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A class of localized solutions of the linear and nonlinear wave equations

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Our aim is to investigate narrow-band and broad-band laser pulses with femto- and attosecond duration in linear and nonlinear regime.

With the progress of laser innovations it is very important to study the localized waves, especially pulses which admit few cycles under envelope only and pulses in half-cycle regime.

One important experimental result is that even in femtosecond region the waist of no modulated initially laser pulse continue to satisfy the Fresnel's low of diffraction. The parabolic diffraction equation governing Fresnel's evolution of a **monochromatic wave** in continuous regime (CW regime) is suggested for firs time from Leontovich and Fock

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 4ik_0 \frac{\partial w}{\partial z} = 0.$$

The equation admits solutions of the kind of **fundamental Gaussian mode**, **higher order modes** such as **Laplace-Gauss**, **Helmholtz-Gauss** and **Bessel-Gauss beams**.
From other hand the optics of laser pulses, especially in the femtosecond (fs) region operates with strongly **polychromatic waves – narrow-band and broad-band pulses**. Additional possibility appear to perform a fs pulse to admits approximately equal size in x, y, z directions – **Light Bullet** – or relatively large transverse and small longitudinal size – **Light Disk.**

Our work is devoted to obtaining and investigating of analytical solutions of the wave equation governing the evolution of ultra-short laser pulses in air and nonlinear vacuum.

LINEAR REGIME OF OPTICAL PULSES

➢ *BASIC EQUATIONS*

The linear Diffraction - Dispersion Equation (DDE) governing the propagation of laser pulses in an approximation up to second order of dispersion for the amplitude function $V = V(x,y,z,t)$ of the electrical field is

$$-2ik_0\left(\frac{\partial V}{\partial z}+\frac{1}{v}\frac{\partial V}{\partial t}\right)=\Delta V-\frac{1+\beta}{v^2}\frac{\partial^2 V}{\partial t^2},$$

where $\beta = k''k_0v^2$ is a number counting the influence of the second order of dispersion, k'' *is* the group velocity dispersion, k_0 is main wave number and v is the group velocity.
In dispersionless media it is obtained also the following Diffraction Equation (DE) for the amplitude function $A = A(x,y,z,t)$

$$-2ik_0\left(\frac{\partial A}{\partial z}+\frac{1}{v}\frac{\partial A}{\partial t}\right)=\Delta A-\frac{1}{v^2}\frac{\partial^2 A}{\partial t^2}.$$

In air $\beta \simeq 2.1\times10^{-5} \simeq 0$, and hence DDE is equal to DE, and at hundred diffraction lengths appear only diffraction problems. This means that we can use an approximation $\beta \simeq 0$ and investigate DE only on these distances.

We note that a simple Courant-Hilbert *ansatz*

$$E(x, y, z, t) = A(x, y, z, t)\exp\left[ik_0(z - vt)\right]$$

applied to the wave equation

$$\Delta E - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0$$

generates the same amplitude equation DE

$$-2ik_0\left(\frac{\partial A}{\partial z} + \frac{1}{v}\frac{\partial A}{\partial t}\right) = \Delta A - \frac{1}{v^2}\frac{\partial^2 A}{\partial t^2}.$$

- The initial problem for a parabolic type equation is correctly posed, while the initial problem for a hyperbolic type equation is problematical ones.

Hence, a solution of the wave equation can be obtained solving the amplitude equation DE and multiplying it with the main phase.

➢ ***FOURIER TRANSFORM***

The equations DDE and DE are solved by applying the spatial Fourier transform to the components of the amplitude functions A and V. The fundamental solutions of the Fourier images $\hat{V}$ and $\hat{A}$ in (k_x, k_y, k_z, t) space are correspondingly

$$\hat{V}(k_x,k_y,k_z,t)=\hat{V}(k_x,k_y,k_z,0)\exp\left\{i\frac{v}{\beta+1}\left(k_0\pm\sqrt{k_0^2+(\beta+1)\left(k_x^2+k_y^2+k_z^2-2k_0k_z\right)}\right)t\right\}$$

and

$$\hat{A}(k_x,k_y,k_z,t)=\hat{A}(k_x,k_y,k_z,0)\exp\left\{iv\left(k_0\pm\sqrt{k_x^2+k_y^2+\left(k_z-k_0\right)^2}\right)t\right\}.$$

When $\beta \simeq 0$ the fundamental solutions are equal. Therefore we will investigate only the second one.

The exact solution of DE can be obtained by applying the backward Fourier transform

$$A = F^{-1}\left[\hat{A}(k_x, k_y, k_z, 0) \exp\left\{ iv\left(k_0 \pm \sqrt{k_x^2 + k_y^2 + \left(k_z - k_0\right)^2} \right) t \right\} \right]$$

or in details

$$A = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{\infty}^{\infty} \hat{A}(k_x, k_y, k_z, 0) \exp\left\{ iv\left(k_0 \pm \sqrt{k_x^2 + k_y^2 + \left(k_z - k_0\right)^2} \right) t \right\} \times$$
$$\exp\left\{ -i\left(xk_x + yk_y + zk_z \right) \right\} dk_x dk_y dk_z .$$

Substituting $k_z - k_0 = \hat{k}_z$ in the latter integral the backward Fourier transform takes the form

$$A = \frac{1}{(2\pi)^3} \exp\left\{ -ik_0 (z - vt) \right\} \int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{\infty}^{\infty} \hat{A}(k_x, k_y, \hat{k}_z + k_0, 0) \exp\left\{ \pm ivt\sqrt{k_x^2 + k_y^2 + \hat{k}_z^2}\, t \right\} \times$$
$$\exp\left\{ -i\left(xk_x + yk_y + z\hat{k}_z \right) \right\} dk_x dk_y d\hat{k}_z .$$

➢ *EXACT SOLUTIONS*

I. Gaussian light bullet

An analytical solution of DE is obtained for first time for initial Gaussian light bullet

$$A(x,y,z,0)=\exp\left\{-\left(x^2+y^2+z^2\right)/2r_0^2\right\}.$$

In this case the 3D backward Fourier transform becomes

$$A=\frac{1}{(2\pi)^3}\exp\left\{-\frac{k_0^2r_0^2}{2}-ik_0\left(z-vt\right)\right\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-\frac{\left(k_x^2+k_y^2+\hat{k}_z^2\right)r_0^2}{2}\right\}\times$$
$$\exp\left\{\pm ivt\sqrt{k_x^2+k_y^2+\hat{k}_z^2}\right\}\exp\left\{-i\left(xk_x+yk_y+\left(z-ir_0^2k_0\right)\hat{k}_z\right)\right\}dk_x dk_y d\hat{k}_z,$$

which in spherical coordinates can be presented in the following way

$$A = \frac{1}{2\pi^2} \exp\left\{ -\frac{k_0^2 r_0^2}{2} - ik_0 \left(z - vt\right) \right\} \frac{1}{\hat{r}} \int_0^\infty \hat{k}_r \exp\left\{ -\frac{\hat{k}_r^2 r_0^2}{2} \right\} \exp\left\{ \pm ivt\hat{k}_r \right\} \sin\left\{ \hat{r}\hat{k}_r \right\} d\hat{k}_r ,$$

where $\hat{r} = \sqrt{x^2 + y^2 + \left(z - ir_0^2 k_0\right)^2}$ and $\hat{k}_r = \sqrt{k_x^2 + k_y^2 + \hat{k}_z^2}$.

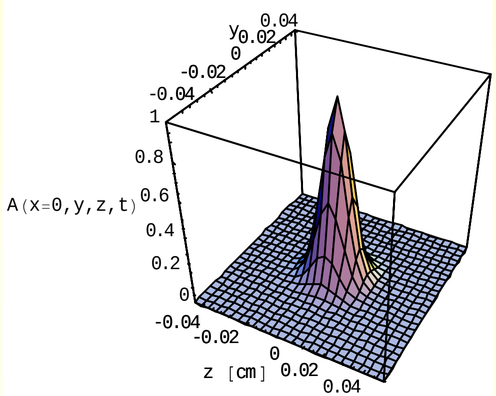
The corresponding exact solution is

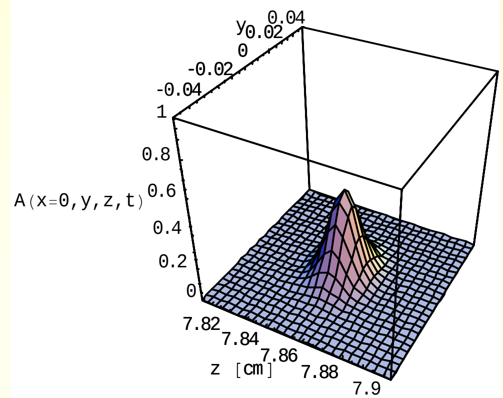
$$A(x, y, z, t) = \frac{i}{2\hat{r}} \exp\left[-\frac{k_0^2 r_0^2}{2} - ik_0 \left(z - vt\right) \right] \times$$

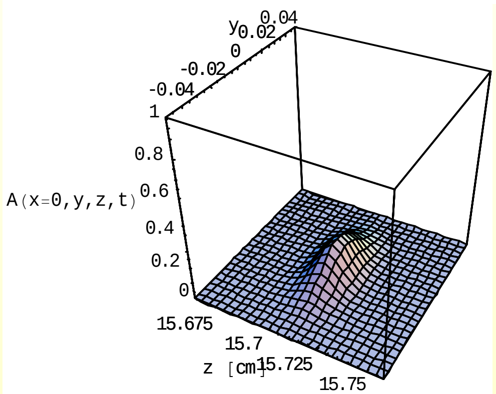
$$\left\{ i\left(vt + \hat{r}\right) \exp\left[-\frac{1}{2r_0^2} \left(vt + \hat{r}\right)^2 \right] erfc\left[\frac{i}{\sqrt{2} r_0} \left(vt + \hat{r}\right) \right] \right.$$

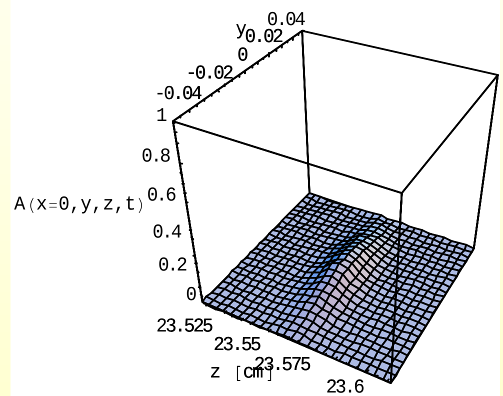
$$\left. - i\left(vt - \hat{r}\right) \exp\left[-\frac{1}{2r_0^2} \left(vt - \hat{r}\right)^2 \right] erfc\left[\frac{i}{\sqrt{2} r_0} \left(vt - \hat{r}\right) \right] \right\}.$$

