ARMEN SERGEEV

HARMONIC SPHERES AND YANG–MILLS FIELDS

Steklov Mathematical Institute
Moscow

HARMONIC MAPS

Harmonic self-maps of the Riemann sphere. Suppose that at any $x = (x_1, x_2) \in \mathbb{R}^2$ we have a unit vector $\varphi(x) \in \mathbb{R}^3$, depending smoothly on x, i.e. we are given with a smooth map

$$\varphi : \mathbb{R}^2 \longrightarrow S^2, \quad x \mapsto \varphi(x).$$

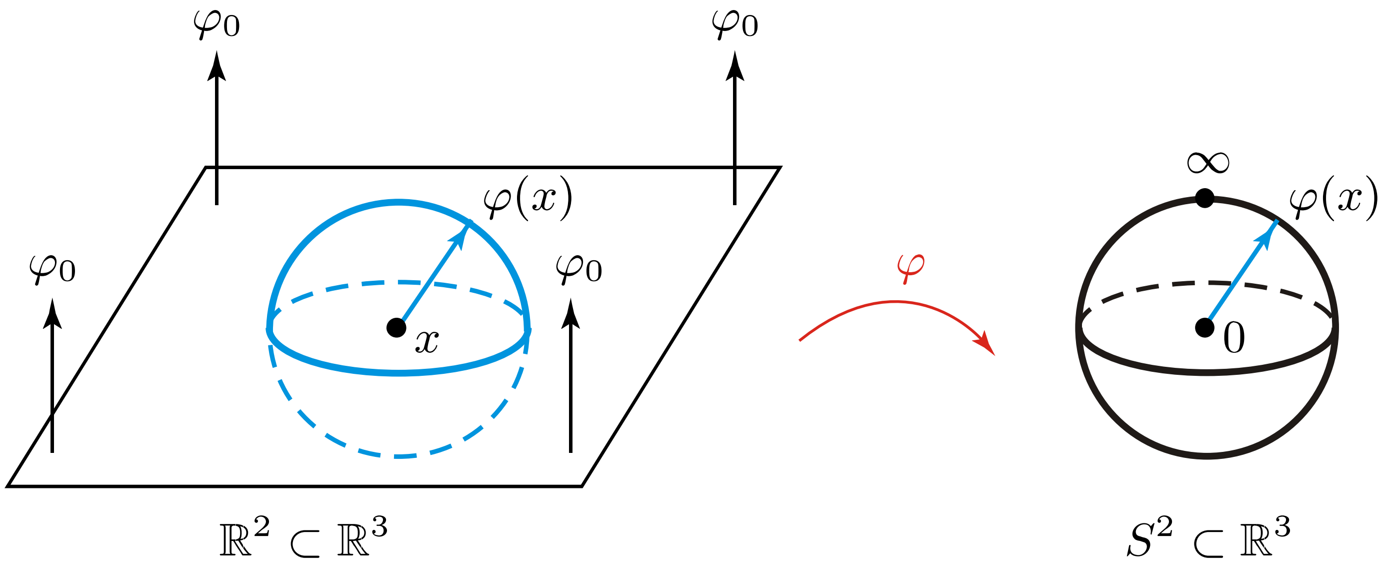
Define the *energy* of φ by the Dirichlet integral

$$\boxed{E(\varphi) = \frac{1}{2} \int_{\mathbb{R}^2} |d\varphi|^2 dx_1 dx_2}$$

where

$$|d\varphi|^2 = \left|\frac{\partial \varphi}{\partial x_1}\right|^2 + \left|\frac{\partial \varphi}{\partial x_2}\right|^2 .$$

Problem. Find all smooth maps $\varphi : \mathbb{R}^2 \to S^2$ with a finite energy $E(\varphi) < \infty$ which are extremal with respect to $E(\varphi)$.



Because of the condition $E(\varphi) < \infty$ it is natural to impose on φ the following *asymptotic condition*

$$\varphi(x) \longrightarrow \varphi_0 \quad \text{uniformly for } |x| \to \infty$$

where φ_0 is a fixed point of S^2.

