Parametric representation of waves propagation in transmission bands of periodic media

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- **Parametric resonance**
  - *in mechanics:* systems with external sources of energy (e.g., the pendulum with oscillating pivot point, periodically varying stiffness, mass, or load),
  - *in fluid or plasma mechanics:* frequency modulation or density fluctuations,
  - *in mathematical biology:* periodic environmental changes.

Hill equation (analysis of the orbit of the Moon — lunar stability problem, modelling of a quadrupole mass spectrometer, as the 1D Schrödinger equation of an electron in a crystal, etc.):

$$\ddot{x} + \left(\omega_0^2 + p(t)\right) x = 0, \tag{1}$$

where ω_0 is a constant, and $p(t)$ is a π-periodic function with zero average.

More generally:

$$\ddot{x} + k\dot{x} + \left(\omega_0^2 + p(t)\right) F(x) = 0, \tag{2}$$

where $k > 0$ is the damping coefficient, and $F(x) = x + bx^2 + cx^3 + \cdots$.

Mathieu equation (stability of railroad rails as trains drive over them, seasonally forced population dynamics, the Floquet theory of the stability of limit cycles, etc.):

$$\ddot{x} + (a - 2q\cos 2t)\, x = 0, \tag{3}$$

where a is a real constant, and q can be complex.

Lamé equation (when we replace circular functions by elliptic ones):

$$\ddot{x} + (A + B\wp(t))\, x = 0, \tag{4}$$

where A, B are some constants, and $\wp(t)$ is the Weierstrass elliptic function. Another form:

$$\ddot{x} + \left(A + B\,\mathrm{sn}^2\, t\right) x = 0, \tag{5}$$

where $\mathrm{sn}(t)$ is the Jacobi elliptic function of the first kind.

• One-dimensional wave equation

Let us consider the following one-dimensional wave equation:

$$w''(x) + q^2(x)w(x) = 0, \qquad q(x) = \frac{\omega}{c}n(x), \tag{6}$$

which describes the harmonic waves $\sim \exp(-i\omega t)$ propagating in a nonuniform dielectric medium with gradually varying dielectric refraction index $n(x)$; c is the speed of light in vacuum and $'$ denotes the differentiation with respect to x.

• Floquet theorem

According to the Floquet theorem, for any periodic refraction index $n(x) = n(x + \lambda)$ (or equivalently for any periodic coefficient $q(x) = q(x + \lambda)$) the one-dimensional wave equation (6) has a quasi-periodic solution

$$w(x) = \widetilde{w}(x) \exp(\pm\mu x), \tag{7}$$

where $\widetilde{w}(x)$ is a periodic function and the characteristic exponent μ can be either (i) real or (ii) purely imaginary. The former case corresponds to a parametric (anti-)resonance in the stop bands of the periodic structure, and the latter one to a periodic modulation of the carrier travelling wave.

• Periodic part of the solution

The one-dimensional wave equation for $\widetilde{w}(x)$ has the following form:

$$\widetilde{w}''(x) \pm 2\mu\widetilde{w}'(x) + \left[q^2(x) + \mu^2\right]\widetilde{w}(x) = 0. \tag{8}$$

• Admittance function

If we introduce an admittance function

$$y(x) = \frac{w'(x)}{q(x)w(x)} \qquad \Rightarrow \qquad w(x) = w_0 \exp\left[\int y(x)q(x)dx\right], \tag{9}$$

then it is easy to observe that Eq. (6) can be equivalently rewritten as follows:

$$q(x)y'(x) + q'(x)y(x) + q^2(x)\left[1 + y^2(x)\right] = 0, \tag{10}$$

i.e.,

$$\int \frac{y'(x)dx}{q(x)\left[1 + y^2(x)\right]} + \int \frac{y(x)q'(x)dx}{q^2(x)\left[1 + y^2(x)\right]} = x_0 - x. \tag{11}$$

• Harmonic oscillator

If $q(x) \equiv q_0$ is a constant, then Eq. (11) reads

$$\frac{1}{q_0}\int \frac{dy}{1 + y^2} = x_0 - x. \tag{12}$$

The integral can be easily integrated with the substitution $y = \operatorname{ctg}\psi$, $dy = -d\psi/\sin^2\psi$, $1 + y^2 = 1/\sin^2\psi$. Then

$$\psi = \operatorname{ctg}^{-1}y = \psi_0 + q_0\left(x - x_0\right) = q_0\left(x - \widetilde{x}_0\right), \qquad \widetilde{x}_0 = x_0 - \frac{\psi_0}{q_0}, \tag{13}$$

and

$$w(x) \;=\; w_0 \exp\left[q_0 \int \operatorname{ctg} q_0\left(x - \widetilde{x}_0\right)dx\right] = w_0 \exp\left[\int \frac{d \sin q_0\left(x - \widetilde{x}_0\right)}{\sin q_0\left(x - \widetilde{x}_0\right)}\right] = \widetilde{w}_0 \sin q_0\left(x - \widetilde{x}_0\right). \tag{14}$$