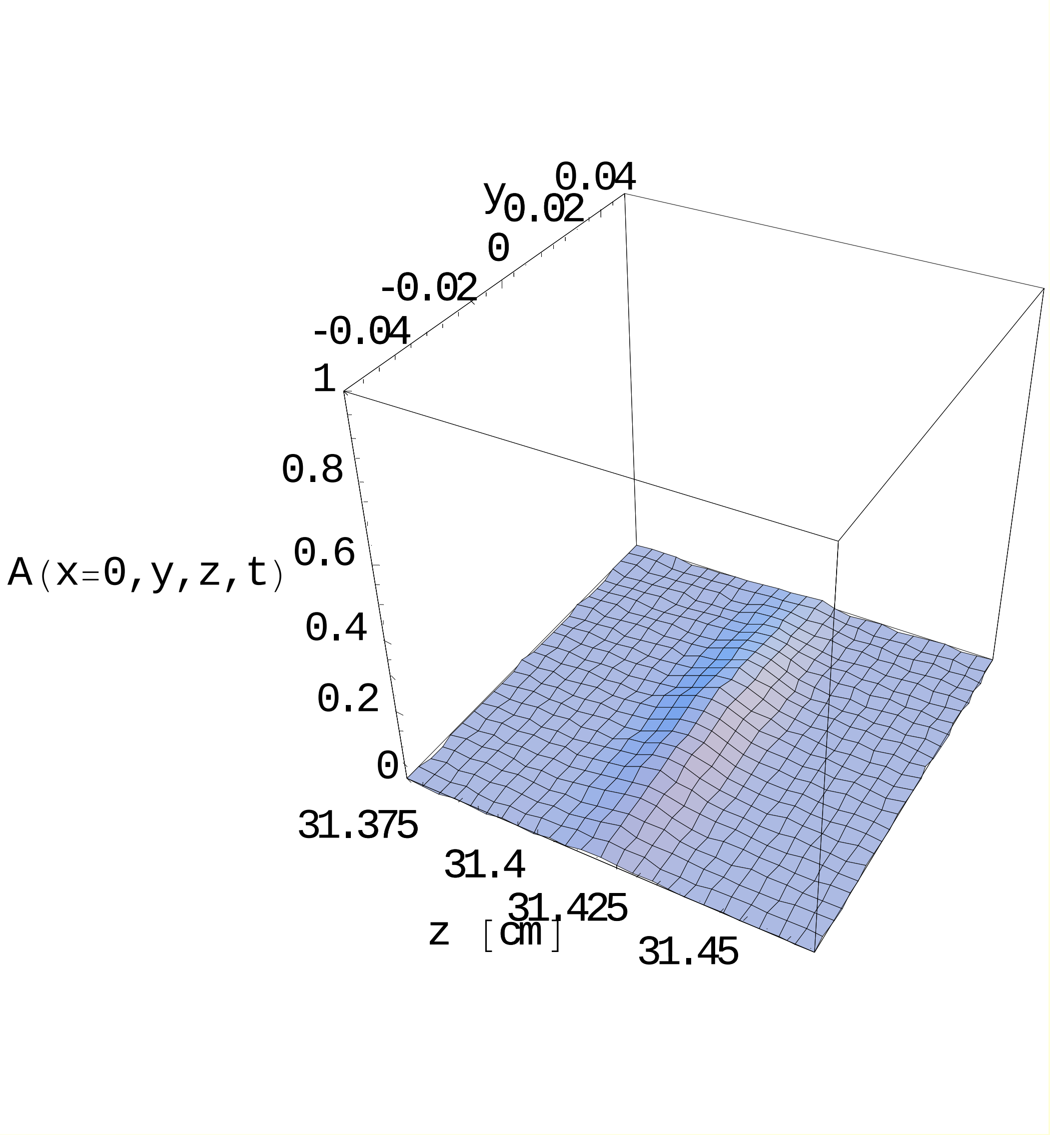
Narrow-band pulses – Fresnel's diffraction





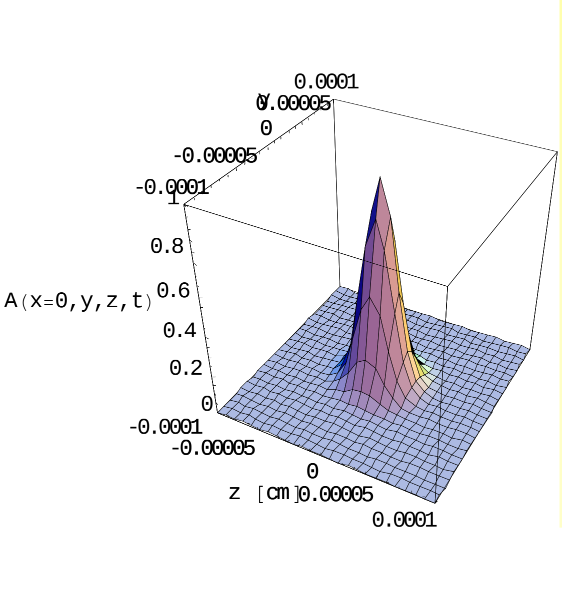


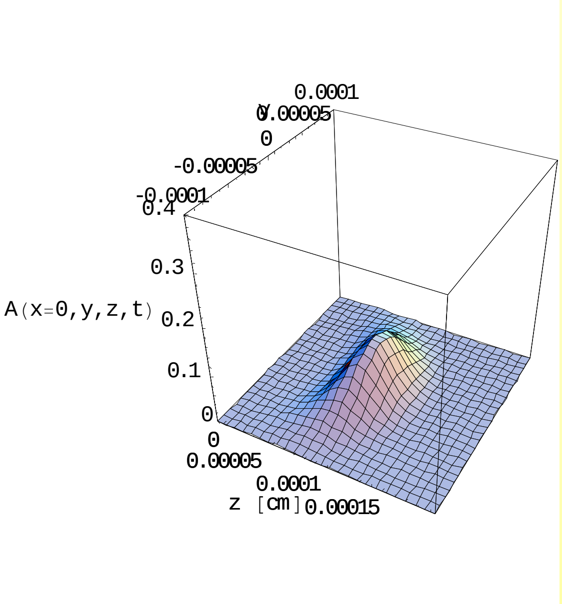


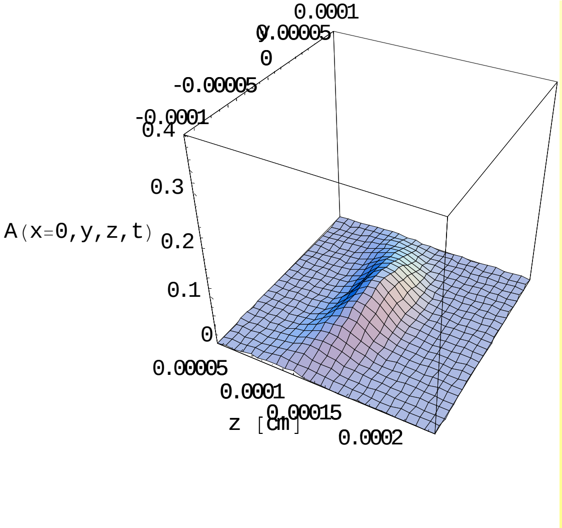


y
0.04
0.02
0
-0.02
-0.04
1
0.8
0.6
0.4
0.2
0
A(x=0,y,z,t)
31.375
31.4
31.425
31.45
z [cm]

Broad-band pulse – semi-spherical diffraction







Multiplying *A*(*x,y,z,t*) with the main phase we find an analytical solution of the wave equation

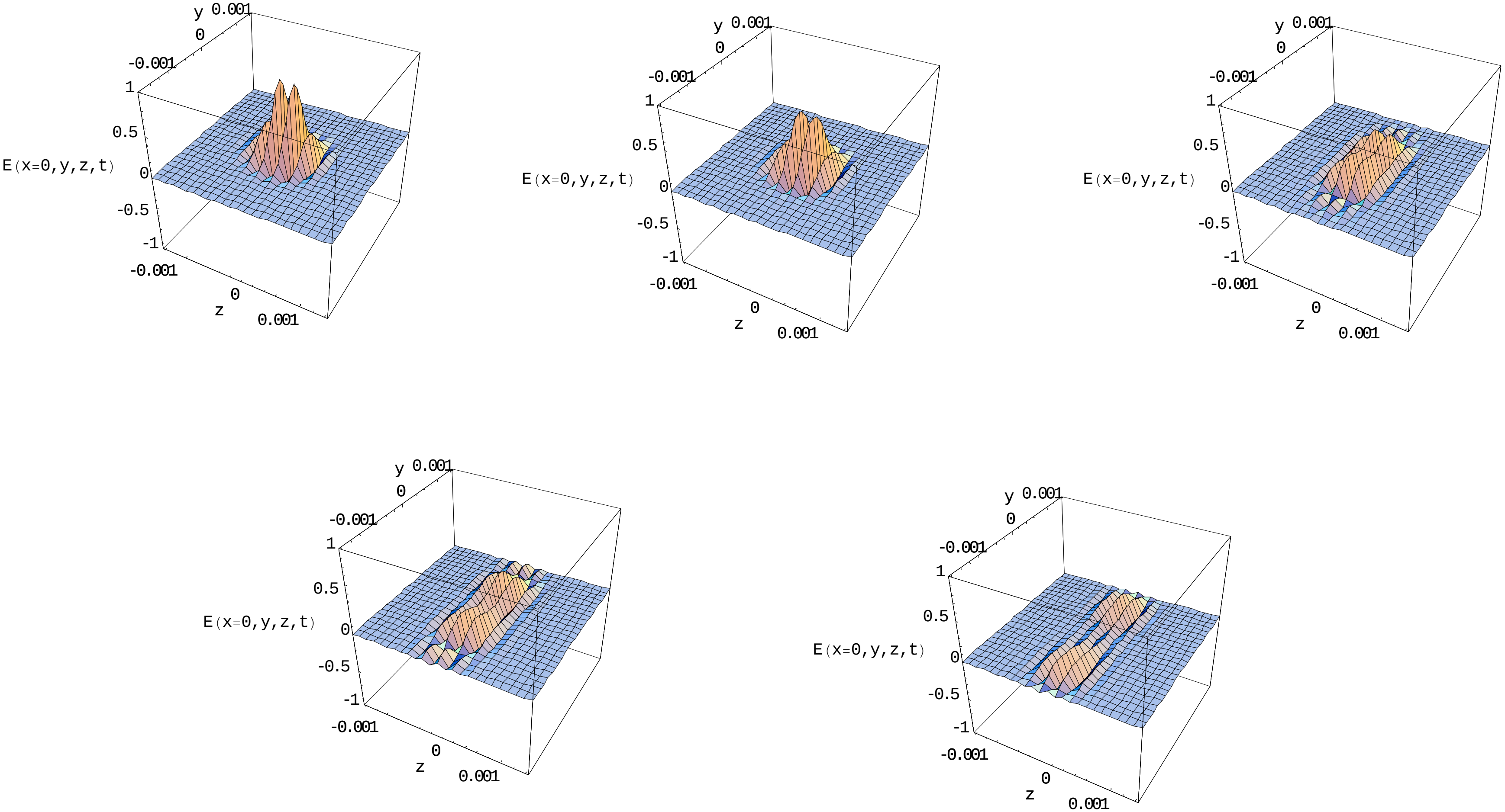
$$E(x,y,z,t)=\frac{i}{2\hat{r}}\exp\left(-\frac{k_0^2r_0^2}{2}\right)\times$$