Under this condition our maps $\varphi : \mathbb{R}^2 \to S^2$ extend to continuous maps

$$\varphi : S^2 = \mathbb{R}^2 \cup \{\infty\} \longrightarrow S^2.$$

Such maps $\varphi : S^2 \to S^2$ have a topological invariant, called the *degree* of the map, given by

$$\boxed{\deg \varphi = \int_{\mathbb{R}^2} \varphi^* \omega}$$

where ω is the normalized volume form on the sphere S^2 and $\varphi^*\omega$ is the preimage of ω under the map φ.

Taking into account this invariant, we can reformulate our original problem as follows:

Problem. Find all extremals of the energy $E(\varphi)$ in the class of smooth maps $\varphi : \mathbb{R}^2 \to S^2$ with finite energy and given degree $k = \deg \varphi$.

To solve this problem, let us introduce the complex coordinate $z = x_1 + ix_2$ in the definition domain $\mathbb{R}^2 \approx \mathbb{C}$ and stereographic complex coordinate w in the image $S^2 \setminus \{\infty\}$.

In these coordinates the energy of $\varphi = w(z)$ is written as

$$E(\varphi) = 2\int_{\mathbb{C}} \frac{|w_z|^2 + |w_{\bar{z}}|^2}{(1+|w|^2)^2} |dz \wedge d\bar{z}|$$

where

$$w_z = \frac{\partial w}{\partial z}, \quad w_{\bar{z}} = \frac{\partial w}{\partial \bar{z}}.$$

The formula for the degree of φ takes the form

$$\deg \varphi = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{|w_z|^2 - |w_{\bar{z}}|^2}{(1+|w|^2)^2} |dz \wedge d\bar{z}|.$$

Comparing these two formulae, we arrive at inequality

$$E(\varphi) \geqslant 4\pi |\deg \varphi|.$$

Moreover, the equality here can be attained only on:

- holomorphic functions $\varphi = w(z)$ for $k = \deg \varphi \geqslant 0$, satisfying $w_{\bar{z}} \equiv 0$;
- anti-holomorphic functions $\varphi = w(z)$ for $k < 0$, satisfying $w_z \equiv 0$.

So holomorphic maps $\varphi = w(z)$ realize the minima of $E(\varphi)$ in topological classes with $k \geqslant 0$, while anti-holomorphic functions $\varphi = w(z)$ realize the minima of $E(\varphi)$ in topological classes with $k < 0$.

For minimizing maps $E(\varphi)$ is equal to $4\pi|k| \Longrightarrow$ is an integer modulo 4π. In other words, the energy in our problem is "quantized" which happens often in classical, but nonlinear physical systems.

We find now find explicit formulas for the minimizing maps. Suppose, for definiteness, that $k > 0$. Using the invariance of $E(\varphi)$ under rotations of S^2 in the image, we fix the asymptotic value φ_0 by setting it equal to $\varphi_0 = w_0 = 1$.

So we have to describe holomorphic maps of the Riemann sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ into itself of degree k, equal to 1 at infinity. Such maps are given by rational functions of the form

$$\varphi = w(z) = \prod_{j=1}^{k} \frac{z - a_j}{z - b_j}$$

where $a_j \neq b_j$ are arbitrary complex numbers.

Note that the space of solutions of our problem depends on $4k$ real parameters (or $4k + 2$ real parameters if we add rotations of S^2 in the image).

Remark. We have described all local minima of $E(\varphi)$. It can be proved that this functional has no other critical points apart from the local minima (which is an effect of two-dimensionality of the target manifold S^2).

General definition of harmonic maps. Let M be an oriented Riemannian manifold of dimension m, provided with a Riemannian metric g with metric tensor (g_{ij}), and N is an oriented Riemannian manifold of dimension n, provided with a Riemannian metric h with metric tensor $(h_{\alpha\beta})$.

For a smooth map $\varphi : (M, g) \to (N, h)$ we define its *energy* by the Dirichlet integral

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi(p)|^2 \mathrm{vol}_g$$

where $d\varphi$ is the differential of φ and vol_g is the volume element of metric g.

