Green's function, wavefunction and Wigner function of the MIC-Kepler problem

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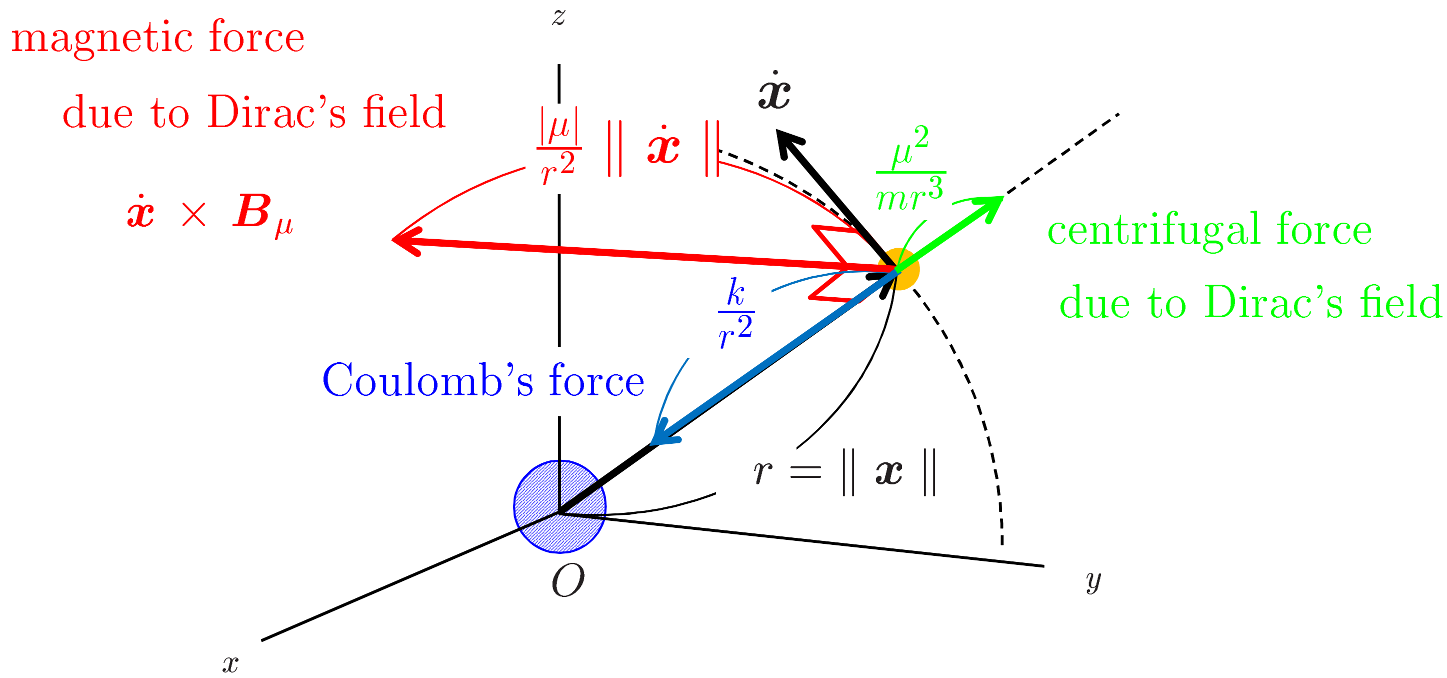
1. Hamiltonian description for the MIC-Kepler problem

In 1970, McIntosh and Cisneros studied the dynamical system describing the motion of a charged particle under the magnetic force due to Dirac's monopole field of strength $-\mu$ and the square inverse centrifugal potential force besides the Coulomb's potential force.

$$\boldsymbol{B}_\mu = \frac{-\mu}{r^3}\boldsymbol{x}$$

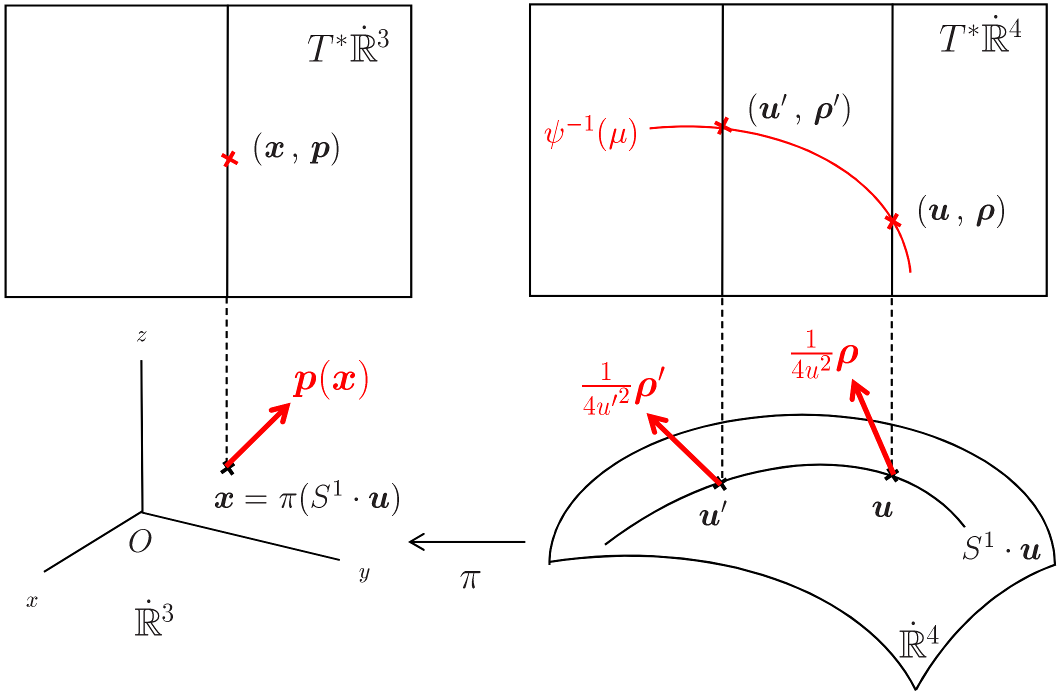
$$\| \boldsymbol{B}_\mu \| = \frac{|\mu|}{r^2}$$



μ is quantized as $\quad \mu \in \frac{\hbar}{2}\mathbb{Z}\,.$

The Hamiltonian description for the MIC-Kepler problem is given by **T. Iwai** and **Y. Uwano** (1986) as follows.