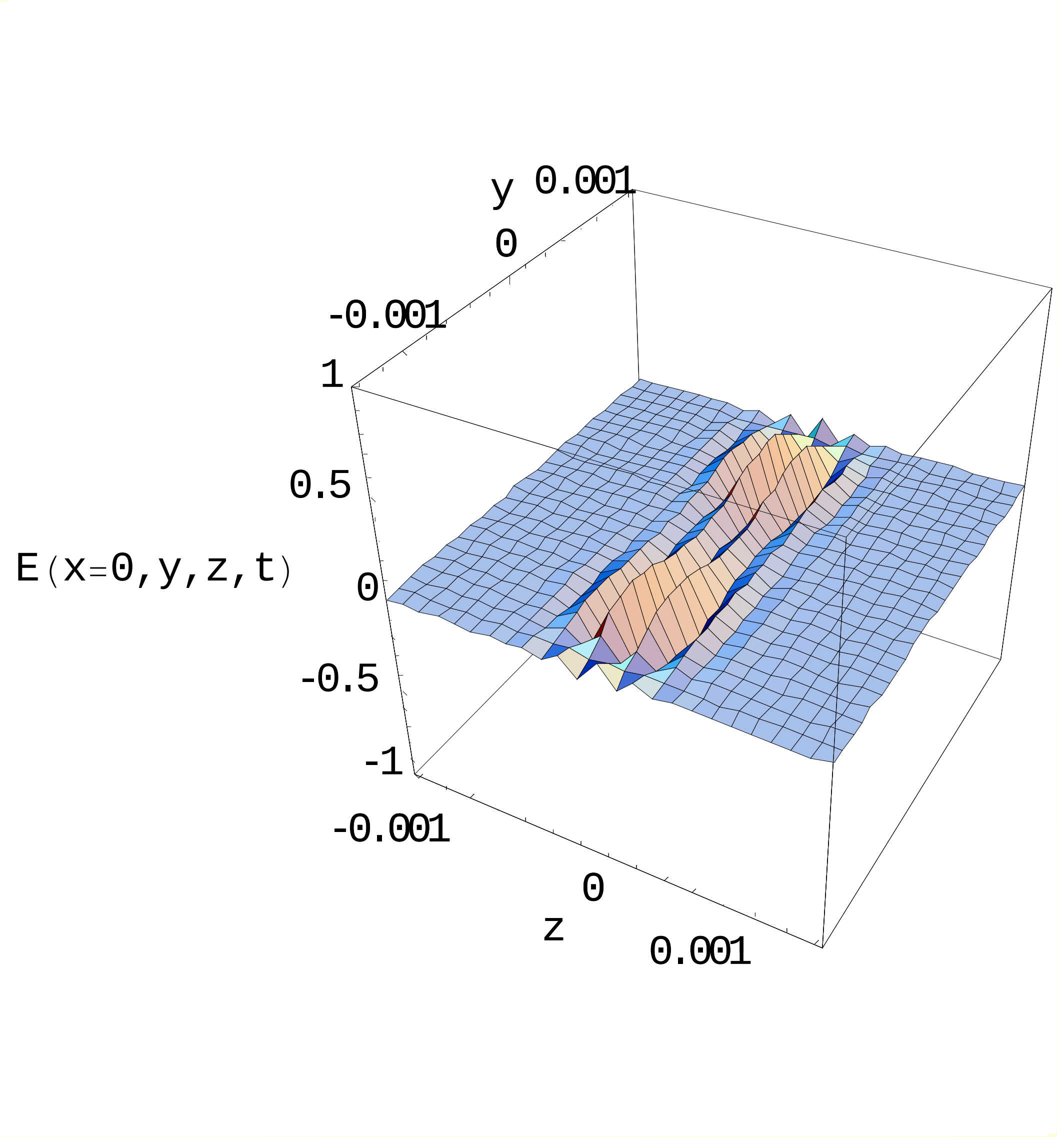
$$\left\{i\left(vt+\hat{r}\right)\exp\left[-\frac{1}{2r_0^2}\left(vt+\hat{r}\right)^2\right]erfc\left[\frac{i}{\sqrt{2}r_0}\left(vt+\hat{r}\right)\right]\right.$$

$$\left.-i\left(vt-\hat{r}\right)\exp\left[-\frac{1}{2r_0^2}\left(vt-\hat{r}\right)^2\right]erfc\left[\frac{i}{\sqrt{2}r_0}\left(vt-\hat{r}\right)\right]\right\}.$$

- $E(x,y,z,0)=\exp(ik_0z)\left[-\left(x^2+y^2+z^2\right)/2r_0^2\right]$
- We can observe a translation of the solution in *z* direction – due to the term $\exp(ik_0z)$
- Not stable with time – the amplitude function decreases and the energy distributes over whole space

Fresnel’s type of diffraction as exact solution of the wave equation

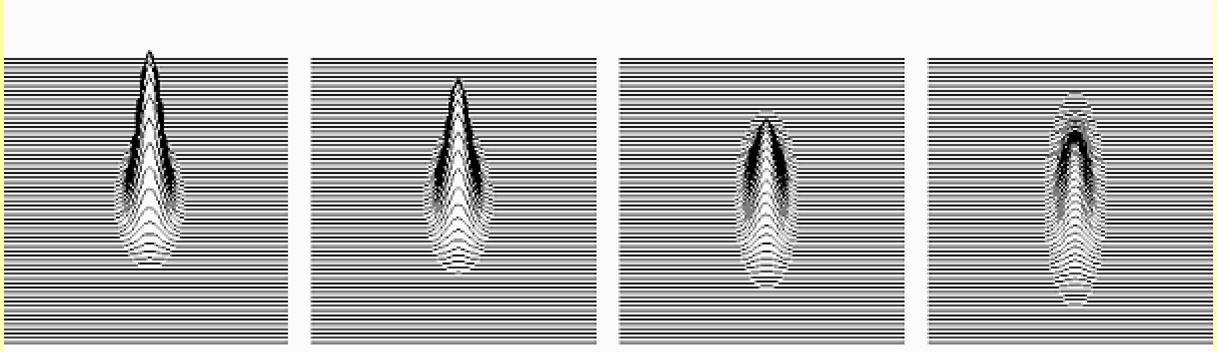



y 0.001
0
-0.001
1
0.5
E(x=0,y,z,t)
0
-0.5
-1
-0.001
0
z
0.001

Typical Fresnel diffraction in air of 100 fs pulse on λ=800 nm; $\Delta k << k_0$
Solution by solving numerically the fundamental solution of AE

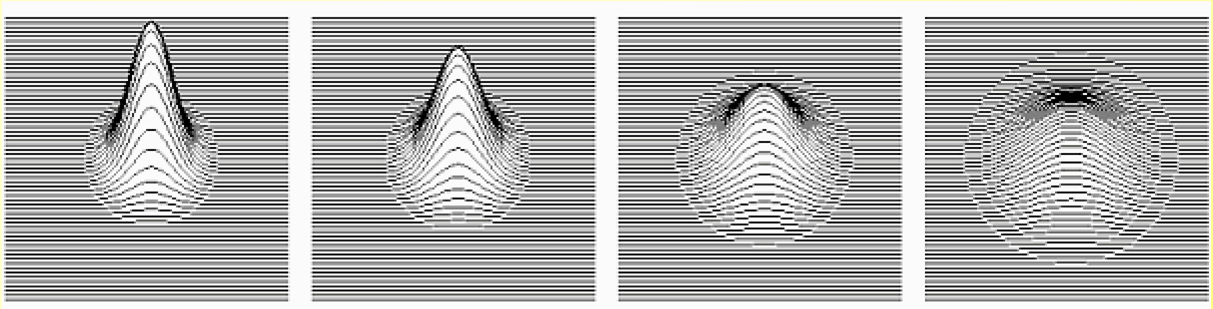
t_0=100 fs; z_0=30μm; $r_0(x,y)$=60 μm ; 37.5 cycles under envelope;

Side (z, y) projection:



$z=0$ $z=z_{diff}/2$ $z=z_{diff}$ $z=3z_{diff}/2$

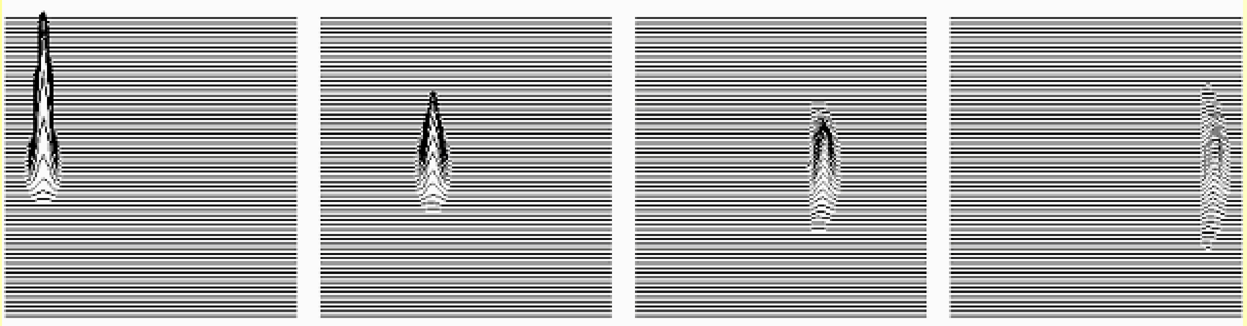
(x,y) projection (spot) :