To compute the norm of $d\varphi$ we choose local coordinates (x^i) at $p \in M$ and (u^α) at $q = \varphi(p) \in N$. Then

$$|d\varphi(p)|^2 = \sum_{i,j} \sum_{\alpha,\beta} g^{ij} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j} h_{\alpha\beta}$$

where $\varphi^\alpha = \varphi^\alpha(x)$ are the components of φ, $(g^{ij}) = (g^{-1})_{ij}$ are the entries of the inverse matrix of (g_{ij}), vol_g is the volume element of g, given by

$$\mathrm{vol}_g = \sqrt{|\det(g_{ij})|}\, dx^1 \wedge \cdots \wedge dx^m.$$

Remark. There is also an invariant description of $d\varphi$. Namely, $\varphi \colon M \to N$ generates the tangent map $\varphi_* : TM \to TN$ which may be identified with a section $d\varphi$ of the bundle

$$T^*M \otimes \varphi^{-1}(TN) \longrightarrow N,$$

where $\varphi^{-1}(TN)$ is the inverse image of TN under the map φ. The fibre of $\varphi^{-1}(TN)$ at $p \in M$ coincides with the fibre T_qN at $q = \varphi(p)$.

The bundle $T^*M \otimes \varphi^{-1}(TN)$ is provided with a natural Riemannian metric, induced by Riemannian metrics g and h. (The local expression for this metric can be read from the local formula for $|d\varphi(p)|^2$.)

Example. Let M be an open subset in $\mathbb{R}^m$ and N is an open subset in $\mathbb{R}^n$. Then the squared norm of the differential of a smooth map $\varphi = (\varphi^1, \ldots, \varphi^n) : M \to N$ is given by

$$|d\varphi(x)|^2 = \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \left| \frac{\partial \varphi^\alpha}{\partial x^i} \right|^2 = \sum_{i=1}^{m} \left| \frac{\partial \varphi}{\partial x^i} \right|^2$$

and the energy is equal to

$$E(\varphi) = \frac{1}{2} \int_M \sum_{i=1}^{m} \left| \frac{\partial \varphi}{\partial x^i} \right|^2 dx^1 \wedge \ldots \wedge dx^m.$$

Extremals of $E(\varphi)$ are given by the maps $\varphi = (\varphi^\alpha)$ with components φ^α being the harmonic functions.

Definition. A smooth map $\varphi : M \to N$ is called *harmonic* if it is extremal for the energy functional $E(\varphi)$ with respect to all smooth variations of φ with compact supports.

Let us write down the Euler–Lagrange equations for $E(\varphi)$ in local coordinates (x^i) on M and (u^α) on N.

Denote by ${}^M\nabla$ the Levi-Civita connection of M, represented locally by the Christoffel symbol ${}^M\Gamma^k_{ij}$, and by ${}^N\nabla$ the Levi-Civita connection of N, represented locally by the Christoffel symbol ${}^N\Gamma^\gamma_{\alpha\beta}$.

In these coordinates the Euler–Lagrange equations take the form

$$\sum_{i,j} g^{ij} \left\{ \frac{\partial^2 \varphi^\gamma}{\partial x_i \partial x_j} - \sum_k {}^M\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x_k} + \sum_{\alpha,\beta} {}^N\Gamma^\gamma_{\alpha\beta}(\varphi) \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_j} \right\} =$$

$$= \Delta_M \varphi^\gamma + \sum_{i,j} g^{ij} \sum_{\alpha,\beta} {}^N\Gamma^\gamma_{\alpha\beta}(\varphi) \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_j} = 0 \,, \quad \gamma = 1, \ldots, n.$$

The operator

$$\Delta_M \varphi^\gamma = \sum_{i,j} g^{ij} \left\{ \frac{\partial^2 \varphi^\gamma}{\partial x_i \partial x_j} - \sum_k {}^M\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x_k} \right\}$$

is the standard *Laplace–Beltrami operator* of M, determined by metric g. Note that it is a linear differential operator of 2nd order in φ^γ. The second term in Euler–Lagrange equations depends on the geometry of the target space N and is quadratic with respect to derivatives of φ^γ.

Example. For $N = \mathbb{R}^n$ the Euler–Lagrange equations reduce to the Laplace–Beltrami equations on the components of φ. Their solutions are given by harmonic functions φ^γ. For $m = \dim M = 1$ harmonic maps $\varphi\colon M \to N$ coincide with geodesics of N, parameterized by the arc length.

Remark. One can write down the Euler–Lagrange equations for $E(\varphi)$ in an invariant form. Recall that $d\varphi$ may be identified with a section of the bundle $T^*M \otimes \varphi^{-1}(TN)$. This bundle can be provided with a natural connection ∇, generated by Levi-Civita connections ${}^M\nabla$ and ${}^N\nabla$.