- (i) **real characteristic exponent**

A wide class of analytical solutions can be found by the method of phase parameter:

$$y(x) = \operatorname{ctg} \psi(x). \tag{15}$$

Then Eq. (10) reads

$$-\frac{q(x)\psi'(x)}{\sin^2 \psi(x)} + q'(x) \operatorname{ctg} \psi(x) + \frac{q^2(x)}{\sin^2 \psi(x)} = 0, \tag{16}$$

i.e.,

$$\psi'(x) - \frac{q'(x)}{2q(x)} \sin 2\psi(x) = q(x). \tag{17}$$

If there exists the inversion $x = X(\psi)$, then we can write $w(x)$, $y(x)$, and $q(x)$ as functions of ψ, i.e.,

$$w\left[X(\psi)\right] = W(\psi), \quad y\left[X(\psi)\right] = Y(\psi) \equiv \operatorname{ctg} \psi, \quad q\left[X(\psi)\right] = Q(\psi). \tag{18}$$

Then Eqs. (9) and (11) can be rewritten as follows:

$$W(\psi) = w_0 \sin \psi \exp\left[-\int \dot{G}(\psi) \cos^2 \psi \, d\psi\right], \tag{19}$$

$$X(\psi) = x_0 + \frac{1}{q_0}\left[\int \frac{d\psi}{\exp G(\psi)} - \frac{1}{2}\int \frac{\dot{G}(\psi) \sin 2\psi \, d\psi}{\exp G(\psi)}\right], \tag{20}$$

where we made a substitution $Q(\psi) = q_0 \exp G(\psi)$; here and below dots denote the differentiation with respect to ψ.

- **Periodic refraction index**

In particular, for any periodic refraction index $n(x) = n(x+\lambda)$ defined implicitly by a Fourier series

$$G(\psi) = a_0 + \sum_{m=1}^{\infty} \left(a_{2m} \cos 2m\psi + b_{2m} \sin 2m\psi\right), \tag{21}$$

we obtain a Floquet solution

$$w(x) = \widetilde{w}(x) \exp(-\mu x), \tag{22}$$

where $\widetilde{w}(x)$ is a periodic function, i.e.,

$$\widetilde{w}(x + 2\lambda) = \widetilde{w}(x), \tag{23}$$

and the characteristic exponent $\mu = \nu/\lambda$ is given by the explicit formulae for the period λ:

$$\lambda = \frac{2}{q_0} \int_0^{\pi} \exp\left[-G(\psi)\right] \sin^2 \psi d\psi, \tag{24}$$

and attenuation per period ν:

$$\nu = \int_0^{\pi} G(\psi) \sin 2\psi d\psi. \tag{25}$$

These analytical relations, giving the very simple description of the wave field attenuation in a periodic structure, are useful for the optimal design of multilayer mirrors and Bragg fiber claddings. However, from the theoretical point of view this solution remains incomplete until a similar parametric representation is found for propagating waves in transmission bands of a periodic medium.

- (ii) **complex characteristic exponent**

For a complex wave also is possible to define a phase parameter $\psi(x)$, which obviously must be a homogeneous function of $w(x)$ and $w'(x)$.

Let us observe that

$$\frac{y(x)+i}{y(x)-i} = \frac{\mathrm{ctg}\,\psi + i}{\mathrm{ctg}\,\psi - i} = \frac{\cos\psi + i\sin\psi}{\cos\psi - i\sin\psi} = \exp\left(2i\psi\right), \tag{26}$$

then Eq. (15) can be equivalently rewritten as follows:

$$\psi(x) = \mathrm{ctg}^{-1} y(x) = \frac{1}{2i}\ln\frac{y(x)+i}{y(x)-i} = \frac{1}{2i}\ln\frac{w'(x)+iq(x)w(x)}{w'(x)-iq(x)w(x)}. \tag{27}$$

Let us define the quasi-phase parameter $\psi(x)$ of a complex wave function $w(x)$ as follows:

$$\psi(x) = \frac{1}{2i}\ln\frac{w'(x)+iq(x)w(x)}{w^{*\prime}(x)-iq(x)w^{*}(x)} = \int\left\{q(x) + \frac{q'(x)}{q(x)}\frac{\mathrm{Re}\left[y(x)\right]}{|y(x)+i|^2}\right\}dx. \tag{28}$$