$z=0$ $z=z_{diff}/2$ $z=z_{diff}$ $z=3z_{diff}/2$

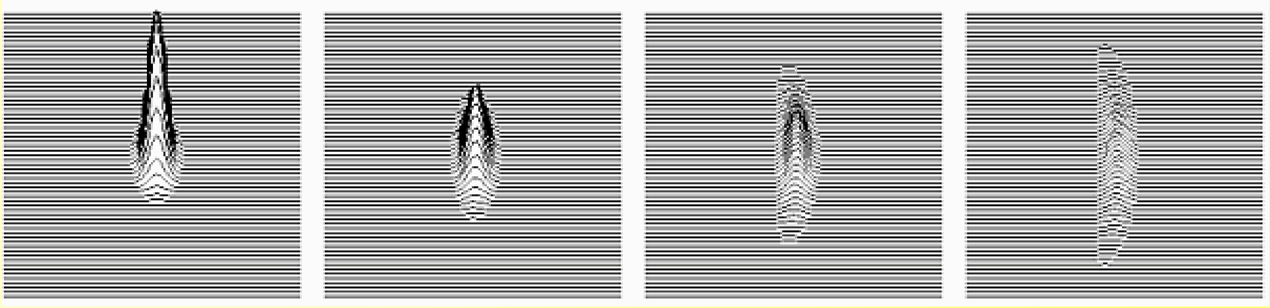
Nonparaxial diffraction of few-cycle optical pulses; Parabolic profile on several diffraction distances!!! **$\Delta k \sim k_0$**

T_0~10 fs; 3 cycle under envelope; Laboratory frame



z=0 z=zdiff z=2zdiff z=3zdiff

t_0=2.666 fs; 1 cycle under envelope; Galilean frame



z=0 $z=z_{diff}$ $z=2z_{diff}$ $z=3z_{diff}$

Why this happens? DDE in local time - Fundamental solution

$$z' = z; t' = t - z/v$$

$$-2ik_0 \frac{\partial \vec{A}}{\partial z'} = \Delta_\perp \vec{A} + \frac{\partial^2 \vec{A}}{\partial z'^2} - \frac{2}{v} \frac{\partial^2 \vec{A}}{\partial t' dz'} - \frac{\beta}{v^2} \frac{\partial^2 \vec{A}}{\partial t'^2}$$

$$\hat{A} = \hat{A}(k_x, k_y, k_z, 0) \exp\left\{ i \left(\left(k_0 - \frac{\Delta\omega}{v_{gr}} \right) \pm \sqrt{\left(k_0 - \frac{\Delta\omega}{v_{gr}} \right)^2 + k_x^2 + k_y^2 - \frac{\beta \Delta\omega^2}{v^2}} \right) z \right\}$$

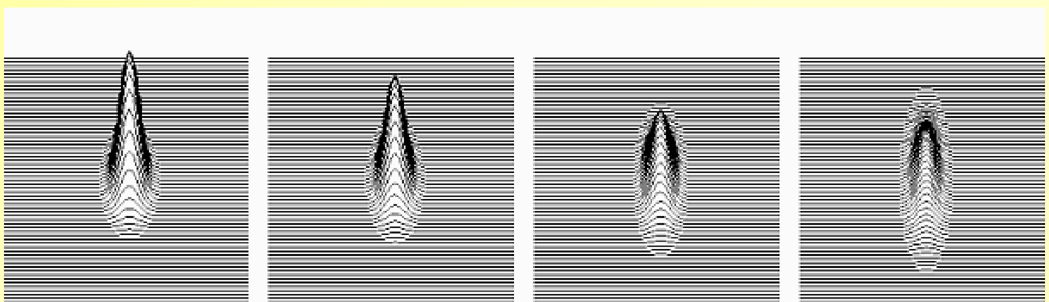
1. Narrow band pulses (50-100 fs) $\Delta k_z = \Delta\omega / v_{gr} << k_0 \quad k_0 - \frac{\Delta\omega}{v_{gr}} \approx k_0$

Using the low order of the Taylor expands the kernel is transformed to:

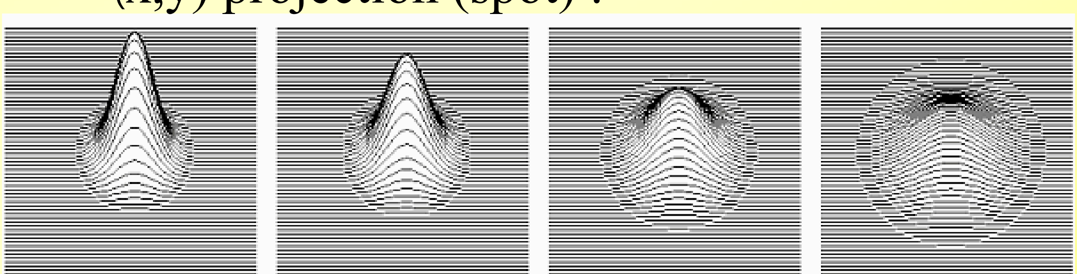
$$\hat{A}(k_x, k_y, \Delta\omega, \zeta) = \hat{A}(k_x, k_y, \Delta\omega, 0) \times \exp\left\{ i \left[\frac{k_x^2 + k_y^2 - \frac{\beta \Delta\omega^2}{v_{gr}^2}}{2k_0} \right] z' \right\}$$

Typical spatio temporal paraxial optics.

Side (x,t) projection



(x,y) projection (spot) :



$$\hat{A} = \hat{A}(k_x, k_y, k_z, 0)\exp\left\{i\left(\left(k_0 - \frac{\Delta\omega}{v_{gr}}\right) \pm \sqrt{\left(k_0 - \frac{\Delta\omega}{v_{gr}}\right)^2 + k_x^2 + k_y^2 - \frac{\beta\Delta\omega^2}{v^2}}\right)z$$

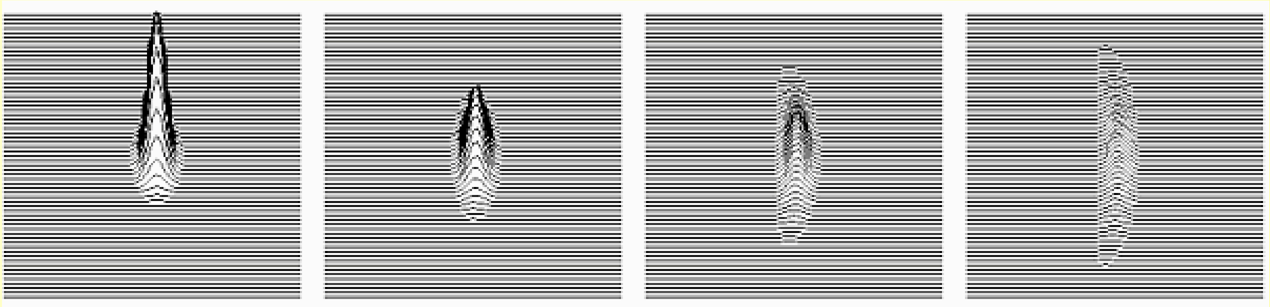
2. **Large band pulses (3-30 fs)** when $\Delta k_z = \Delta\omega / v_{gr} \sim k_0$

and

$$\left(k_0 - \frac{\Delta\omega}{v_{gr}}\right)^2 \approx k_x^2 \approx k_y^2$$

.

Nonparaxial soluion.



Parabolic profile of diffraction of attosecond pulses

(G. Mourou, Presentation in Bulgarian Academy of Sciences, 2008)

II. Spherically symmetric finite energy solutions of the wave equation

1. The Transition theorem $F[g(x)\exp(i\alpha x)] = F[g](k_x + \alpha)$

Our main purpose is to find exact solutions of the DE and wave equations. Let us consider again the solution of the amplitude function A presented as integral in $3D$ Fourier space

$$A = \frac{1}{(2\pi)^3}\exp\{-ik_0(z - vt)\}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{\infty}^{\infty}\hat{A}(k_x, k_y, \hat{k}_z + k_0, 0)\exp\left\{\pm ivt\sqrt{k_x^2 + k_y^2 + \hat{k}_z^2}\, t\right\}\times$$

$$\exp\left\{-i\left(xk_x + yk_y + z\hat{k}_z\right)\right\}dk_x dk_y d\hat{k}_z.$$

All terms except the Fourier image of the initial conditions $\hat{A}(k_x, k_y, \hat{k}_z + k_0, 0)$ depend on the translated wave number $\hat{k}_z$. We can use the Transition theorem to present $\hat{A}$ as a function of $\hat{k}_z$ only. Let the initial conditions are in form

$$A(x, y, z, 0) = A^*(x, y, z, 0)\exp\{-ik_0 z\}.$$

Then, applying the Transition theorem we have

$$F[A(x, y, z, 0)] = \hat{A}(k_x, k_y, k_z - k_0, 0) = \hat{A}^*(k_x, k_y, \hat{k}_z, 0).$$

Thus, all functions in the backward Fourier transform depend on translated wave number only

$$A = \frac{1}{(2\pi)^3} \exp\{-ik_0(z-vt)\} \int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{\infty}^{\infty} \hat{A}^*(k_x, k_y, \hat{k}_z, 0) \exp\left\{\pm ivt\sqrt{k_x^2 + k_y^2 + \hat{k}_z^2}\; t\right\} \times$$

$$\exp\left\{-i\left(xk_x + yk_y + z\hat{k}_z\right)\right\} dk_x dk_y d\hat{k}_z .$$

In our study we shall consider spherically-symmetric functions $A^*(x,y,z,0)$, i.e.

$$A^*(x, y, z, 0) = A^*(r, 0), \qquad r = \sqrt{x^2 + y^2 + z^2} .$$

The Fourier image of a spherically-symmetric function is also spherically-symmetric, i.e.

$$F\left[A^*(x, y, z, 0)\right] = F\left[A^*(r, 0)\right] = \hat{A}^*(k_r, 0), \qquad k_r = \sqrt{k_x^2 + k_y^2 + k_z^2} .$$