The MIC-Kepler problem
is the
reduced Hamiltonian system
of the 4-dimensional
conformal Kepler problem
by an S^1 action,
if the associated
momentum mapping
$\psi(\boldsymbol{u}, \boldsymbol{\rho}) \equiv \mu \neq 0$.



$\psi(\boldsymbol{u}, \boldsymbol{\rho})$ is invariant under the S^1 action, then
let $\psi^{-1}(\mu) \subset T^*_{\boldsymbol{u}}\dot{\mathbb{R}}^4$ be a subset s.t.

$$\psi^{-1}(\mu) = \left\{ (\boldsymbol{u}, \boldsymbol{\rho}) \in T^*_{\boldsymbol{u}}\dot{\mathbb{R}}^4 \;\middle|\; \psi(\boldsymbol{u}, \boldsymbol{\rho}) = \frac{1}{2}(-u_2\rho_1 + u_1\rho_2 - u_4\rho_3 + u_3\rho_4) = \mu \right\}.$$

The MIC-Kepler problem is a triple $(T^*\dot{\mathbb{R}}^3, \sigma_\mu , \boxed{H_\mu})$ where

$$\sigma_\mu = dp_x\wedge dx + dp_y\wedge dy + dp_z\wedge dz - \frac{\mu}{r^3}\left(x\, dy\wedge dz + y\, dz\wedge dx + z\, dx\wedge dy\right),$$

$$\boxed{H_\mu(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2m}({p_x}^2 + {p_y}^2 + {p_z}^2) + \frac{\mu^2}{2mr^2} - \frac{k}{r}}.$$

Its energy hyper surface : $\boxed{H_\mu} = E \Leftrightarrow \Phi(\boldsymbol{x}, \boldsymbol{p}) \equiv r\,(H_\mu - E) = 0$
is equal to

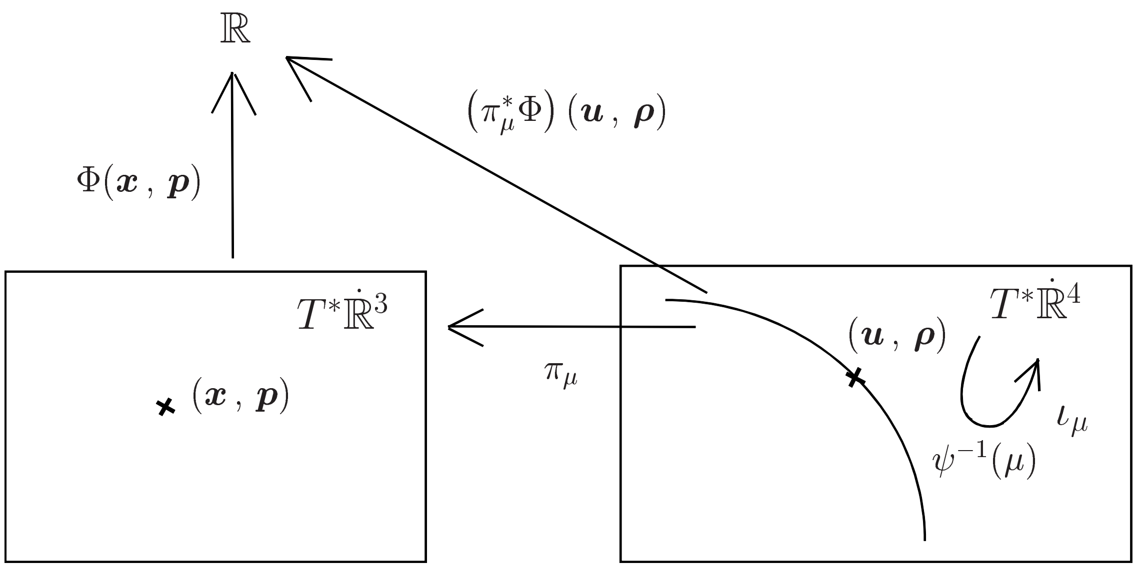
$$\left(\pi_\mu^*\Phi\right)(\boldsymbol{u}, \boldsymbol{\rho}) = 0$$

where

$$\psi^{-1}(\mu) \xrightarrow{\pi_\mu} T^*\dot{\mathbb{R}}^3,$$

$$\pi_\mu^*\Phi = u^2\left(\boxed{H} - E\right),$$

$$u^2 \equiv {u_1}^2 + {u_2}^2 + {u_3}^2 + {u_4}^2 .$$



The conformal Kepler problem is a triple $(T^*\dot{\mathbb{R}}^4\,,\, d\boldsymbol{\rho}\wedge d\boldsymbol{u}\,,\, \boxed{H})$ where

$$d\boldsymbol{\rho}\wedge d\boldsymbol{u} \equiv \sum_{j=1}^{4} d\rho_j \wedge du_j \quad , \quad \boxed{H(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \frac{1}{2m}\left(\frac{1}{4u^2}\sum_{j=1}^{4}{\rho_j}^2\right) - \frac{k}{u^2}}\,.$$

Since $u^2 = r > 0$ (invariant under the S^1 action), $\pi_\mu^*\Phi = 0$ is equal to $\boxed{H} - E = 0$.
The energy hyper surface $\boxed{H} = E$ is equivalent to $\underline{K(\boldsymbol{u}\,,\,\boldsymbol{\rho})} = \epsilon$ where $K(\boldsymbol{u},\,\boldsymbol{\rho})$ is the Hamiltonian of 4-dimensional harmonic oscillator:

$$\underline{K(\boldsymbol{u},\,\boldsymbol{\rho}) = \frac{1}{2m}\sum_{j=1}^{4}{\rho_j}^2 + \frac{1}{2}m\omega^2\sum_{j=1}^{4}{u_j}^2} \qquad \begin{cases} m > 0 & \text{mass of pendulum} \\ \omega > 0 & \text{angular frequency} \end{cases}$$

considering only the case where the real parameter $E < 0$ and putting both the constant $m\omega^2 \equiv -8E$ and a real parameter $\epsilon \equiv 4k$.

We solved the harmonic oscillator by means of **the Moyal product**, which brought the following functions.