The complex-valued admittance $y(x)$ as a function of ψ reads

$$y\left[X(\psi)\right] = Y(\psi) = \frac{\dot{W}(\psi)}{\dot{X}(\psi)Q(\psi)W(\psi)}, \tag{29}$$

and then Eq. (1) can be rewritten as a pair of nonlinear differential equations

$$\dot{X} = \frac{1}{Q}\left(1 - \frac{\dot{G}\,\mathrm{Re}\,Y}{|Y+i|^2}\right), \qquad \dot{Y} = \frac{\dot{G}\,\mathrm{Im}\,Y\left[i\left(Y^2-1\right)-2Y\right]}{|Y+i|^2} - \left(1+Y^2\right). \tag{30}$$

Proof. The second part of Eq. (28) can be obtained from the first one by the direct calculation of the integral representation of the logarithm, i.e.,

$$\mathrm{Re}\left[\frac{1}{i}\int\frac{d(w'+iqw)}{w'+iqw}\right]=\mathrm{Re}\left[\frac{1}{i}\int\frac{w''+i(qw'+q'w)}{w'+iqw}dx\right] \tag{31}$$

and now, using Eq. (6) and the facts that $\mathrm{Re}\,(iz)=-\mathrm{Im}(z)$, $\mathrm{Im}\,(iz)=\mathrm{Re}(z)$, we finally obtain that Eq. (31) can be rewritten as follows:

$$\mathrm{Im}\left[\int\frac{iq\,(w'+iqw)+iq'w}{w'+iqw}dx\right]=\int\left(q+\frac{q'}{q}\mathrm{Re}\left[\frac{1}{y(x)+i}\right]\right)dx. \tag{32}$$

Let us also note that

$$|y(x)+i|^2=|y(x)|^2+2\mathrm{Im}\,[y(x)]+1. \tag{33}$$

As for the first of Eqs. (30), it follows directly from Eq. (28), i.e.,

$$\frac{d\psi}{dx}\equiv\frac{1}{\dot{X}(\psi)}=Q(\psi)+\frac{\dot{G}(\psi)}{\dot{X}(\psi)}\frac{\mathrm{Re}\,[Y(\psi)]}{|Y(\psi)+i|^2}. \tag{34}$$

And the second of Eqs. (30) is obtained inserting $w''(x)=[q(x)w(x)h(x)]'$ into Eq. (6), then

$$w''\,[X(\psi)]=Q^2W\left(\frac{\dot{Y}+Y\dot{G}}{Q\dot{X}}+Y^2\right)\equiv-Q^2W=-q^2w, \tag{35}$$

what provides us also with the compatibility condition (cf. Eq. (10))

$$\dot{Y}+Y\dot{G}+\left(1+Y^2\right)Q\dot{X}=0 \tag{36}$$

imposing constraints on choosing the complex admittance $Y(\psi)$. □

• $\mathcal{R}, \mathcal{Y}$-variables

For the sake of convenience, let us denote $Y = \mathcal{R}\exp(i\mathcal{Y})$ and separate real and imaginary parts of the second of Eqs. (30), then as a result we obtain the following pair of nonlinear differential equations:

$$\dot{\mathcal{R}} = -\dot{G}\,\mathcal{S}\left(\mathcal{R},\mathcal{Y}\right)\left[2\mathcal{R} + \left(1+\mathcal{R}^2\right)\sin\mathcal{Y}\right] - \left(1+\mathcal{R}^2\right)\cos\mathcal{Y} = -\dot{G}R + \left(\mathcal{R}^2+1\right)\left[\dot{G}\,\mathcal{C}\left(\mathcal{R},\mathcal{Y}\right) - 1\right]\cos\mathcal{Y}, \tag{37}$$

$$\dot{\mathcal{Y}} = \frac{\mathcal{R}^2 - 1}{\mathcal{R}}\left[\dot{G}\,\mathcal{C}\left(\mathcal{R},\mathcal{Y}\right) - 1\right]\sin\mathcal{Y}, \tag{38}$$

where

$$\mathcal{S}\left(\mathcal{R},\mathcal{Y}\right) = \frac{\mathcal{R}\sin\mathcal{Y}}{1+\mathcal{R}^2+2\mathcal{R}\sin\mathcal{Y}}, \quad \mathcal{C}\left(\mathcal{R},\mathcal{Y}\right) = \frac{\mathcal{R}\cos\mathcal{Y}}{1+\mathcal{R}^2+2\mathcal{R}\sin\mathcal{Y}}. \tag{39}$$

Let us also note that in new variables we have that

$$\operatorname{Re}Y(\psi) = \mathcal{R}(\psi)\cos\mathcal{Y}(\psi), \qquad \operatorname{Im}Y(\psi) = \mathcal{R}(\psi)\sin\mathcal{Y}(\psi) \tag{40}$$

and

$$|Y(\psi) + i|^2 = 1 + \mathcal{R}^2(\psi) + 2\mathcal{R}(\psi)\sin\mathcal{Y}(\psi). \tag{41}$$