The backward radial Fourier integral is

$$A = \frac{1}{2\pi^2} \exp\{-ik_0(z-vt)\} \frac{1}{r} \int_0^{\infty} \hat{k}_r \hat{A}^*\left(\hat{k}_r, 0\right) \exp\left\{\pm ivt\hat{k}_r\right\} \sin\left\{r\hat{k}_r\right\} d\hat{k}_r .$$

2. Spherically-symmetric exact solutions with finite energy

A. *Localized algebraic function of kind* $A^* = 1/\left[1 + r^2 / r_0^2\right]$

$$\Rightarrow \ A(x, y, z, 0) = \exp\{-ik_0 z\} / \left[1 + \frac{r^2}{r_0^2}\right]$$

The Fourier image of the initial data is

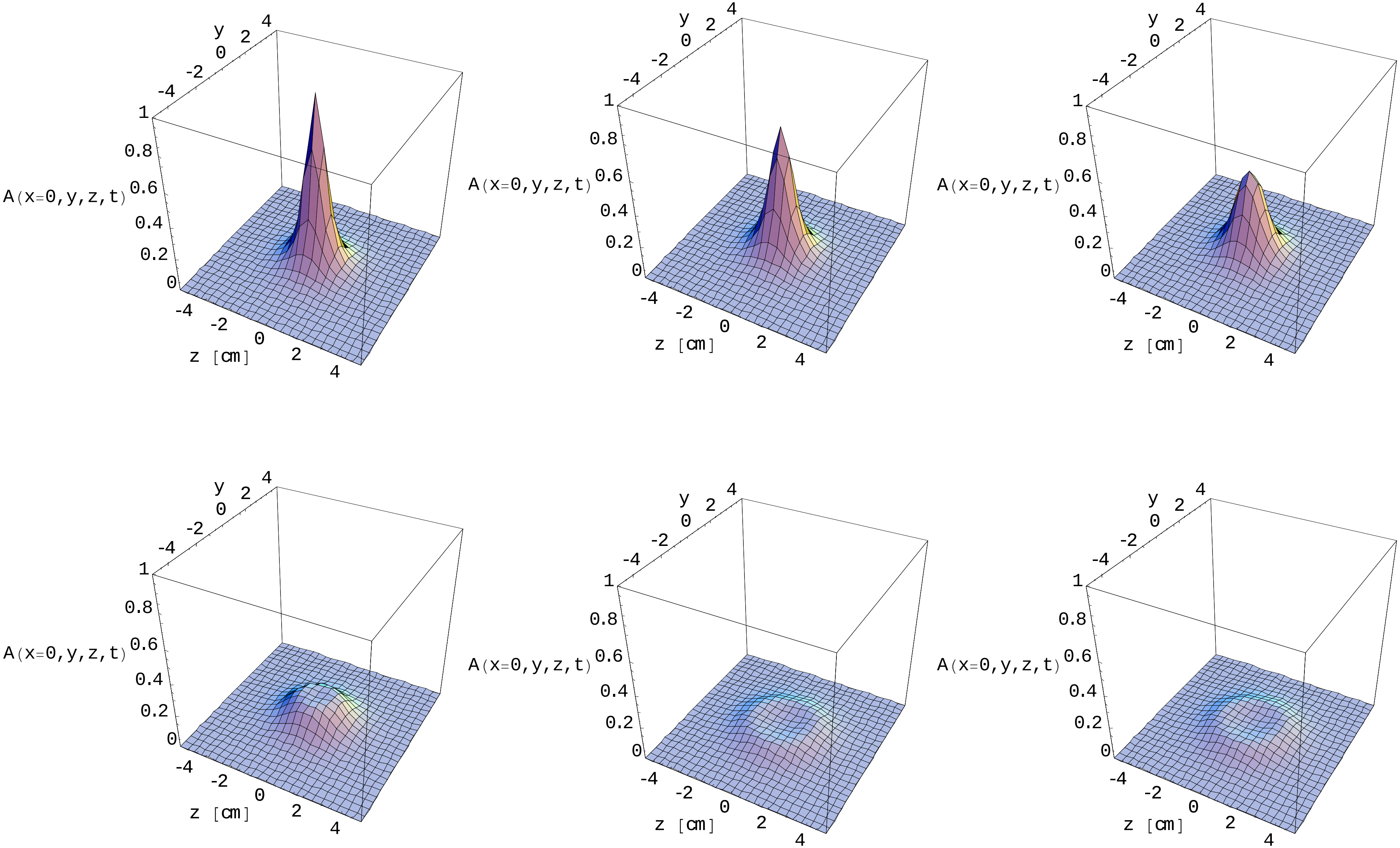
$$\hat{A}^*(k_r, 0) = \frac{\pi}{2k_r} \exp\{-r_0 k_r\} \qquad \Rightarrow \hat{A}(\hat{k}_r, 0) = \frac{\pi}{2\hat{k}_r} \exp\{-r_0 \hat{k}_r\}.$$

Hence, the corresponding solutions of the amplitude and wave equations are

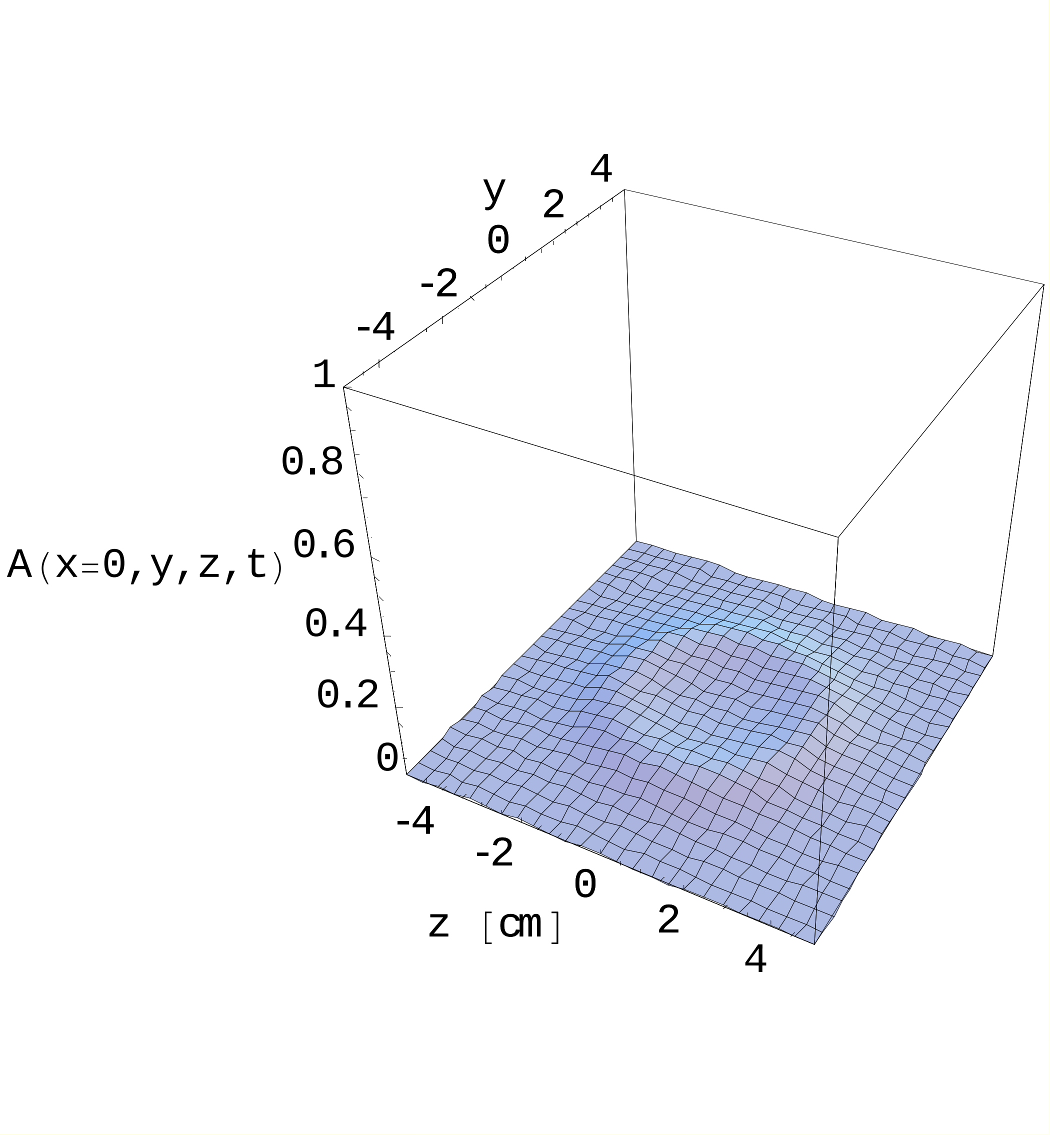
$$A(x, y, z, t) = \exp\{-ik_0 (z - vt)\} / \left[\frac{r^2}{r_0^2} + \left(1 + \frac{ivt}{r_0}\right)^2\right]$$

$$E(x, y, z, t) = 1 / \left[\frac{r^2}{r_0^2} + \left(1 + \frac{ivt}{r_0}\right)^2\right].$$

$$E(x, y, z, 0) = 1 / \left[1 + \frac{r^2}{r_0^2}\right] = A^*$$



A(x=0,y,z,t)
y
z [cm]
-4
-2
0
2
4
0
0.2
0.4
0.6
0.8
1



y
4
2
0
-2
-4
1
0.8
0.6
0.4
0.2
0
A(x=0,y,z,t)
-4
-2
0
2
4
z [cm]

B. *Localized algebraic function of kind* $A^* = 1/\left[\left(1 + r^2/r_0^2\right)^2\right]$

$$\Rightarrow \quad A(x, y, z, 0) = \exp\{-ik_0 z\} / \left[\left(1 + \frac{r^2}{r_0^2}\right)^2\right]$$

The Fourier image of the initial data is

$$\hat{A}^*(k_r, 0) = \frac{\pi}{4} r_0 \exp\{-r_0 k_r\} \qquad \Rightarrow \hat{A}\left(\hat{k}_r, 0\right) = \frac{\pi}{4} r_0 \exp\left\{-r_0 \hat{k}_r\right\}.$$

Hence, the corresponding solutions of the amplitude and wave equations are

$$A(x, y, z, t) = \exp\{-ik_0 (z - vt)\} \left[\frac{2(r_0 + itv)}{\left(r^2 + (r_0 + itv)^2\right)^2}\right]$$

$$E(x, y, z, t) = \left[\frac{2(r_0 + itv)}{\left(r^2 + (r_0 + itv)^2\right)^2}\right]$$