Then Euler–Lagrange equations in terms of this connection are written as

$$\mathrm{tr}(\nabla d\varphi) = 0$$

where $\tau_\varphi = \mathrm{tr}(\nabla d\varphi)$ is called the *stress tensor* of φ.

Harmonic maps of almost complex manifolds. Let M be an almost complex Riemannian manifold, provided with an almost complex structure ${}^{M}J$, compatible with Riemannian metric g, and N is an almost complex Riemannian manifold, provided with an almost complex structure ${}^{N}J$, compatible with Riemannian metric h.

Recall that an *almost complex structure* J on M is a smooth family $\{J_p\}_{p\in M}$ of endomorphisms $J_p : T_pM \to T_pM$ such that $J_p^2 = -I$. This structure J is *integrable* if it generates the $\bar{\partial}_J$-operator, satisfying the integrability condition $\bar{\partial}_J^2 = 0$.

The *compatibility of J with Riemannian metric g* means that the 2-form ω on M, defined by

$$\omega(X, Y) := g(X, JY),$$

is symplectic and the metric g is Hermitian. A manifold (M, g, J, ω) with such an almost complex structure is called *almost Kähler* and it is called *Kähler* if J is integrable.

Definition. Let $\varphi : M \to N$ be a smooth map of almost Kähler manifolds. It is *holomorphic* if the tangent map $\varphi_*: TM \to TN$ commutes with almost complex structures ${}^MJ$ and ${}^NJ$, i.e.

$$\varphi_* \circ {}^MJ = {}^NJ \circ \varphi_*.$$

It is called *anti-holomorphic* if φ_* anti-commutes with ${}^MJ$ and ${}^NJ$.

Theorem (Lichnerowicz theorem). *Holomorphic and anti-holomorphic maps $\varphi: M \to N$ realize local minima of the energy functional $E(\varphi)$ among smooth maps in a given topological class.*

However, in general, the energy functional $E(\varphi)$ has also non-minimal critical points (harmonic maps).

We are going to describe harmonic spheres $\varphi : \mathbb{P}^1 \to N$, i.e. harmonic maps of the Riemann sphere $\mathbb{P}^1 = S^2$ to a given Riemannian manifold N, by reducing this problem to the description of holomorphic spheres in almost Kähler manifolds.

INSTANTONS AND YANG–MILLS FIELDS

Yang–Mills equations on $\mathbb{R}^4$. Let G be a compact Lie group (*gauge group*).

A *gauge G-potential* on $\mathbb{R}^4$ is a connection in a principal G-bundle over $\mathbb{R}^4$, identified with a 1-form A on $\mathbb{R}^4$ with values in the Lie algebra $\mathfrak{g}$ of G.

If G coincides with the group $\mathrm{U}(n)$ of unitary $(n \times n)$-matrices then this form may be written as

$$A = \sum_{\mu=1}^{4} A_\mu(x) dx_\mu$$

where $x = (x_1, x_2, x_3, x_4)$ are coordinates on $\mathbb{R}^4$, $A_\mu(x)$ are smooth functions on $\mathbb{R}^4$ with values in skew-Hermitian $(n \times n)$-matrices.

For $n = 1$ the gauge potential is the Euclidean analogue of the electromagnetic 4-potential.

A *gauge G-field* F is the curvature of connection A, given by a 2-form on $\mathbb{R}^4$ with values in $\mathfrak{g}$ of the form

$$F = DA = dA + \frac{1}{2}[A, A]$$

where $D : \Omega^1(\mathbb{R}^4, \mathfrak{g}) \to \Omega^2(\mathbb{R}^4, \mathfrak{g})$ is the covariant exterior derivative, generated by the connection A. In the case $G = \mathrm{U}(n)$ this form is equal to

$$F = \sum_{\mu,\nu=1}^{4} F_{\mu\nu}(x)\, dx_\mu \wedge dx_\nu \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

with $\partial_\mu := \partial/\partial x_\mu$, $\mu = 1, 2, 3, 4$. For $n = 1$ the form $\{F_{\mu\nu}\}$ coincides with the Euclidean analogue of the Maxwell tensor of electromagnetic field.

