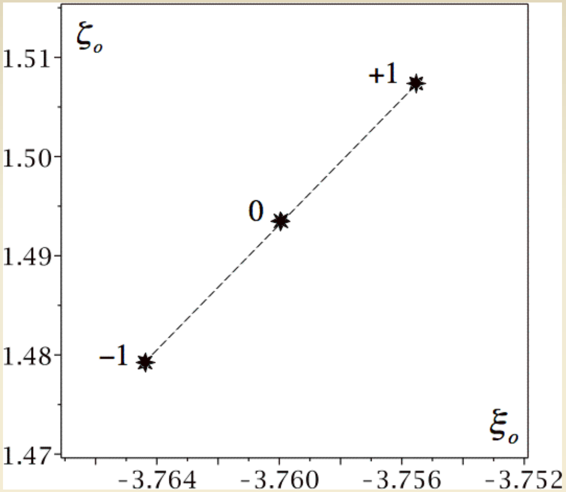


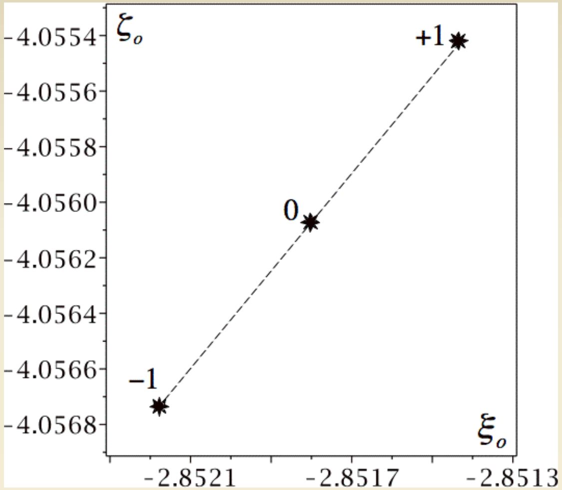
Shift in the bending angle: $\alpha = \Delta(\text{angle}) / \varepsilon$

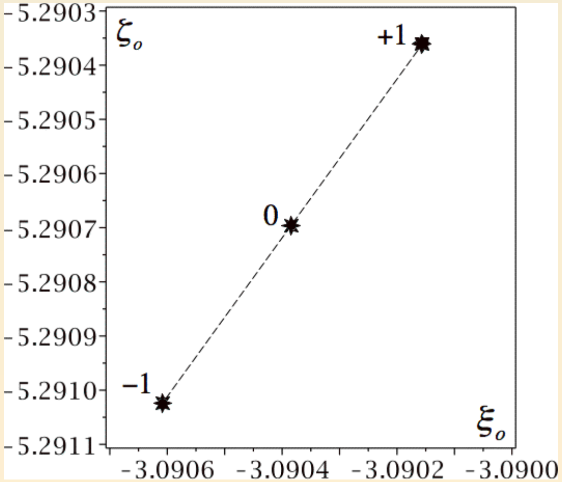


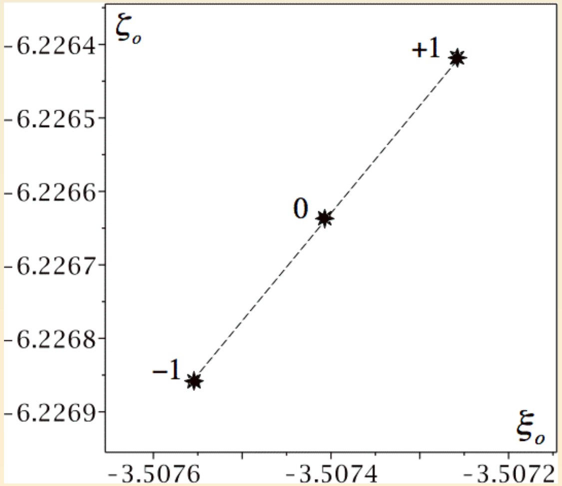
$$a = M = 1, \cos\theta = \pi/10, \pi/6, \pi/3, \pi/2, \quad |\vec{L}|/(\omega M) = 7.0$$