C. *Localized algebraic function of kind* $A^* = 1/\left[\left(1+r^2\right)^4\right]$

$$\Rightarrow \; A(x,y,z,0) = \exp\{-ik_0 z\} / \left[\left(1+r^2\right)^4\right]$$

The Fourier image of the initial data is

$$\hat{A}^*(k_r,0) = \frac{\pi}{96}\exp\{-k_r\}\left(3+3k_r+k_r^2\right) \qquad \Rightarrow \hat{A}\left(\hat{k}_r,0\right) = \frac{\pi}{96}\exp\left\{-\hat{k}_r\right\}\left(3+3\hat{k}_r+\hat{k}_r^2\right)$$

Hence, the corresponding solutions of the amplitude and wave equations are

$$A(x,y,z,t) = 6\exp\{-ik_0(z-vt)\}\times \frac{[8+29ivt]+tv\left[-r^2+t^2v^2\right]\left[-ir^2+tv(8+itv)\right]-2tv\left[-3ir^2+tv(20+13itv)\right]}{\left(r^2+(r_0+itv)^2\right)^4}$$

$$E(x,y,z,t) = 6\frac{[8+29ivt]+tv\left[-r^2+t^2v^2\right]\left[-ir^2+tv(8+itv)\right]-2tv\left[-3ir^2+tv(20+13itv)\right]}{\left(r^2+(r_0+itv)^2\right)^4}$$

D. *Localized algebraic function of kind* $A^{*} = \left(1-r^{2}\right)/\left(1+r^{2}\right)^{4}$

$$\Rightarrow \quad A(x,y,z,0) = \exp\{-ik_0 z\}\left(1-r^{2}\right)/\left(1+r^{2}\right)^{4}$$

The Fourier image of the initial data is

$$\hat{A}^{*}(k_r,0) = \frac{\pi}{48}\exp\{-k_r\}k_r^{2} \qquad \Rightarrow \hat{A}\left(\hat{k}_r,0\right) = \frac{\pi}{48}\exp\left\{-\hat{k}_r\right\}\hat{k}_r^{2} \ .$$

Hence, the corresponding solutions of the amplitude and wave equations are

$$A(x,y,z,t) = \frac{3i}{32}\exp\{-ik_0(z-vt)\}\left[\frac{1}{(vt+r-i)^{4}} - \frac{1}{(-vt+r+i)^{4}}\right]$$

$$E(x,y,z,t) = \frac{3i}{32}\left[\frac{1}{(vt+r-i)^{4}} - \frac{1}{(-vt+r+i)^{4}}\right]$$

E. Localized algebraic function of kind $A^* = r\left(3 - r^2\right) / \left(1 + r^2\right)^3$

$$\Rightarrow \quad A(x, y, z, 0) = \exp\{-ik_0 z\}\, r\left(3 - r^2\right) / \left(1 + r^2\right)^3$$

The Fourier image of the initial data is

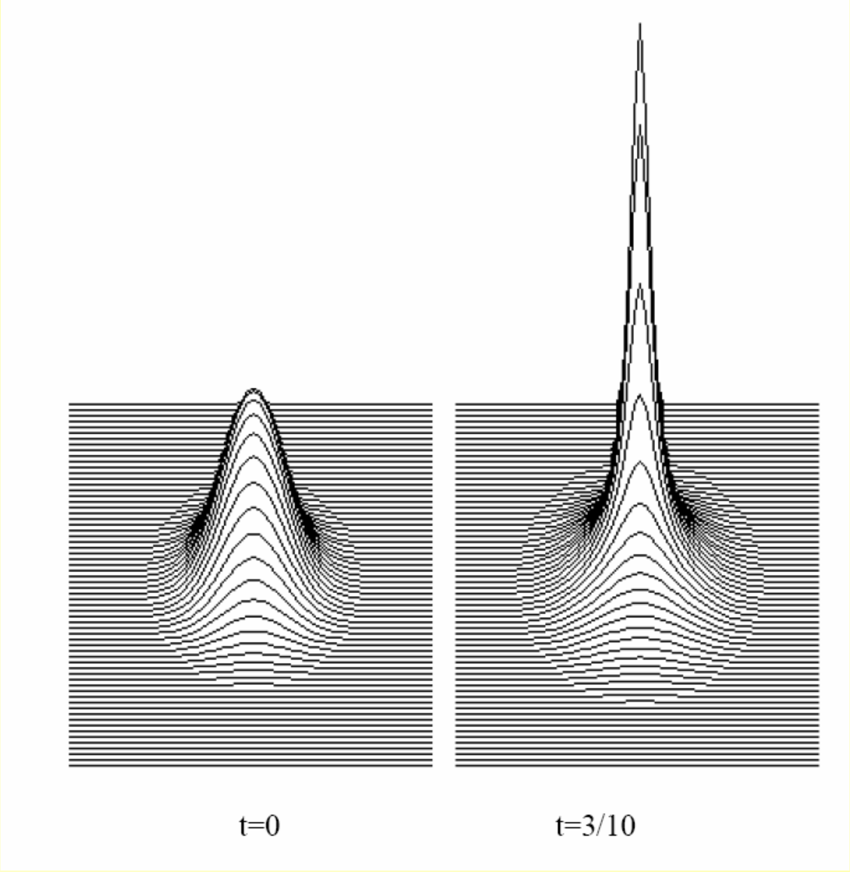
$$\hat{A}^*(k_r, 0) = \frac{\pi}{4} \exp\{-k_r\} k_r \qquad \Rightarrow \hat{A}\left(\hat{k}_r, 0\right) = \frac{\pi}{48} \exp\left\{-\hat{k}_r\right\} \hat{k}_r \ .$$

Hence, the corresponding solutions of the amplitude and wave equations are

$$A(x, y, z, t) = \frac{1}{2r} \exp\{-ik_0 (z - vt)\} \left[\frac{1}{(vt - r - i)^3} - \frac{1}{(vt + r - i)^3}\right]$$

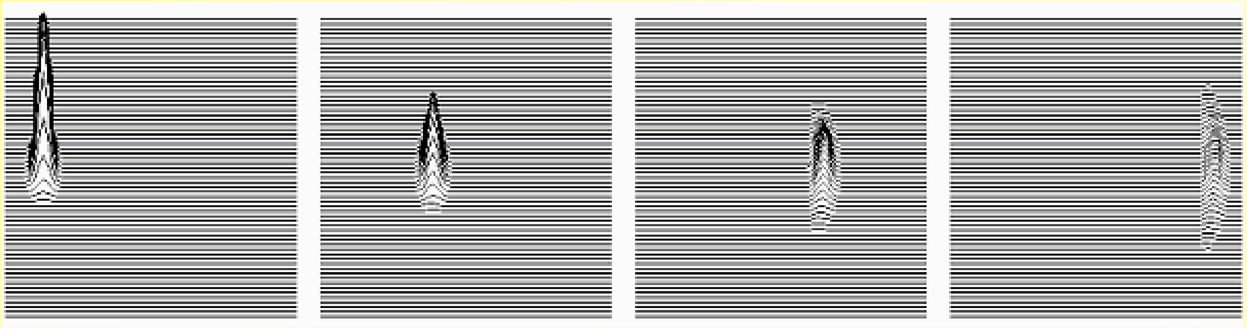
$$E(x, y, z, t) = \frac{1}{2r} \left[\frac{1}{(vt - r - i)^3} - \frac{1}{(vt + r - i)^3}\right]$$

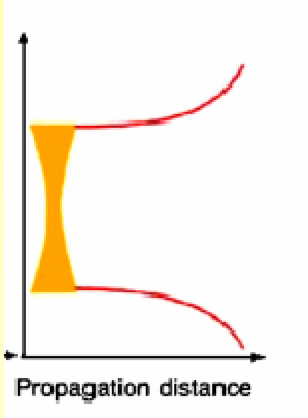
Self-focusing of narrow band (nano- and ps) pulses

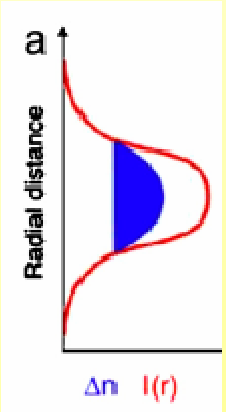


$$2ik_0 \frac{\partial A}{\partial z} + \Delta_{\perp} A + \gamma |A|^2 A = 0$$

•Main physical idea – nonparaxial parabolic diffraction + nonlinearity→ stable soliton propagation.







Nonparaxial (diverges) diffraction + Cubic nonlinearity = Soliton regime

NONLINEAR REGIME OF BROAD-BAND OPTICAL PULSES

- ***NONLINEAR PROPAGATION OF BROAD-BAND PULSES IN AIR. LORENTZ TYPE SOLITON.***

After neglecting two small perturbations terms the corresponding nonlinear amplitude equation for broad-band femtosecond pulses can be reduced to