Then the functions $\mathcal{S}\left(\mathcal{R},\mathcal{Y}\right)$ and $\mathcal{C}\left(\mathcal{R},\mathcal{Y}\right)$ can be also defined as follows:

$$\mathcal{S}\left(\mathcal{R},\mathcal{Y}\right) = \frac{\operatorname{Im}Y(\psi)}{|Y(\psi)+i|^2}, \qquad \mathcal{C}\left(\mathcal{R},\mathcal{Y}\right) = \frac{\operatorname{Re}Y(\psi)}{|Y(\psi)+i|^2} \tag{42}$$

with the compatibility condition

$$\mathcal{S}^2\left(\mathcal{R},\mathcal{Y}\right) + \mathcal{C}^2\left(\mathcal{R},\mathcal{Y}\right) = \frac{|Y(\psi)|^2}{|Y(\psi)+i|^4}. \tag{43}$$

- **Another form of equations**

Let us note that Eqs. (37) and (38) can be equivalently rewritten as follows:

$$\frac{\dot{\mathcal{R}} + \dot{G}\mathcal{R}}{\mathcal{R}^2 + 1} = \left[\dot{G}\,\mathcal{C}\left(\mathcal{R}, \mathcal{Y}\right) - 1\right] \cos \mathcal{Y}, \tag{44}$$

$$\frac{\mathcal{R}\dot{\mathcal{Y}}}{\mathcal{R}^2 - 1} = \left[\dot{G}\,\mathcal{C}\left(\mathcal{R}, \mathcal{Y}\right) - 1\right] \sin \mathcal{Y}. \tag{45}$$

- **Solution**

In a general complex case Eqs. (37) and (38) can be integrated with respect to $\mathcal{Y}(\psi)$:

$$\mathcal{Y}(\psi) = \arcsin\left\{\frac{1 + \mathcal{R}^2(\psi)}{\mathcal{R}(\psi)}\, Q(\psi) \exp\left[-2\int \frac{\dot{G}(\psi) d\psi}{1 + \mathcal{R}^2(\psi)}\right]\right\}. \tag{46}$$

Proof. Let us denote

$$\mathcal{J}\left(\mathcal{R}, \mathcal{Y}\right) = \frac{\mathcal{R} \sin \mathcal{Y}}{\mathcal{R}^2 + 1}. \tag{47}$$

Then using Eqs. (37) and (38) we can calculate its derivative with respect to ψ:

$$\dot{\mathcal{J}}\left(\mathcal{R}, \mathcal{Y}\right) = \frac{\left(1 - \mathcal{R}^2\right) \dot{\mathcal{R}} \sin \mathcal{Y} + \left(1 + \mathcal{R}^2\right) \mathcal{R}\dot{\mathcal{Y}} \cos \mathcal{Y}}{\left(\mathcal{R}^2 + 1\right)^2} = \frac{\mathcal{R}^2 - 1}{\mathcal{R}^2 + 1}\, \dot{G} \mathcal{J}\left(\mathcal{R}, \mathcal{Y}\right). \tag{48}$$

We see that Eq. (48) can be easily integrated. □

- **Second-order nonlinear differential equation**

For the function $\mathcal{C}(\psi) = \mathcal{C}\left[\mathcal{R}(\psi), \mathcal{Y}(\psi)\right]$ we obtain a nice nonlinear second-order differential equation

$$\ddot{\mathcal{C}}(\psi) + 4\mathcal{C}(\psi) = \frac{\dot{G}(\psi)}{2}\left[\dot{\mathcal{C}}^2(\psi) + 4\mathcal{C}^2(\psi) - 1\right] \tag{49}$$

with the eigenfrequency 2 and modulation determined by the variable refraction index

$$n(x) = n_0 \exp\left[G\left[\psi(x)\right]\right], \tag{50}$$

where $n_0 = (c/\omega)\, q_0$.

- **Parametric solutions**

Therefore, there are two ways of constructing sought parametric solutions:

(i) to define $G(\psi)$ and then solve Eq. (49) with respect to $\mathcal{C}(\psi)$ or

(ii) to define $\mathcal{C}(\psi)$ and then find $G(\psi)$ by integration:

$$G(\psi) = 2\int \frac{\ddot{\mathcal{C}}(\psi) + 4\mathcal{C}(\psi)}{\dot{\mathcal{C}}^2(\psi) + 4\mathcal{C}^2(\psi) - 1}\, d\psi. \tag{51}$$

Remark: If we take that $\dot{\mathcal{C}}(\psi) \equiv 0$, then the function $\mathcal{C}(\psi)$ is constant and from Eq. (51) we obtain that

$$G(\psi) = \frac{8\mathcal{C}}{4\mathcal{C}^2 - 1}\left(\psi - \psi_0\right). \tag{52}$$