- ♣ **Moyal propagator ($*$-exponential)**

$$e_*^{\frac{i}{\hbar}(t+iy')K\left(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\boldsymbol{\rho}\right)} = \left(\cos\frac{\omega z'}{2}\right)^{-4} \exp\left[i\,\frac{2}{\hbar\omega}K\left(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\,\boldsymbol{\rho}\right)\tan\frac{\omega z'}{2}\right]$$

where $\boldsymbol{u}_i = (u_1^i,\, u_2^i,\, u_3^i,\, u_4^i) \in \dot{\mathbb{R}}^4$ and $\boldsymbol{u}_f = (u_1^f,\, u_2^f,\, u_3^f,\, u_4^f) \in \dot{\mathbb{R}}^4$ denote initial point and final point of motion respectively.

- ♣ **Feynman's propagator**

$$\begin{aligned}
&\mathscr{K}(\boldsymbol{u}_f,\,\boldsymbol{u}_i\,;\,z'=t+iy')\\
&= \frac{-m^2\omega^2}{4\pi^2\hbar^2}\,\frac{1}{\sin^2(\omega z')}\exp\left[-i\,\frac{m\omega}{2\hbar}\frac{1}{\sin(\omega z')}\left\{(u_i^2+u_f^2)\cos(\omega z') - 2\boldsymbol{u}_i\cdot\boldsymbol{u}_f\right\}\right]\\
&= \sum_{n=-\infty}^{\infty} C_n\, e^{in\omega t}
\end{aligned}$$

where $$C_n \equiv \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega}\mathscr{K}(\boldsymbol{u}_f,\,\boldsymbol{u}_i\,;\,\tau+iy')\,e^{-in\omega\tau}\,d\tau\,.$$

♣ Green's function

$G(\boldsymbol{u}_f, \boldsymbol{u}_i\,; \epsilon)$

$$= \lim_{y' \to +0} \frac{i}{\hbar} \int_0^\infty \left(\sum_{n=-\infty}^{\infty} C_n \, e^{in\omega t} \right) e^{-\frac{y'+i\epsilon}{\hbar}(t+iy')} dt$$

$$= \frac{m^2\omega^2}{\pi^2\hbar^2} \exp\left[-\frac{m\omega}{2\hbar}(u_i^2 + u_f^2) \right] \sum_{N=0}^{\infty} \sum_{l_1+l_2+l_3+l_4=N} \frac{1}{\epsilon - (N+2)\hbar\omega}$$

$$\frac{1}{2^N \, l_1! l_2! l_3! l_4!} H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}} u_1^i\right) H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}} u_1^f\right) H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}} u_2^i\right) H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}} u_2^f\right)$$

$$H_{l_3}\left(\sqrt{\frac{m\omega}{\hbar}} u_3^i\right) H_{l_3}\left(\sqrt{\frac{m\omega}{\hbar}} u_3^f\right) H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}} u_4^i\right) H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}} u_4^f\right)$$

where $n - 2 \equiv N = l_1 + l_2 + l_3 + l_4$ $(l_1, l_2, l_3, l_4 \in \mathbb{N} \cup \{0\})$,
$H_l(X)$ is the Hermite polynomial :

$$H_l(X) = (-1)^l \, e^{X^2} \frac{d^l}{dX^l} \, e^{-X^2} \, .$$

Moreover, we denote by $\Psi_N(\boldsymbol{u})$ **the wave function** of 4-dimensional harmonic oscillator on $\dot{\mathbb{R}}^4$

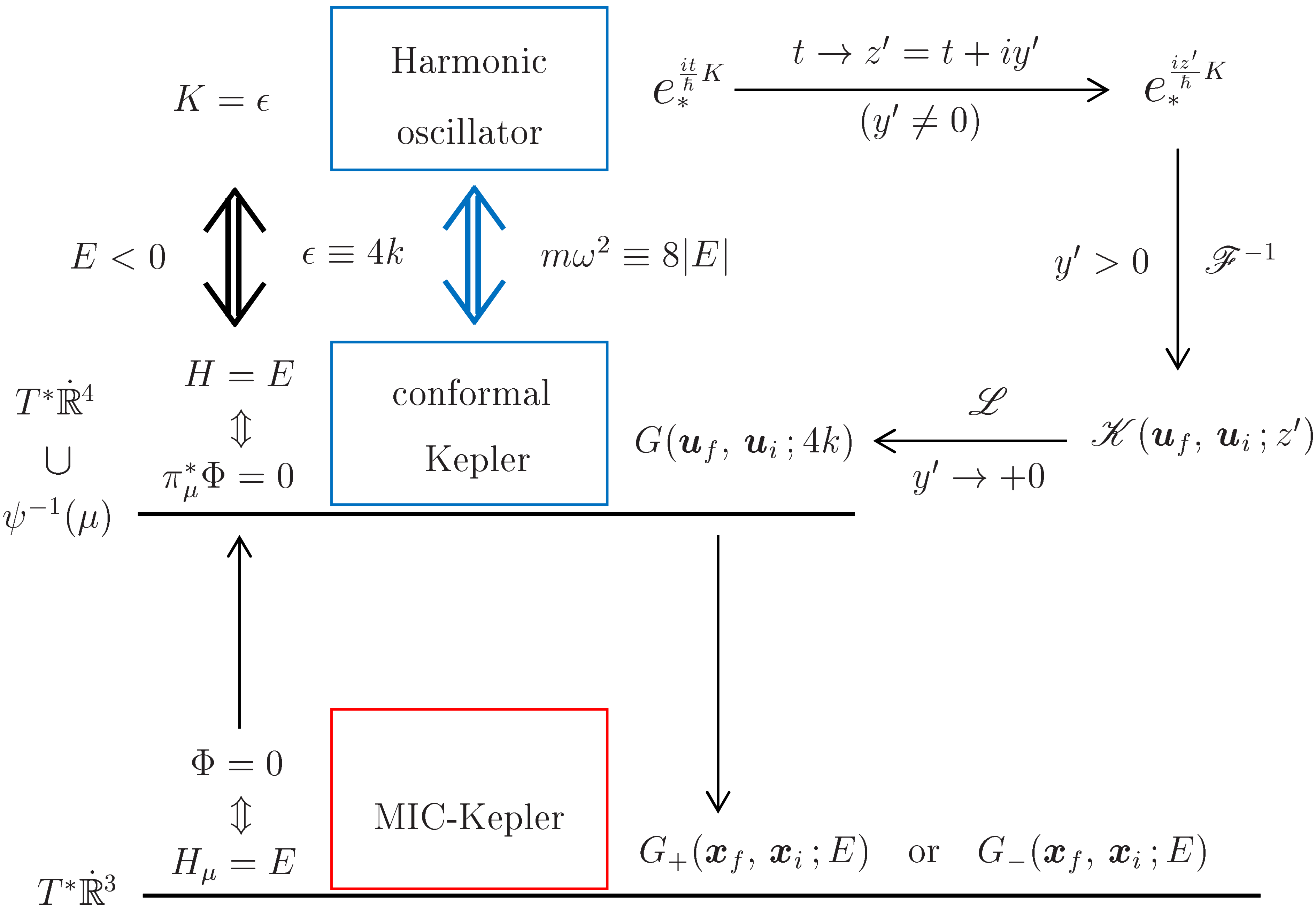
$$\Psi_N(\boldsymbol{u}) \;\; \underset{\longleftarrow}{=} \;\; \frac{m\omega}{\pi\hbar}\frac{1}{\sqrt{2^N l_1!l_2!l_3!l_4!}}\exp\left[-\frac{m\omega}{2\hbar}(u_1^2+u_2^2+u_3^2+u_4^2)\right]$$
$$H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1\right)H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}}u_2\right)H_{l_3}\left(\sqrt{\frac{m\omega}{\hbar}}u_3\right)H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4\right)$$