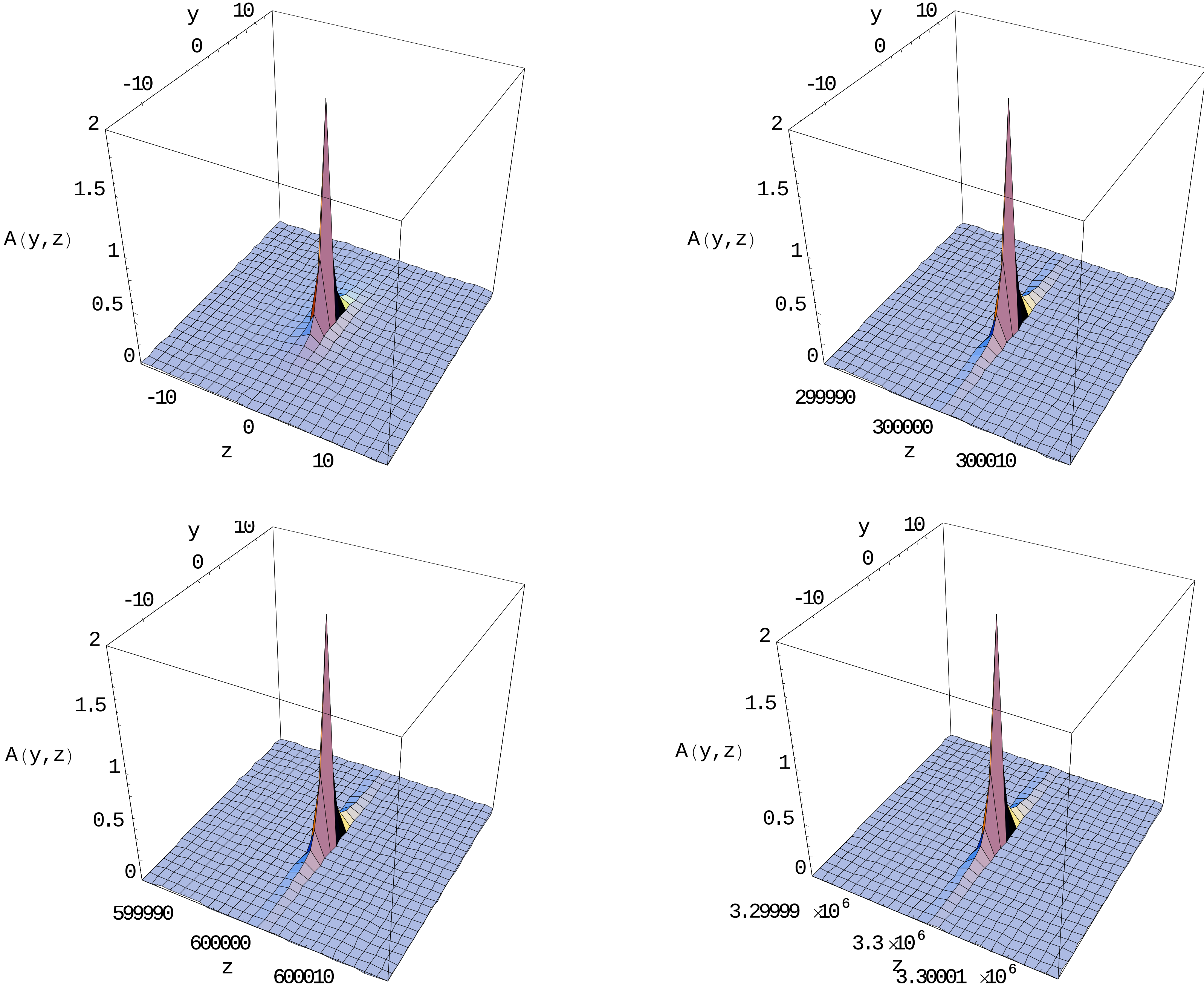
$$\Delta C - \frac{1}{v^2}\frac{\partial^2 C}{\partial t^2} + \gamma C^3 = 0, \qquad \gamma = C_0^2 k_0^2 n_2,$$

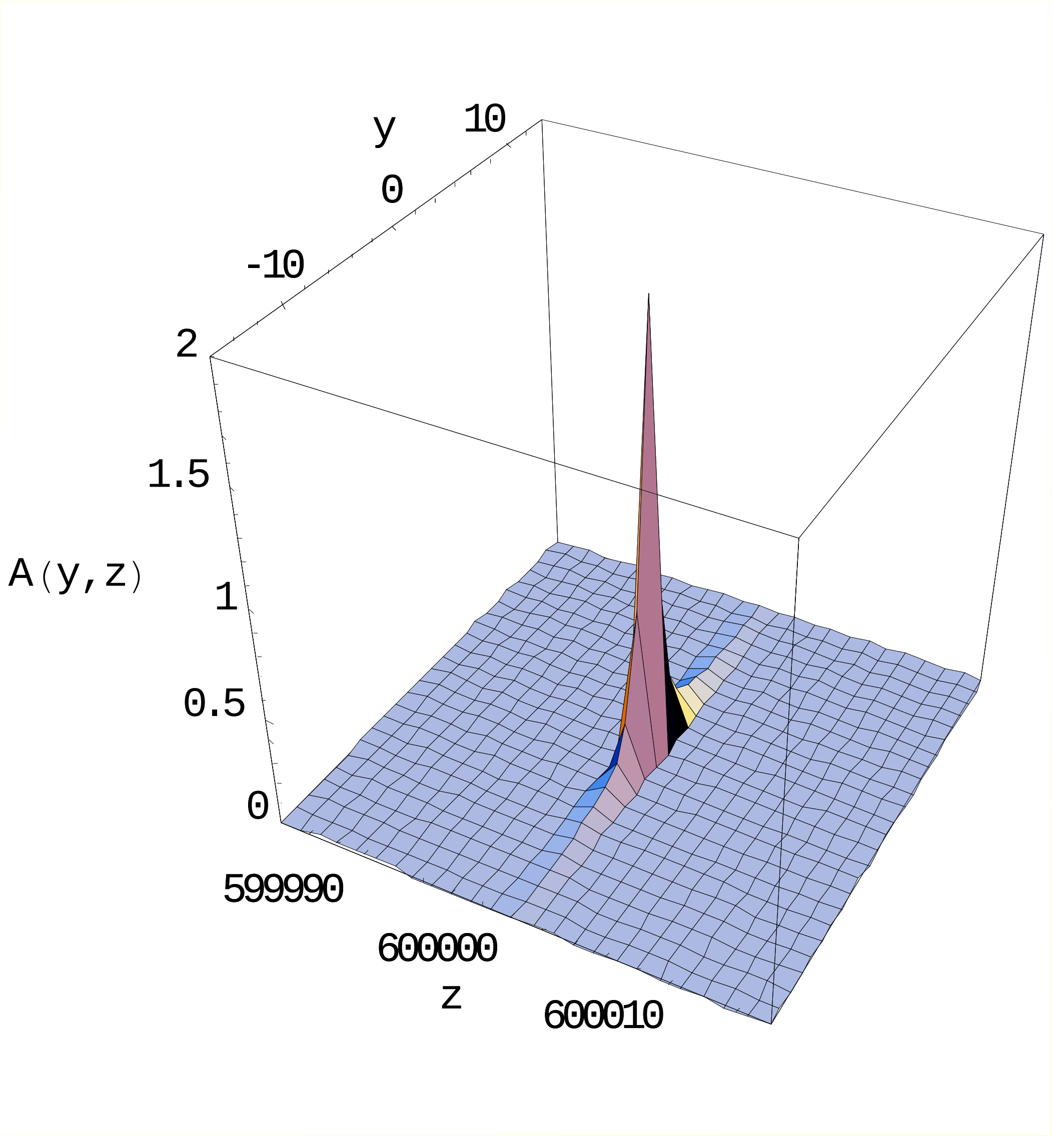
where γ is the nonlinear coefficient, C_0 is the amplitude maximum and n_2 is the nonlinear refractive index. The equation admits exact soliton solution propagating in forward direction only

$$C = \frac{\sec h(\ln \tilde{r})}{\tilde{r}} = \frac{2}{1+\tilde{r}^2}, \qquad \tilde{r} = \sqrt{x^2 + y^2 + (z+ia)^2 - v(t+ia/v)^2},$$

where $\gamma = 2$.

- *The 3D+1 soliton solution has Lorentz shape with asymmetric k_z spectrum*
- *The solution preserves its spatial and spectral shape in time*


y
10
0
-10
2
1.5
A(y,z)
1
0.5
0
-10
0
z
10
y
10
0
-10
2
1.5
A(y,z)
1
0.5
0
299990
300000
z
300010
y
10
0
-10
2
1.5
A(y,z)
1
0.5
0
599990
600000
z
600010
y
10
0
-10
2
1.5
A(y,z)
1
0.5
0
3.29999 ×10^6
3.3 ×10^6
z
3.30001 ×10^6



y
10
0
-10
2
1.5
A(y,z)
1
0.5
0
599990
600000
z
600010

➢ *PROPAGATION OF BROAD-BAND PULSES IN NONLINEAR VACUUM. VORTEX LOCALIZED SOLUTION.*

In 1935 Euler and Kockel predict one intrinsic nonlinearity to the electromagnetic vacuum due to electron-positron nonlinear polarization. This leads to field-dependent dielectric tensor in form

$$\varepsilon_{ik} = \delta_{ik} + \frac{7e^4\hbar}{45\pi m^4 c^7}\left[2\left(\left|\vec{E}\right|^2 - \left|\vec{B}\right|^2\right) + 7B_i B_k\right],$$

where complex form of presentation of the electrical E_i and magnetic B_i components is used. The term containing $B_i B_k$ vanishes when localized electromagnetic wave with only one magnetic component B_l is investigated. Thus, the dielectric response relevant to such pulse is

$$\varepsilon_{ik} = \delta_{ik} + \frac{14e^4\hbar}{45\pi m^4 c^7}\left(\left|\vec{E}\right|^2 - \left|\vec{B}\right|^2\right).$$

The magnetic field, rather than the electrical filed, appears in the expression for the dielectric response and the nonlinear addition to the intensity profile (effective mass density) of one electromagnetic wave in nonlinear vacuum can be expressed in electromagnetic units as $I_{nl} = (|E|^2 - |B|^2)$. When the spectral width of one pulse Δk_z exceeds the values of the main wave vector $\Delta k_z \simeq k_0$ the system of amplitude equations in nonlinear vacuum becomes

$$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \gamma\left(\left|\vec{E}\right|^2 - \left|\vec{B}\right|^2\right)\vec{E} = 0$$

$$\Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} + \gamma\left(\left|\vec{E}\right|^2 - \left|\vec{B}\right|^2\right)\vec{B} = 0, \qquad \gamma = \frac{7k_0^2 e^4 \hbar}{90\pi m^4 c^7}.$$

We present the components of the electrical and magnetic fields as a vector sum of circular and linear components

$$E_z$$
$$E_c = iE_x - E_y$$
$$B_l = -B_z.$$

Hence we get the following scalar system of equations

$$\Delta E_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} + \gamma\left(\left|E_z\right|^2 + \left|E_c\right|^2 - \left|B_l\right|^2\right)E_z = 0$$

$$\Delta E_c - \frac{1}{c^2}\frac{\partial^2 E_c}{\partial t^2} + \gamma\left(\left|E_z\right|^2 + \left|E_c\right|^2 - \left|B_l\right|^2\right)E_c = 0$$

$$\Delta B_l - \frac{1}{c^2}\frac{\partial^2 B_l}{\partial t^2} + \gamma\left(\left|E_z\right|^2 + \left|E_c\right|^2 - \left|B_l\right|^2\right)B_l = 0.$$

Let us now parameterize the space-time by pseudo-spherical coordinates (r, τ, θ, φ)

$$ct = r\sinh\tau$$

$$z = r\cosh\tau\cos\theta$$

$$y = r\cosh\tau\sin\theta\sin\varphi$$

$$x = r\cosh\tau\sin\theta\cos\varphi,$$

where $r = \sqrt{x^2 + y^2 + z^2 - c^2t^2}.$

The d'Alambert operator in pseudo-spherical coordinates is

$$\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial \tau^2} - 2\frac{\tanh \tau}{r^2}\frac{\partial}{\partial \tau} + \frac{1}{r^2 \cosh^2 \tau}\Delta_{\theta,\varphi},$$

Where $\Delta_{\theta,\varphi}$ is the angular part of the usual Laplace operator

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

We solve the system of equations using the method of separation of the variables

$$E_i(r,\tau,\theta,\varphi) = R(r)T_i(\tau)Y_i(\theta,\varphi), \qquad i = z,c,$$
$$B_l(r,\tau,\theta,\varphi) = R(r)T_l(\tau)Y_l(\theta,\varphi),$$

with an additional constrain on the angular and "spherical" time parts

$$|T_z|^2|Y_z(\theta,\varphi)|^2 + |T_c|^2|Y_c(\theta,\varphi)|^2 - |T_l|^2|Y_l(\theta,\varphi)|^2 = const,$$

which separates the variables. The nonlinear terms appear in radial part only.