• Relations

The variables $\mathcal{R}$ and $\mathcal{Y}$ can be expressed through $\mathcal{C}$ and its first derivative $\dot{\mathcal{C}}$ as follows:

$$\operatorname{ctg}\mathcal{Y} = \frac{\mathcal{C}}{\mathcal{S}} = \frac{4\mathcal{C}}{1 - 4\mathcal{C}^2 - \dot{\mathcal{C}}^2}, \qquad \mathcal{R}^2 = \frac{4\mathcal{C}^2 + \left(1 + \dot{\mathcal{C}}\right)^2}{4\mathcal{C}^2 + \left(1 - \dot{\mathcal{C}}\right)^2}. \tag{53}$$

Therefore, the complex admittance $Y = \mathcal{R}\exp(i\mathcal{Y})$ can be expressed through $\mathcal{C}$ and its first derivative $\dot{\mathcal{C}}$ as follows:

$$Y = \left(1 + \mathcal{R}^2\right)\left[\frac{\mathcal{R}\cos\mathcal{Y}}{1 + \mathcal{R}^2} + i\,\frac{\mathcal{R}\sin\mathcal{Y}}{1 + \mathcal{R}^2}\right] = \frac{4\mathcal{C} + i\left(1 - 4\mathcal{C}^2 - \dot{\mathcal{C}}^2\right)}{4\mathcal{C}^2 + \left(1 - \dot{\mathcal{C}}\right)^2}. \tag{54}$$

If the functions $Q(\psi)$ and/or $\mathcal{C}(\psi)$ are given, then the following expressions for $X(\psi)$ and the complex-valued wave function $W(\psi)$ can be written:

$$X(\psi) = \int \left(1 - \dot{G}(\psi)\mathcal{C}(\psi)\right)\frac{d\psi}{Q(\psi)}, \tag{55}$$

$$W(\psi) = w_0 \exp\left[\int \left(1 - \dot{G}(\psi)\mathcal{C}(\psi)\right) Y(\psi) d\psi\right] = w_0 \exp\left[\int \frac{\left(1 - \dot{G}\mathcal{C}\right)\left[4\mathcal{C} + i\left(1 - 4\mathcal{C}^2 - \dot{\mathcal{C}}^2\right)\right]}{4\mathcal{C}^2 + \left(1 - \dot{\mathcal{C}}\right)^2}\, d\psi\right]. \tag{56}$$

- **Partial solutions**

Let us note that for any function $\dot{G}(\psi)$ there are two particular solutions of Eq. (49), namely,

$$\mathcal{C}_1(\psi) = \alpha \sin \beta\psi, \qquad \mathcal{C}_2(\psi) = \alpha \cos \beta\psi, \tag{57}$$

where

$$\alpha = \pm\frac{1}{\beta}, \qquad \beta = \pm 2. \tag{58}$$

Proof. It is easy to check that the first particular solution $\mathcal{C}_1(\psi)$ is the solution of Eq. (49) by direct calculations of the following terms:

$$\ddot{\mathcal{C}}_1(\psi) + 4\mathcal{C}_1(\psi) = \left(4 - \beta^2\right) \alpha \sin \beta\psi, \tag{59}$$

$$\dot{\mathcal{C}}_1^2(\psi) + 4\mathcal{C}_1^2(\psi) - 1 = \left(4 - \beta^2\right) \alpha^2 \sin^2 \beta\psi + \left(\alpha^2\beta^2 - 1\right). \tag{60}$$

Therefore, if we suppose that α and β fulfil the following conditions:

$$\alpha^2\beta^2 - 1 = 0, \qquad 4 - \beta^2 = 0, \tag{61}$$

then for any function $\dot{G}(\psi)$ the left- and right-hand sides of Eq. (49) are equal to zero separately.

The same is true for the second particular solution $\mathcal{C}_2(\psi)$. □

• Real-valued admittance

For the partial solutions of Eq. (49), the admittance $Y(\psi)$ is purely real, i.e.,

$$Y(\psi) = \begin{cases} \text{“}+\text{”}: & \operatorname{ctg}(\psi - \psi_0), \\ \text{“}-\text{”}: & -\operatorname{tg}(\psi - \psi_0), \end{cases} \tag{62}$$

therefore, for any given function $\dot{G}(\psi)$ (equivalently $Q(\psi)$) we obtain the following expressions for $X(\psi)$:

$$X(\psi) = \int \left[1 \mp \dot{G}(\psi) \sin(\psi - \psi_0) \cos(\psi - \psi_0)\right] \frac{d\psi}{Q(\psi)}, \tag{63}$$

and the complex-valued wave function $W(\psi)$:

$$W(\psi) = w_0 \sqrt{1 - Z^2(\psi)} \exp\left[-\int \dot{G}(\psi) Z^2(\psi) d\psi\right], \tag{64}$$

where

$$Z(\psi) = \begin{cases} \text{“}+\text{”}: & \cos(\psi - \psi_0), \\ \text{“}-\text{”}: & \sin(\psi - \psi_0). \end{cases} \tag{65}$$