satisfying

$$\hat{K}\,\Psi_N(\boldsymbol{u}) = \epsilon\,\Psi_N(\boldsymbol{u}) \quad \text{where} \;\; \hat{K} = -\frac{\hbar^2}{2m}\left(\sum_{j=1}^{4}\frac{\partial^2}{\partial u_j^2}\right) + \frac{1}{2}m\omega^2\sum_{j=1}^{4}u_j^2\,.$$

Then we verify

$$G(\boldsymbol{u}_f,\,\boldsymbol{u}_i\,;\epsilon) = \sum_{N=0}^{\infty}\frac{1}{\epsilon-(N+2)\hbar\omega}\,\Psi_N(\boldsymbol{u}_f)\,\overline{\Psi_N(\boldsymbol{u}_i)}\,.$$



Harmonic oscillator
$K = \epsilon$
$e_*^{\frac{it}{\hbar}K}$
$t \to z' = t + iy'$
$(y' \neq 0)$
$e_*^{\frac{iz'}{\hbar}K}$
$E < 0$
$\epsilon \equiv 4k$
$m\omega^2 \equiv 8|E|$
$y' > 0$
$\mathscr{F}^{-1}$
$T^*\dot{\mathbb{R}}^4$
$\cup$
$\psi^{-1}(\mu)$
$H = E$
$\Updownarrow$
$\pi_\mu^* \Phi = 0$
conformal Kepler
$G(\boldsymbol{u}_f, \boldsymbol{u}_i; 4k)$
$\mathscr{L}$
$y' \to +0$
$\mathscr{K}(\boldsymbol{u}_f, \boldsymbol{u}_i; z')$
$\Phi = 0$
$\Updownarrow$
$H_\mu = E$
MIC-Kepler
$G_+(\boldsymbol{x}_f, \boldsymbol{x}_i; E)$ or $G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E)$
$T^*\dot{\mathbb{R}}^3$

2. Green's function of the MIC-Kepler problem

We suppose $E \neq \dfrac{-2mk^2}{\hbar^2 (N+2)^2}$ $\quad (N = 0, 1, 2, \cdots)$, then reduce the Green's function of the conformal Kepler problem $G(\boldsymbol{u}_f, \boldsymbol{u}_i\,; \epsilon \equiv 4k)$ assumed $m\omega^2 \equiv -8E$ to the Green's function of the MIC-Kepler problem $G_+(\boldsymbol{x}_f, \boldsymbol{x}_i\,; E)$ or $G_-(\boldsymbol{x}_f, \boldsymbol{x}_i\,; E)$ by an S^1 action.

$G_+(\boldsymbol{x}_f, \boldsymbol{x}_i\,; E)$ and $G_-(\boldsymbol{x}_f, \boldsymbol{x}_i\,; E)$ denote the Green's functions in the following local coordinates τ_+ and τ_- respectively.

$$\tau_+ : \pi^{-1}(U_+) \ni \boldsymbol{u} \longmapsto (\pi(\boldsymbol{u}), \varphi_+(\boldsymbol{u})) = (\boldsymbol{x}(r, \theta, \phi), \exp(i\nu/2)) \in U_+ \times S^1$$

$$\left\{ \begin{array}{l} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{array} \right. , \quad \left\{ \begin{array}{l} u_1 = \sqrt{r} \cos\dfrac{\theta}{2} \cos\dfrac{\nu+\phi}{2} \ , \quad u_2 = \sqrt{r} \cos\dfrac{\theta}{2} \sin\dfrac{\nu+\phi}{2} \\ \\ u_3 = \sqrt{r} \sin\dfrac{\theta}{2} \cos\dfrac{\nu-\phi}{2} \ , \quad u_4 = \sqrt{r} \sin\dfrac{\theta}{2} \sin\dfrac{\nu-\phi}{2} \end{array} \right.$$

where

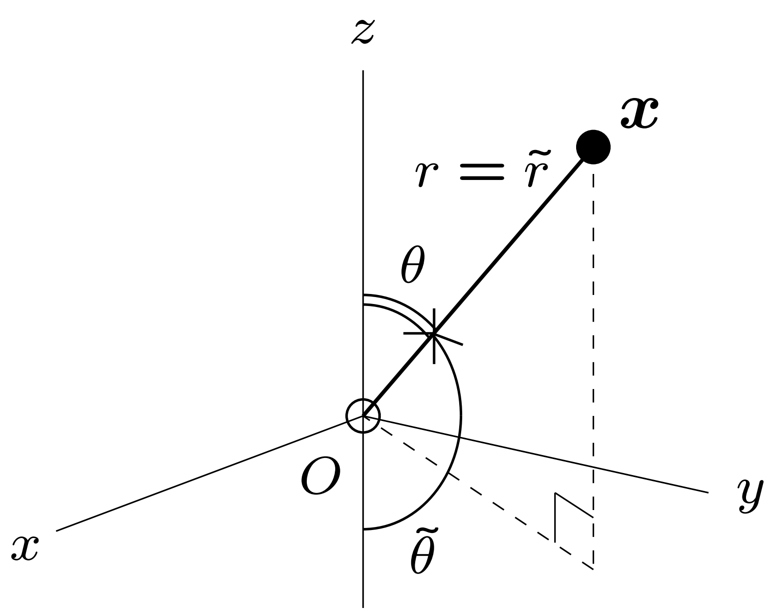
$$U_+ \underset{\longleftarrow}{\equiv} \left\{ \boldsymbol{x}(r, \theta, \phi) \in \dot{\mathbb{R}}^3\,; r > 0\,, 0 \le \theta < \pi\,, 0 \le \phi < 2\pi \right\}, 0 \le \nu < 4\pi\,.$$