The radial parts obey the equations

$$\frac{3}{r}\frac{\partial R}{\partial r} + \frac{\partial^2 R}{\partial r^2} - \frac{A_i}{r^2} R + \gamma |R|^2 R = 0, \qquad i = z, c, l,$$

where A_i are separation constants. We look for localized solutions of the kind

$$R = \frac{\sec h\left(\ln r^{\alpha}\right)}{r}.$$

The separation constants A_i, α and γ satisfy the relations

$$\alpha^2 - 1 = A_i;\ 2\alpha^2 = \gamma.$$

The corresponding τ – dependent parts of the equations are linear

$$\cosh^2\tau \frac{d^2 T_i}{d\tau^2} + 2\sinh\tau\cosh\tau \frac{dT_i}{d\tau} + \left(C_i - A_i \cosh^2\tau\right)T_i = 0, \qquad i = z, c, l,$$

where C_i are another separation constants connected with the angular part of the Laplace operator $Y_i(\theta,\varphi)$.

The solutions of the τ – equations which satisfy the constrain condition are

$$T_z = \cosh \tau; \quad T_c = \cosh \tau; \quad T_l = \sinh \tau$$

with separation constants for the electrical part $A_z = A_c = 3$; $C_z = C_c = 2$ and for the magnetic part $A_l = 3$; $C_l = 0$ correspondingly. The magnetic part of the system of equations do not depend from the angular components, i.e. $Y_l(\theta,\varphi) = 0$, while for the electrical part we have the following linear system of equations

$$\frac{\Delta_{\theta,\varphi} Y_i}{Y_i}. = -2, \qquad i = z, c.$$

The solutions which satisfy the above system and the constrain condition are

$$Y_z = \cos \theta; \quad T_c = \sin \theta \exp(i\varphi).$$

Using the relation between the separation constants A_i and the real number α we have

$$\alpha^2 = 4; \quad \alpha = \pm 2; \quad \gamma = 8.$$

Finally we can write the exact localized solution of the system of nonlinear equations representing the propagation of electromagnetic wave in vacuum

$$E_z(r,\tau,\theta,\varphi) = \frac{\sec h\left(\ln r^{\pm 2}\right)}{r}\cosh \tau \cos \theta$$

$$E_c(r,\tau,\theta,\varphi) = \frac{\sec h\left(\ln r^{\pm 2}\right)}{r}\cosh \tau \sin \theta \exp(i\varphi)$$

$$B_l(r,\tau,\theta,\varphi) = \frac{\sec h\left(\ln r^{\pm 2}\right)}{r}\sinh \tau .$$

The solution can be rewritten in Cartesian coordinates

$$E_z(x,y,z,t) = \frac{2z}{r^4+1}$$

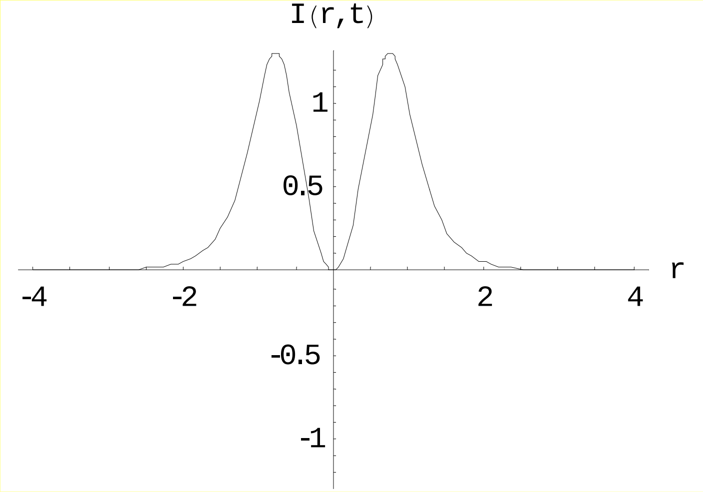
$$E_c(r,\tau,\theta,\varphi) = \frac{2(x+iy)}{r^4+1}$$

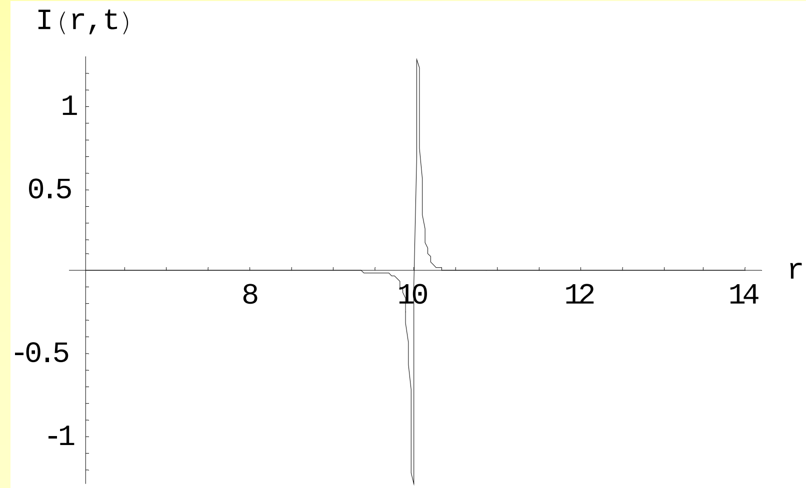
$$B_l(r,\tau,\theta,\varphi) = \frac{2ct}{r^4+1}, \qquad r = \sqrt{x^2+y^2+z^2-c^2t^2}.$$

The linear part of the energy density (intensity) of the solution can be expressed as

$$I(x,y,z,t) = \frac{4r^2 + 8c^2t^2}{\left(r^4+1\right)^2} = \frac{4\left(x^2+y^2+z^2+c^2t\right)}{\left[\left(x^2+y^2+z^2-c^2t\right)^2+1\right]}.$$

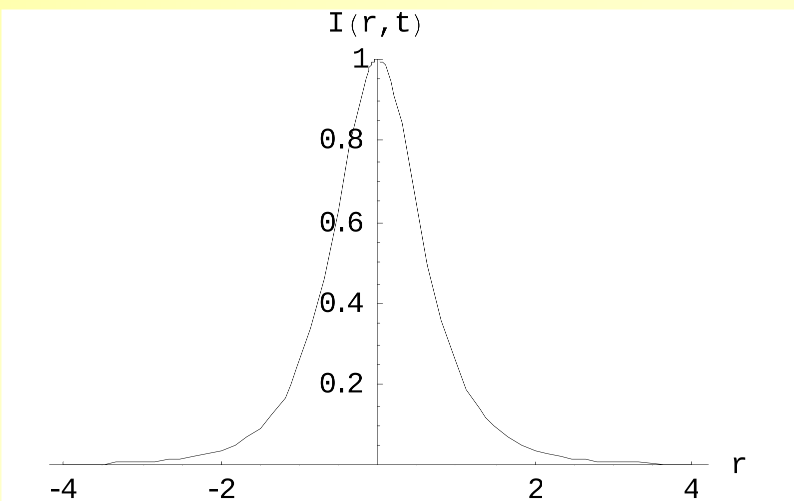
- *The solution admits own orbital momentum l = 1 for the electrical components*
- *The solution is with finite energy and presents nonlinear spherical shock wave*

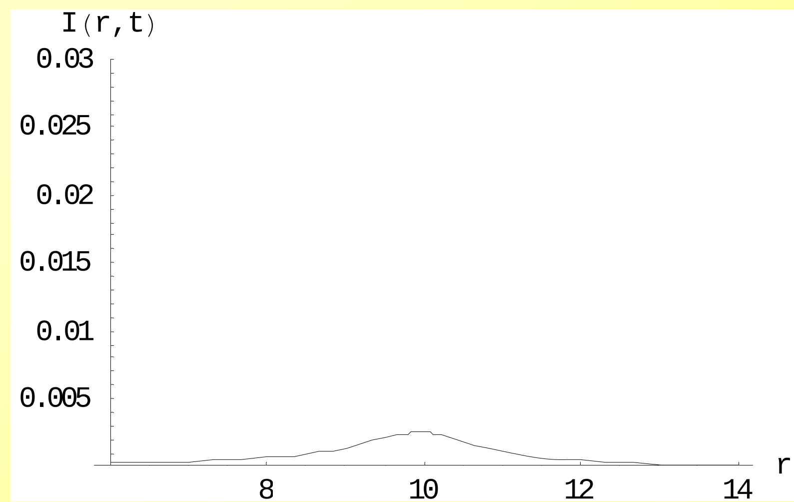

I(r,t)
1
0.5
-0.5
-1
r
-4
-2
2
4


I(r,t)
1
0.5
-0.5
-1
r
8
10
12
14

$t = 0$

$t = 10$


I(r,t)
1
0.8
0.6
0.4
0.2
r
-4
-2
2
4


I(r,t)
0.03
0.025
0.02
0.015
0.01
0.005
r
8
10
12
14

Conclusions

1) *New exact localized solutions of the linear amplitude and wave equations are presented. The amplitude function decreases and the energy distributes over whole space for finite time.*

2) The soliton solution of the nonlinear wave equation admits Lorentz' shape and propagates preserving the initial form and spectrum.

3) The system of nonlinear vacuum equations is solved in 3D+1 Minkowski time-space. The solution admits own angular momentum of the electrical part. The solution is with finite energy and presents nonlinear spherical wave.