- **Special solutions**

Though it is hardly possible to find the exact solution of Eq. (49) in a general case, the above analysis clarifies the nature of quasi-periodic Bloch waves in the transmission band and allows one to construct a wide class of special analytical solutions. A continual set of integrable wave equations can be obtained if we choose

$$\dot{G}\left[\psi(\mathcal{C})\right] = \frac{d\ln M(\mathcal{C})}{d\mathcal{C}} = \frac{1}{M(\mathcal{C})}\frac{dM(\mathcal{C})}{d\mathcal{C}}, \tag{66}$$

where $M(\mathcal{C})$ is an arbitrary real-valued function. In this case Eq. (49) has an energy integral

$$\dot{\mathcal{C}}^2 = 1 - 4\mathcal{C}^2 + M(\mathcal{C}) \tag{67}$$

and a periodic solution $\mathcal{C}(\psi) = \mathcal{C}(\psi+\tau)$ given by the following expressions:

$$\psi = \pm\int \frac{d\mathcal{C}}{\sqrt{1-4\mathcal{C}^2+M(\mathcal{C})}}, \qquad \tau = 2\int_{\mathcal{C}_-}^{\mathcal{C}_+} \frac{d\mathcal{C}}{\sqrt{1-4\mathcal{C}^2+M(\mathcal{C})}}, \tag{68}$$

where the turning points $\mathcal{C}_\pm$ are the roots of the radical.

Proof. Let us notice that using Eq. (66) we can calculate the complete derivative of $M\left[\mathcal{C}(\psi)\right]$ with respect to ψ as follows:

$$\frac{\dot{M}(\mathcal{C})}{M(\mathcal{C})} = \frac{1}{M(\mathcal{C})}\frac{dM(\mathcal{C})}{d\mathcal{C}}\frac{d\mathcal{C}}{d\psi} = \dot{G}\dot{\mathcal{C}}. \tag{69}$$

Then rewriting Eq. (49) in the form

$$\dot{G}\dot{\mathcal{C}} = \frac{2\dot{\mathcal{C}}\left(\ddot{\mathcal{C}}+4\mathcal{C}\right)}{\dot{\mathcal{C}}^2+4\mathcal{C}^2-1} = \frac{d}{d\psi}\ln\left[\dot{\mathcal{C}}^2+4\mathcal{C}^2-1\right] \equiv \frac{d}{d\psi}\ln M(\mathcal{C}) \tag{70}$$

and integrating Eq. (70) we obtain Eq. (67). □

- **General reasoning**

Eq. (49) can be written in the following form:

$$\ddot{\mathcal{C}} = f\left(\dot{\mathcal{C}}, \mathcal{C}, \psi\right), \tag{71}$$

where the direct dependence on ψ is realized only through the function $\dot{G}(\psi)$. Let us suppose that in some way we have rewritten it as a function of $\mathcal{C}$, i.e., $\dot{G}(\mathcal{C}) = \dot{G}\left[\psi(\mathcal{C})\right]$. Then in Eq. (71) the direct dependence on ψ is missing, and therefore, we can take $\mathcal{C}$ as an independent variable. Then we obtain that $\dot{\mathcal{C}} = y(\mathcal{C})$, $\ddot{\mathcal{C}} = y(\mathcal{C})y'(\mathcal{C})$, and Eq. (71) reads

$$2y(\mathcal{C})y'(\mathcal{C}) - \dot{G}(\mathcal{C})y^2(\mathcal{C}) = \dot{G}(\mathcal{C})\left(4\mathcal{C}^2 - 1\right) - 8\mathcal{C}, \tag{72}$$

where $\prime$ denotes the derivative with respect to $\mathcal{C}$. Let us note that

$$\left(\frac{y^2}{M(\mathcal{C})}\right)' = \frac{1}{M(\mathcal{C})}\left[2yy' - \frac{M'(\mathcal{C})}{M(\mathcal{C})}y^2\right], \tag{73}$$

and we see that to integrate Eq. (72) it is enough to suppose that the connection between the functions $\dot{G}(\mathcal{C})$ and $M(\mathcal{C})$ is given by Eq. (66). Then we obtain the following first-order differential equation:

$$\dot{\mathcal{C}}^2 = y^2 = M(\mathcal{C})\left[\int \frac{4\mathcal{C}^2 - 1}{M^2(\mathcal{C})}\, dM(\mathcal{C}) - \int \frac{8\mathcal{C}}{M(\mathcal{C})}\, d\mathcal{C}\right]. \tag{74}$$