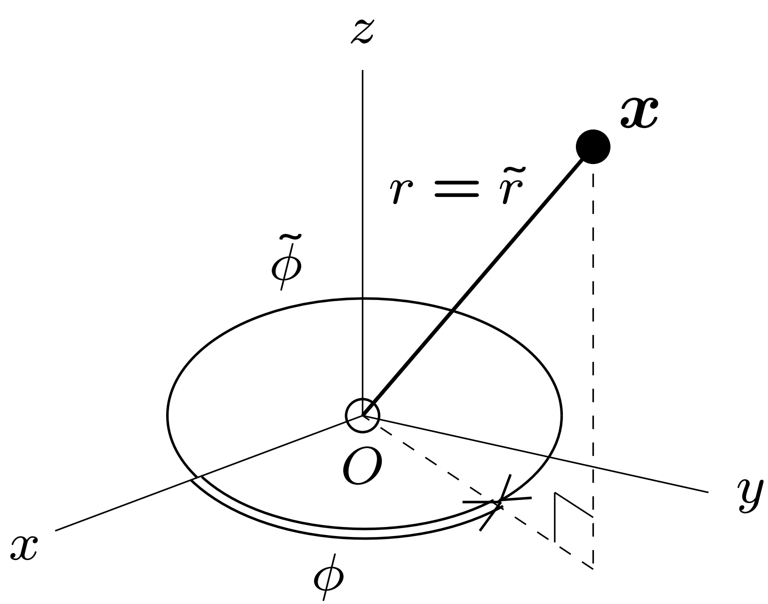
$$\tau_- : \pi^{-1}(U_-) \ \ni \ \boldsymbol{u} \longmapsto (\pi(\boldsymbol{u}),\, \varphi_-(\boldsymbol{u})) = (\boldsymbol{x}(\tilde r,\, \tilde\theta,\, \tilde\phi),\, \exp(i\tilde\nu/2)) \ \in \ U_- \times S^1$$

$$\begin{cases} x = \tilde r \sin\tilde\theta \cos\tilde\phi \\ y = -\tilde r \sin\tilde\theta \sin\tilde\phi \\ z = -\tilde r \cos\tilde\theta \end{cases} , \quad \begin{cases} u_1 = -\sqrt{\tilde r} \sin\dfrac{\tilde\theta}{2} \cos\dfrac{\tilde\nu + \tilde\phi}{2} \,,\ u_2 = -\sqrt{\tilde r} \sin\dfrac{\tilde\theta}{2} \sin\dfrac{\tilde\nu + \tilde\phi}{2} \\ u_3 = -\sqrt{\tilde r} \cos\dfrac{\tilde\theta}{2} \cos\dfrac{\tilde\nu + 3\tilde\phi}{2} \,,\ u_4 = -\sqrt{\tilde r} \cos\dfrac{\tilde\theta}{2} \sin\dfrac{\tilde\nu + 3\tilde\phi}{2} \end{cases}$$

where

$$U_- \ \overset{\equiv}{\longleftarrow} \ \left\{\boldsymbol{x}(\tilde r,\, \tilde\theta,\, \tilde\phi) \in \dot{\mathbb{R}}^3 \,;\, \tilde r > 0 \,,\, 0 \le \tilde\theta < \pi \,,\, 0 < \tilde\phi \le 2\pi\right\} ,\ 0 \le \tilde\nu < 4\pi \,.$$





Iwai and **Uwano** also investigated the "quantized" system (1988) :

The "quantized" conformal Kepler problem is defined as a pair $(L^2(\mathbb{R}^4; 4u^2 d\boldsymbol{u}), \boxed{\hat{H}})$ where

$$\begin{cases} L^2(\mathbb{R}^4; 4u^2 d\boldsymbol{u}) : \text{ The Hilbert space of square integrable complex-valued functions on } \mathbb{R}^4, \\ \boxed{\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{1}{4u^2}\sum_{j=1}^{4}\frac{\partial^2}{\partial u_j^2}\right) - \frac{k}{u^2}} : \text{ The Hamiltonian operator.} \end{cases}$$

- They introduce **complex line bundles** L_l $(l \in \mathbb{Z})$ on which the linear connection is induced from a connection on the principal fibre bundle $\pi : \dot{\mathbb{R}}^4 \to \dot{\mathbb{R}}^3$.

- By an S^1 action, $L^2(\mathbb{R}^4; 4u^2 d\boldsymbol{u})$ is reduced to the Hilbert space, denoted by Γ_l, of square integrable **cross sections** in L_l over $\dot{\mathbb{R}}^3$.

The quantized MIC-Kepler problem is the reduced quantum system $(\Gamma_l,\ \hat{H}_l)$ where ∇_j stands for the covariant derivation with respect to the linear connection whose curvature gives Dirac's monopole field of strength $-l\hbar/2$,

$$\boxed{\hat{H}_l = -\frac{\hbar^2}{2m}\sum_{j=1}^{3}\nabla_j^2 + \frac{(l\hbar/2)^2}{2mr^2} - \frac{k}{r}}\,.$$

- Cross section in L_l corresponds uniquely to an eigenfunction of the momentum operator $\hat{N}$

$$\hat{N} = \frac{i\hbar}{2}\left(-u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_4}\right),$$

$$\hat{N}\,\Psi(\boldsymbol{u}) = -\frac{l}{2}\hbar\,\Psi(\boldsymbol{u}) \qquad \Psi(\boldsymbol{u}) \in L^2(\mathbb{R}^4; 4u^2 d\boldsymbol{u})\ ,\ \ l \in \mathbb{Z}\,.$$

- Accordingly, the introduction of L_l is understood as a geometric consequence of **the conservation of the angular momentum** associated with the $U(1) \simeq S^1$ action.