Image splitting









Time Delay

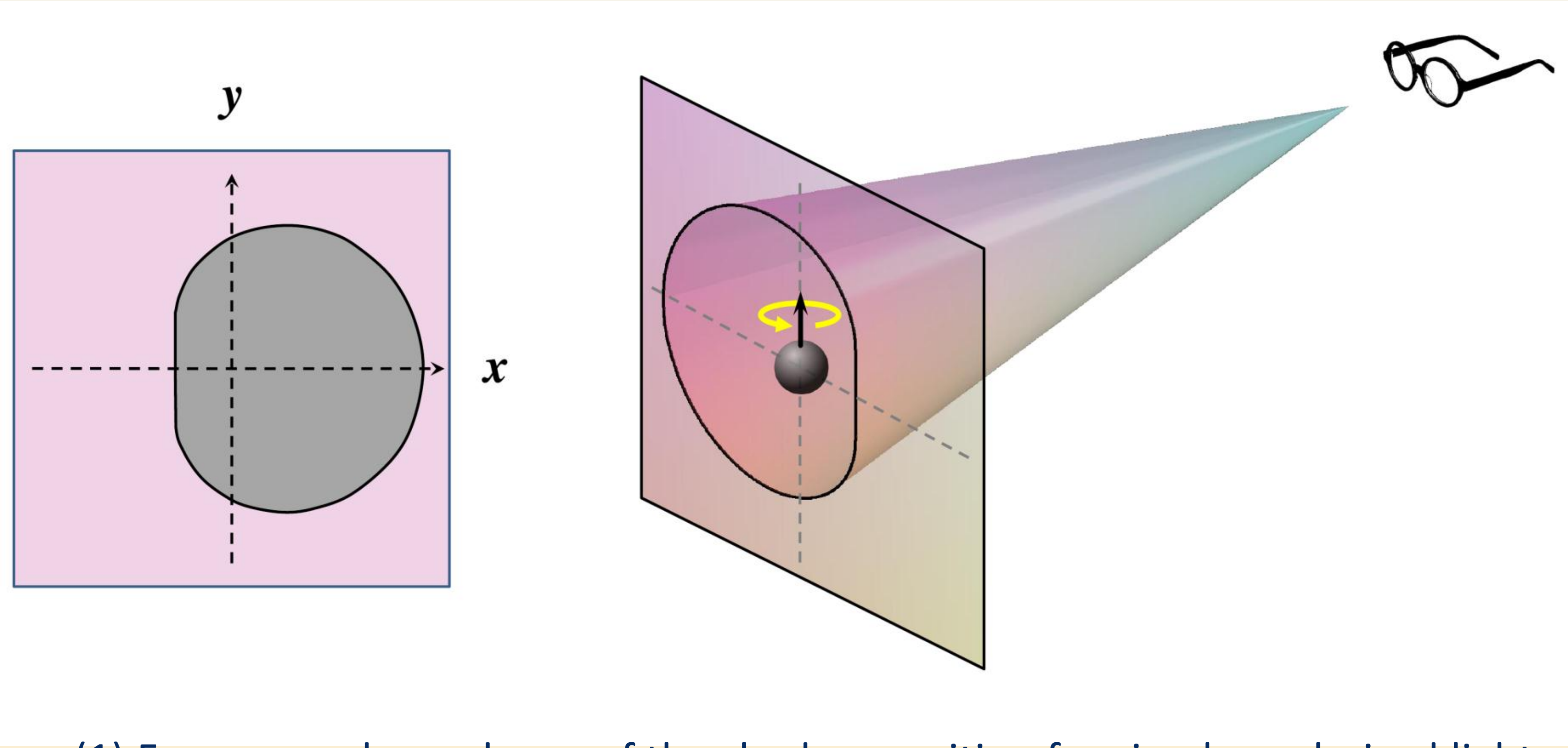
Effect of the second order in ε;

Fermat principle in gravitational field

[Landau & Lifshits "Classical Field Theory";

Brill in "Relativity, Astrophysics and Cosmology, 1973]

Rainbow effect for BH shadow



(1) Frequency dependence of the shadow position for circular polarized light;

(2) For given frequency shadow position depends on the polarization

SUMMARY

(1) Standard GO picture: In the Kerr ST a linearly polarized photon moves a null geodesic and its polarization vector is parallel propagated.
(2) Modified GO picture: Linear polarized photon beam splits into two circular polarized beams.
(3) Right and left polarized photons have different trajectories.
(4) In a stationary ST their motion can be obtained by introducing frequency dependent effective metric.
(5) Effects: Shift in bending angle, shift of images, time delay.