If we compare Eq. (74) with Eq. (67), we obtain the compatibility condition

$$\frac{4\mathcal{C}^2 - 1}{M(\mathcal{C})} + \int \frac{4\mathcal{C}^2 - 1}{M^2(\mathcal{C})}\, dM(\mathcal{C}) = 1 + \int \frac{8\mathcal{C}}{M(\mathcal{C})}\, d\mathcal{C}. \tag{75}$$

- $M(\mathcal{C}) = const \qquad \Rightarrow \qquad \sin$

If we suppose that the function $M(\mathcal{C})$ is constant, i.e.,

$$M(\mathcal{C}) = c, \qquad c > -1, \qquad c \neq 0, \tag{76}$$

then $\dot{G} = 0$ and we obtain that

$$\psi(\mathcal{C}) = \pm \int \frac{d\mathcal{C}}{\sqrt{1 + c - 4\mathcal{C}^2}} = \pm\frac{1}{2} \int \frac{dy}{\sqrt{1 - y^2}}, \qquad y = \frac{2\mathcal{C}}{\sqrt{1+c}}, \tag{77}$$

$$\mathcal{C}(\psi) = \pm\frac{\sqrt{1+c}}{2} \sin 2\left(\psi - \psi_0\right), \qquad \psi_0 = \psi(0). \tag{78}$$

- $M(\mathcal{C}) = c + 8e\mathcal{C} \qquad \Rightarrow \qquad \sin$

If we suppose that

$$M(\mathcal{C}) = c + 8e\mathcal{C}, \qquad c > -1 - 4e^2, \qquad c \neq 0, \tag{79}$$

where c and e are constants, then

$$\psi(\mathcal{C}) = \pm \int \frac{d\mathcal{C}}{\sqrt{1 + c + 8e\mathcal{C} - 4\mathcal{C}^2}} = \pm\frac{1}{2} \int \frac{dy}{\sqrt{1 - y^2}}, \qquad y = \frac{2\left(\mathcal{C} - e\right)}{\sqrt{1 + c + 4e^2}}. \tag{80}$$

$$\mathcal{C}(\psi) = e \pm \frac{\sqrt{1 + c + 4e^2}}{2} \sin 2\left(\psi - \psi_0\right). \tag{81}$$

- $M(\mathcal{C}) = c + 8e\mathcal{C} - d^2\mathcal{C}^2 \qquad \Rightarrow \qquad \sin$

If we suppose that

$$M(\mathcal{C}) = c + 8e\mathcal{C} - d^2\mathcal{C}^2, \qquad c > -1 - \frac{16e^2}{d^2+4}, \qquad c \neq 0, \tag{82}$$

where c, e, and d are constants, then

$$\psi(\mathcal{C}) = \pm\int \frac{d\mathcal{C}}{\sqrt{1+c+8e\mathcal{C}-(d^2+4)\,\mathcal{C}^2}} = \pm\frac{1}{2}\int\frac{dy}{\sqrt{1-y^2}}, \qquad y = \frac{(d^2+4)\,\mathcal{C}-4e}{\sqrt{(1+c)\,(d^2+4)+16e^2}}. \tag{83}$$

$$\mathcal{C}(\psi) = \frac{1}{d^2+4}\left\{4e \pm \sqrt{(1+c)\,(d^2+4)+16e^2}\,\sin\left[\sqrt{d^2+4}\,(\psi-\psi_0)\right]\right\}. \tag{84}$$

- $M(\mathcal{C}) = c + 8e\mathcal{C} + \left(k^2+4\right)\mathcal{C}^2 \qquad \Rightarrow \qquad \mathrm{sh}$

If we suppose that

$$M(\mathcal{C}) = c + 8e\mathcal{C} + \left(k^2+4\right)\mathcal{C}^2, \qquad c > -1 + \frac{16e^2}{k^2}, \qquad k > 0, \qquad c \neq 0, \tag{85}$$

where c, e, and k are constants, then

$$\psi(\mathcal{C}) = \pm\int \frac{d\mathcal{C}}{\sqrt{1+c+8e\mathcal{C}+k^2\mathcal{C}^2}} = \pm\frac{1}{2}\int\frac{dy}{\sqrt{1+y^2}}, \qquad y = \frac{k^2\mathcal{C}+4e}{\sqrt{(1+c)k^2-16e^2}}. \tag{86}$$

$$\mathcal{C}(\psi) = \frac{1}{k^2}\left\{-4e \pm \sqrt{(1+c)k^2-16e^2}\,\mathrm{sh}\,k\,(\psi-\psi_0)\right\}. \tag{87}$$

- $M(\mathcal{C}) = \left(4a^2 - 1\right) + b^2\mathcal{C}^4 \qquad \Rightarrow \qquad \mathrm{sn}$