We can obtain **the wave function** of the quantized MIC-Kepler problem, denoted by $\Psi_N(\boldsymbol{x}) \in \Gamma_l$, as the following cross section with either of the local coordinates.

$$\text{i)} \ {}^\forall \boldsymbol{x} \in U_+ \ , \ \ \Psi_N^+(\boldsymbol{x}) \equiv \frac{\sqrt{\pi}}{2} e^{-i\,l\,\nu/2}\, \Psi_{N,\,l}(\boldsymbol{u})$$

$$\text{ii)} \ {}^\forall \boldsymbol{x} \in U_- \ , \ \ \Psi_N^-(\boldsymbol{x}) \equiv \frac{\sqrt{\pi}}{2} e^{-i\,l\,\tilde{\nu}/2}\, \Psi_{N,\,l}(\boldsymbol{u})$$

where $L^2(\mathbb{R}^4; 4u^2 d\boldsymbol{u}) \ni \Psi_{N,\,l}(\boldsymbol{u})$ satisfies

$$\begin{cases} \hat{H}\,\Psi(\boldsymbol{u}) = E\,\Psi(\boldsymbol{u}) \Leftrightarrow \hat{K}\,\Psi(\boldsymbol{u}) = \epsilon\,\Psi(\boldsymbol{u}) \\ \qquad\qquad\qquad\qquad\qquad \text{s.t. } \epsilon \equiv 4k \text{ and } m\omega^2 \equiv -8E \\ \hat{N}\,\Psi(\boldsymbol{u}) = -\dfrac{l}{2}\,\hbar\,\Psi(\boldsymbol{u})\,. \end{cases}$$

Finally, we can calculate the Green's function of the MIC-Kepler problem by the following infinite series consists of $\Psi_N(\boldsymbol{x})$ with either of the local coordinates.

$$G\,(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E = -m\omega^2/8) \ = \ \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega}\, \Psi_N(\boldsymbol{x}_f)\, \overline{\Psi_N(\boldsymbol{x}_i)}$$

Proposition 1-1. [The Green's function of the MIC-Kepler problem]

(i) When $\boldsymbol{x}_i\,,\ \boldsymbol{x}_f\ \in U_+\,,$

$$
\begin{aligned}
&G_+(\boldsymbol{x}_f,\boldsymbol{x}_i;E=-m\omega^2/8)\\
&=\frac{m^2\omega^2}{4\pi\hbar^2}\,e^{-\frac{m\omega}{2\hbar}(r_i+r_f)}\sum_{N=0}^{\infty}\frac{1}{4k-(N+2)\hbar\omega}\left(\frac{m\omega}{\hbar}\right)^N\\
&\quad\frac{1}{k_1!k_2!k_3!k_4!}\left(\sqrt{r_ir_f}\cos\frac{\theta_i}{2}\cos\frac{\theta_f}{2}\right)^{k_1+k_3}\left(\sqrt{r_ir_f}\sin\frac{\theta_i}{2}\sin\frac{\theta_f}{2}\right)^{k_2+k_4}\\
&\quad\mathscr{P}\left(r_i\cos^2\frac{\theta_i}{2}\,,\ r_i\sin^2\frac{\theta_i}{2}\right)\mathscr{P}\left(r_f\cos^2\frac{\theta_f}{2}\,,\ r_f\sin^2\frac{\theta_f}{2}\right)\ e^{i(k_1-k_2-k_3+k_4)(\phi_i-\phi_f)/2}
\end{aligned}
$$

where

$$
k_1,\,k_2,\,k_3,\,k_4\in\mathbb{N}\cup\{0\}\ \text{ s.t. }\begin{cases}k_1+k_2+k_3+k_4=N\\ k_1+k_2-k_3-k_4=-l\qquad(\mathbb{Z}\ni l=2\mu/\hbar)\,,\end{cases}
$$

$\mathscr{P}\,(X\,,\,Y)$ is the following polynomial.

$$
\mathscr{P}\,(X\,,\,Y)=\sum_{j=0}^{k_1}\sum_{s=0}^{k_2}j!s!\left(-\frac{\hbar}{m\omega}\right)^{j+s}{}_{k_1}C_j\cdot{}_{k_3}C_j\cdot{}_{k_2}C_s\cdot{}_{k_4}C_s\;X^{-j}\,Y^{-s}
$$

Proposition 1-2. [The Green's function of the MIC-Kepler problem]

(ii) When $\boldsymbol{x}_i$, $\boldsymbol{x}_f \in U_-$,

$$G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E = -m\omega^2/8)$$

$$= \frac{m^2\omega^2}{4\pi\hbar^2} e^{-\frac{m\omega}{2\hbar}(\tilde{r}_i+\tilde{r}_f)} \sum_{N=0}^{\infty} \frac{1}{4k-(N+2)\hbar\omega} \left(\frac{m\omega}{\hbar}\right)^N$$

$$\frac{1}{k_1!k_2!k_3!k_4!} \left(\sqrt{\tilde{r}_i\tilde{r}_f}\sin\frac{\tilde{\theta}_i}{2}\sin\frac{\tilde{\theta}_f}{2}\right)^{k_1+k_3} \left(\sqrt{\tilde{r}_i\tilde{r}_f}\cos\frac{\tilde{\theta}_i}{2}\cos\frac{\tilde{\theta}_f}{2}\right)^{k_2+k_4}$$

$$\mathscr{P}\left(\tilde{r}_i\sin^2\frac{\tilde{\theta}_i}{2}, \tilde{r}_i\cos^2\frac{\tilde{\theta}_i}{2}\right) \mathscr{P}\left(\tilde{r}_f\sin^2\frac{\tilde{\theta}_f}{2}, \tilde{r}_f\cos^2\frac{\tilde{\theta}_f}{2}\right) e^{i(k_1+3k_2-k_3-3k_4)(\tilde{\phi}_i-\tilde{\phi}_f)/2}$$

(iii) When $\boldsymbol{x}_i$, $\boldsymbol{x}_f \in U_+ \cap U_-$,
the following correlation of G_- with G_+ is shown
by $\tilde{r} = r$, $\tilde{\theta} = \pi - \theta$ and $\tilde{\phi} = 2\pi - \phi$.

$$G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E) = G_+(\boldsymbol{x}_f, \boldsymbol{x}_i; E)\, e^{i\,l\,(\phi_i-\phi_f)}$$

Incidentally, we can also find the following proposition.