Let us also consider an instructive example of modulated waves in a periodic dielectric medium, determined by the following potential:

$$M(\mathcal{C}) = \left(4a^2 - 1\right) + b^2\mathcal{C}^4, \qquad a, b > 0, \qquad ab < 1, \qquad 4a^2 \neq 1. \tag{88}$$

Then we obtain that

$$\psi(\mathcal{C}) = \psi_0 \pm \int_0^{\mathcal{C}} \frac{d\mathcal{C}}{\sqrt{4a^2 - 4\mathcal{C}^2 + b^2\mathcal{C}^4}} = \psi_0 \pm \frac{1}{b}\int_0^{\mathcal{C}} \frac{d\mathcal{C}}{\sqrt{\left(\mathcal{C}_+^2 - \mathcal{C}^2\right)\left(\mathcal{C}_-^2 - \mathcal{C}^2\right)}}, \tag{89}$$

where the roots of the radical are given as follows:

$$\mathcal{C}_\pm^2 = \frac{2}{b^2}\left(1 \pm \sqrt{1 - a^2b^2}\right). \tag{90}$$

If we take that $\mathcal{C}_+ > \mathcal{C}_- > \mathcal{C} > 0$ (the roots $\mathcal{C}_\pm$ are real for $ab \leq 1$; additionally the condition $\mathcal{C}_+ > \mathcal{C}_-$ imposes $ab \neq 1$), then the auxiliary function $\mathcal{C}(\psi)$ is expressed through the Jacobi elliptic functions of the first kind:

$$\mathcal{C}(\psi) = \pm a\sqrt{1 + p^2}\ \mathrm{sn}\left[\frac{2\left(\psi - \psi_0\right)}{\sqrt{1 + p^2}}, p\right], \tag{91}$$

where

$$\frac{\mathcal{C}_-}{\mathcal{C}_+} = p = \frac{\sqrt{1 - \sqrt{1 - a^2b^2}}}{\sqrt{1 + \sqrt{1 - a^2b^2}}} = \frac{1 - \sqrt{1 - a^2b^2}}{ab}, \qquad \sqrt{1 + p^2} = \frac{\sqrt{2}}{ab}\sqrt{1 - \sqrt{1 - a^2b^2}} = \frac{\mathcal{C}_-}{a}, \qquad \mathcal{C}_+\mathcal{C}_- = \frac{2a}{b}, \tag{92}$$

• Complex-valued wave function

For any given function $M\left[\mathcal{C}(\psi)\right]$ the complex-valued wave function $W(\psi)$ can be rewritten as follows:

$$W = w_0 \exp\left[\pm\int \frac{4\mathcal{C} - iM}{4\mathcal{C}^2 + \left(1 \mp \sqrt{1 - 4\mathcal{C}^2 + M}\right)^2} \frac{(M - \mathcal{C}M')\, d\mathcal{C}}{M\sqrt{1 - 4\mathcal{C}^2 + M}}\right]. \tag{93}$$

In particular, for the complex increment we obtain that

$$\chi + i\eta = \ln\left[\frac{W(\psi + \tau)}{W(\psi)}\right] = 2\int_{\mathcal{C}_-}^{\mathcal{C}_+} \frac{2 + M}{M^2 + 16\mathcal{C}^2} \left\{\frac{4\mathcal{C}}{M} - i\right\} \frac{(M - \mathcal{C}M')\, d\mathcal{C}}{\sqrt{1 - 4\mathcal{C}^2 + M}}. \tag{94}$$

• Even functions $M(\mathcal{C})$

Let us take an arbitrary even function $M(\mathcal{C})$, then the function

$$G(\psi) = \int \dot{G}(\psi) d\psi = \pm\int \frac{dM(\mathcal{C})}{M(\mathcal{C})\sqrt{1 - 4\mathcal{C}^2 + M(\mathcal{C})}} \tag{95}$$

will be periodic. Moreover, for any even function $M(\mathcal{C})$ we have that

$$\chi = 2\int_{\mathcal{C}_-}^{\mathcal{C}_+} \frac{2 + M}{M^2 + 16\mathcal{C}^2} \left\{1 - \frac{\mathcal{C}M'}{M}\right\} \frac{4\mathcal{C} d\mathcal{C}}{\sqrt{1 - 4\mathcal{C}^2 + M}} = 0, \tag{96}$$

which means that $|W(\psi)|$ is periodic, while the phase advance per period τ, i.e.,

$$\eta = 4\int_{0}^{\mathcal{C}_+} \frac{2 + M}{M^2 + 16\mathcal{C}^2} \frac{(\mathcal{C}M' - M)\, d\mathcal{C}}{\sqrt{1 - 4\mathcal{C}^2 + M}}, \tag{97}$$

determines the modulation period $T = (2\pi/\eta)\,\tau$ of the quasi-periodic solution $W(\psi)$ predicted by the Floquets theory.

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Thank you for your attention!

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