Proposition 2. [The wave function of the MIC-Kepler problem]

(i) When $\boldsymbol{x} \in U_+$,

$$\Psi_N^+(\boldsymbol{x}) = \frac{m\omega}{2\sqrt{\pi}\hbar}\left(\sqrt{\frac{m\omega}{\hbar}}\right)^N \frac{\mathscr{P}\left(r\cos^2\frac{\theta}{2}, \, r\sin^2\frac{\theta}{2}\right)}{\sqrt{k_1!k_2!k_3!k_4!}} e^{-\frac{m\omega}{2\hbar}r} \left(\sqrt{r}\cos\frac{\theta}{2}\right)^{k_1+k_3}\left(\sqrt{r}\sin\frac{\theta}{2}\right)^{k_2+k_4} \exp\left[-i(k_1-k_2-k_3+k_4)\frac{\phi}{2}\right].$$

(ii) When $\boldsymbol{x} \in U_-$,

$$\Psi_N^-(\boldsymbol{x}) = \frac{m\omega}{2\sqrt{\pi}\hbar}\left(-\sqrt{\frac{m\omega}{\hbar}}\right)^N \frac{\mathscr{P}\left(\tilde{r}\sin^2\frac{\tilde{\theta}}{2}, \, \tilde{r}\cos^2\frac{\tilde{\theta}}{2}\right)}{\sqrt{k_1!k_2!k_3!k_4!}} e^{-\frac{m\omega}{2\hbar}\tilde{r}} \left(\sqrt{\tilde{r}}\sin\frac{\tilde{\theta}}{2}\right)^{k_1+k_3}\left(\sqrt{\tilde{r}}\cos\frac{\tilde{\theta}}{2}\right)^{k_2+k_4} \exp\left[-i(k_1+3k_2-k_3-3k_4)\frac{\tilde{\phi}}{2}\right].$$

(iii) When $\boldsymbol{x} \in U_+ \cap U_-$, $\qquad \Psi_N^-(\boldsymbol{x}) = \Psi_N^+(\boldsymbol{x})\, e^{-i\,l\,\phi}$

where $\quad N = 0, 1, 2, \cdots$,

the combination of non-negative integers $(k_1,\, k_2,\, k_3,\, k_4)$ and the polynomial $\mathscr{P}$ are the same as those shown in Proposition 1.

3. Wigner function of the MIC-Kepler problem

We showed the energy-eigenspace of the MIC-Kepler problem in our proceeding as follows.

Theorem [The eigenspace of the MIC-Kepler problem]

Its eigenspace associated with the negative energy

$$E_N = \frac{-2mk^2}{\hbar^2 (N+2)^2} \quad (N = 0, 1, 2, \cdots) \text{ is spanned by}$$

$$f_N(\boldsymbol{u}, \boldsymbol{\rho}) = \frac{(-1)^N}{(\pi\hbar)^4} e^{-2(N+2)} L_{n_a}(4b_3^+ b_3) L_{n_b}(4b_1^+ b_1) L_{n_c}(4b_2^+ b_2) L_{n_d}(4b_4^+ b_4)$$

where $n_a, n_b, n_c, n_d \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{Z}$ s.t.

$$\begin{cases} 2(n_a + n_d) \equiv N + l \\ 2(n_b + n_c) \equiv N - l \end{cases} \quad \text{i.e.} \quad \begin{cases} |\, l \,| \le N \\ N \text{ and } l \text{ are simultaneously even or odd.} \end{cases}$$

The dimension is

$$\left(\frac{N+l}{2} + 1\right)\left(\frac{N-l}{2} + 1\right) = \frac{(N+l+2)(N-l+2)}{4}.$$

In the above-mentioned theorem, $L_n(X)$ denotes the Laguerre polynomial of degree n s.t.

$$L_n(X) = \sum_{\alpha=0}^{n} (-1)^\alpha \frac{n!}{(\alpha!)^2 (n-\alpha)!} X^\alpha , \sum_{n=0}^{\infty} L_n(X)\, \xi^n = \frac{1}{1-\xi} \exp\left(-\frac{\xi}{1-\xi} X \right).$$

Further,

$$\begin{cases} 4b_3^+ b_3 = \dfrac{m\omega}{\hbar}({u_1}^2 + {u_2}^2) + \dfrac{1}{m\hbar\omega}({\rho_1}^2 + {\rho_2}^2) + \dfrac{2}{\hbar}(u_1\rho_2 - u_2\rho_1) \\ 4b_1^+ b_1 = \dfrac{m\omega}{\hbar}({u_1}^2 + {u_2}^2) + \dfrac{1}{m\hbar\omega}({\rho_1}^2 + {\rho_2}^2) - \dfrac{2}{\hbar}(u_1\rho_2 - u_2\rho_1) \\ 4b_2^+ b_2 = \dfrac{m\omega}{\hbar}({u_3}^2 + {u_4}^2) + \dfrac{1}{m\hbar\omega}({\rho_3}^2 + {\rho_4}^2) - \dfrac{2}{\hbar}(u_3\rho_4 - u_4\rho_3) \\ 4b_4^+ b_4 = \dfrac{m\omega}{\hbar}({u_3}^2 + {u_4}^2) + \dfrac{1}{m\hbar\omega}({\rho_3}^2 + {\rho_4}^2) + \dfrac{2}{\hbar}(u_3\rho_4 - u_4\rho_3) . \end{cases}$$

We reduce the eigenfunction $f_N(\boldsymbol{u}, \boldsymbol{\rho})$ on $T^*\dot{\mathbb{R}}^4$ to that on $T^*\mathbb{R}^3$ with the above-mentioned local polar coordinates.

Proposition 3-1. [The Wigner function of the MIC-Kepler problem]

Suppose $\boldsymbol{x} \in U_+ \cap U_-$,

(i) $f_N(r,\, \theta,\, \phi,\, p_r,\, p_\theta,\, p_\phi)$

$$= \frac{(-1)^N}{(\pi\hbar)^4} e^{-2(N+2)}$$

$$L_{n_a}\left(\frac{N+2}{2mk}\left[\mathcal{A}^2 + \frac{\mathcal{C}^2}{r(1+\cos\theta)}\right]\right) L_{n_b}\left(\frac{N+2}{2mk}\left[\mathcal{A}^2 + \frac{\mathcal{D}^2}{r(1+\cos\theta)}\right]\right)$$

$$L_{n_c}\left(\frac{N+2}{2mk}\left[\mathcal{B}^2 + \frac{\mathcal{E}^2}{r(1-\cos\theta)}\right]\right) L_{n_d}\left(\frac{N+2}{2mk}\left[\mathcal{B}^2 + \frac{\mathcal{F}^2}{r(1-\cos\theta)}\right]\right)$$

(ii) $f_N(\tilde{r},\, \tilde{\theta},\, \tilde{\phi},\, p_{\tilde{r}},\, p_{\tilde{\theta}},\, p_{\tilde{\phi}})$

$$= \frac{(-1)^N}{(\pi\hbar)^4} e^{-2(N+2)}$$

$$L_{n_a}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{A}}^2 + \frac{\tilde{\mathcal{C}}^2}{\tilde{r}(1-\cos\tilde{\theta})}\right]\right) L_{n_b}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{A}}^2 + \frac{\tilde{\mathcal{D}}^2}{\tilde{r}(1-\cos\tilde{\theta})}\right]\right)$$

$$L_{n_c}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{B}}^2 + \frac{\tilde{\mathcal{E}}^2}{\tilde{r}(1+\cos\tilde{\theta})}\right]\right) L_{n_d}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{B}}^2 + \frac{\tilde{\mathcal{F}}^2}{\tilde{r}(1+\cos\tilde{\theta})}\right]\right)$$

where $p_x\, dx + p_y\, dy + p_z\, dz = p_r\, dr + p_\theta\, d\theta + p_\phi\, d\phi = p_{\tilde{r}}\, d\tilde{r} + p_{\tilde{\theta}}\, d\tilde{\theta} + p_{\tilde{\phi}}\, d\tilde{\phi}$.

Proposition 3-2. [The Wigner function of the MIC-Kepler problem]

Functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ on $T^*(U_+ \cap U_-)$ are as follows.

$$\mathcal{A}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) \equiv p_r\sqrt{r(1+\cos\theta)} - p_\theta\sqrt{\frac{1-\cos\theta}{r}}$$

$$\mathcal{B}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) \equiv p_r\sqrt{r(1-\cos\theta)} + p_\theta\sqrt{\frac{1+\cos\theta}{r}}$$

$$\mathcal{C}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) \equiv p_\phi + r(1+\cos\theta)\left\{\frac{2mk}{\hbar(N+2)} + \frac{\mu}{r}\right\}$$

$$\mathcal{D}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) \equiv p_\phi - r(1+\cos\theta)\left\{\frac{2mk}{\hbar(N+2)} - \frac{\mu}{r}\right\}$$

$$\mathcal{E}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) \equiv p_\phi + r(1-\cos\theta)\left\{\frac{2mk}{\hbar(N+2)} - \frac{\mu}{r}\right\}$$

$$\mathcal{F}(r,\,\theta,\,\phi,\,p_r,\,p_\theta,\,p_\phi) \equiv p_\phi - r(1-\cos\theta)\left\{\frac{2mk}{\hbar(N+2)} + \frac{\mu}{r}\right\}$$

Proposition 3-3. [The Wigner function of the MIC-Kepler problem]

Functions $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{D}}$, $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ on $T^*(U_+ \cap U_-)$ are as follows.

$$\tilde{\mathcal{A}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) \equiv p_{\tilde{r}}\sqrt{\tilde{r}(1-\cos\tilde{\theta})} + p_{\tilde{\theta}}\sqrt{\frac{1+\cos\tilde{\theta}}{\tilde{r}}}$$

$$\tilde{\mathcal{B}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) \equiv p_{\tilde{r}}\sqrt{\tilde{r}(1+\cos\tilde{\theta})} - p_{\tilde{\theta}}\sqrt{\frac{1-\cos\tilde{\theta}}{\tilde{r}}}$$

$$\tilde{\mathcal{C}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) \equiv p_{\tilde{\phi}} - \tilde{r}(1-\cos\tilde{\theta})\left\{\frac{2mk}{\hbar(N+2)} + \frac{\mu}{\tilde{r}}\right\}$$

$$\tilde{\mathcal{D}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) \equiv p_{\tilde{\phi}} + \tilde{r}(1-\cos\tilde{\theta})\left\{\frac{2mk}{\hbar(N+2)} - \frac{\mu}{\tilde{r}}\right\}$$

$$\tilde{\mathcal{E}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) \equiv p_{\tilde{\phi}} - \tilde{r}(1+\cos\tilde{\theta})\left\{\frac{2mk}{\hbar(N+2)} - \frac{\mu}{\tilde{r}}\right\}$$

$$\tilde{\mathcal{F}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) \equiv p_{\tilde{\phi}} + \tilde{r}(1+\cos\tilde{\theta})\left\{\frac{2mk}{\hbar(N+2)} + \frac{\mu}{\tilde{r}}\right\}$$

Proposition 3-4. [The Wigner function of the MIC-Kepler problem]

Using the following equivalences :

$$\tilde{r} = r \ , \ \cos\tilde{\theta} = -\cos\theta \, ,$$

$$p_{\tilde{r}} = p_r \ , \ p_{\tilde{\theta}} = -p_\theta \ , \ p_{\tilde{\phi}} = -p_\phi$$

we verify

$$\tilde{\mathcal{A}} = \mathcal{A} \ , \ \tilde{\mathcal{B}} = \mathcal{B}$$

$$\tilde{\mathcal{C}} = -\mathcal{C} \ , \ \tilde{\mathcal{D}} = -\mathcal{D} \ , \ \tilde{\mathcal{E}} = -\mathcal{E} \ , \ \tilde{\mathcal{F}} = -\mathcal{F} \, .$$

Then we have

$$f_N(r, \theta, \phi, p_r, p_\theta, p_\phi) = f_N(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) \, .$$

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