A *gauge transform* is a smooth map $g : \mathbb{R}^4 \to G$, acting on gauge potentials and fields by the formula

$$A \longmapsto A_g := g^{-1}dg + g^{-1}Ag, \qquad g : F \longmapsto F_g := g^{-1}Fg$$

where G acts on $\mathfrak{g}$ by the adjoint representation. In the case $G = \mathrm{U}(1)$ the gauge transform coincides with the multiplication by the factor $g(x) = e^{i\theta(x)}$ so that $A \mapsto A - id\theta$ and F does not change under this map.

Define the *Yang–Mills action* functional by the formula

$$S(A) = \frac{1}{2}\int_{\mathbb{R}^4} \|F\|^2 \, d^4x \qquad \text{where} \quad \|F\|^2 = \sum_{\mu,\nu=1}^{4} \|F_{\mu\nu}\|^2$$

and the norm $\|F_{\mu\nu}\|$ is computed with the help of an invariant inner product on $\mathfrak{g}$. In the case $G = \mathrm{U}(n)$ one can take for such a product $\langle X, Y, \rangle := \mathrm{tr}(XY)$. Then the formula for $S(A)$ will rewrite as

$$S(A) = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}(F \wedge *F)$$

where $*$ is the Hodge star-operator on $\mathbb{R}^4$.

The functional $S(A)$ is invariant under gauge transformations so that $S(A)$ depends on the class of connection A modulo gauge transformations rather than A itself.

Definition. *Yang–Mills fields* are the gauge fields F with finite Yang–Mills action $S(A) < \infty$, realizing the extremals of $S(A)$. The corresponding gauge potentials A are called the *Yang–Mills connections.*

Yang–Mills fields satisfy the Euler–Lagrange equations for $S(A)$ which have the form

$$D^*F = 0$$

where $D^* : \Omega^2(\mathbb{R}^4, \mathfrak{g}) \to \Omega^1(\mathbb{R}^4, \mathfrak{g})$ is the formal adjoint of D. It is equal to $D^* = - * D *$ and the Euler–Lagrange equations for $S(A)$ may be rewritten as

$$D * F = 0.$$

This equation is called the *Yang–Mills equation* and is often supplemented with the *Bianchi identity*

$$\boxed{DF = 0}$$

which is automatically satisfied for gauge fields F.

Instantons. A gauge field F is called *selfdual* (resp. *anti-selfdual*) if

$$*F = F \quad \text{(resp. } *F = -F\text{)}.$$

Bianchi identity implies that solutions of *duality equations*

$$*F = \pm F$$

satisfy Yang–Mills equations.

If we write down the form F as a sum

$$F = F_+ + F_-$$

with $F_\pm = \frac{1}{2}(*F \pm F)$ then Yang–Mills action will be rewritten as

$$S(A) = \frac{1}{2}\int_{\mathbb{R}^4} \left(\|F_+\|^2 + \|F_-\|^2\right) d^4x.$$

For gauge fields F with finite Yang–Mills action the quantity

$$k(A) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \left(-\|F_+\|^2 + \|F_-\|^2\right) d^4x = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \operatorname{tr}(F \wedge F)$$

is an integer-valued topological invariant, called the *topological charge* of F.

Evidently,

$$\boxed{S(A) \geqslant 4\pi^2|k(A)|.}$$

The minimum of $S(A)$, equal to $4\pi^2|k|$ in the topological class of gauge potentials with finite Yang–Mills action and fixed topological charge $k(A) = k$, may be attained for $k > 0$ only on anti-selfdual fields and for $k < 0$ only on selfdual ones.

Definition. Anti-selfdual fields with finite action $S(A) < \infty$ are called the *instantons* while selfdual fields with finite action $S(A) < \infty$ are called the *anti-instantons.*

Instantons and anti-instantons realize local minima of the action $S(A)$, however, there exist also non-minimal critical points of this functional.

One of the main objects in Yang–Mills theory is the *moduli space of Yang–Mills fields* which is the quotient of the space of all Yang–Mills fields modulo gauge transforms. The structure of this space is far from being understood and one of our goals is to approach this problem on the base of harmonic spheres conjecture.

However, an analogous problem for instantons, i.e. the description of the moduli space of instantons on $\mathbb{R}^4$, was solved by Atiyah–Drinfeld–Hitchin–Manin with the help of the twistor approach.

Comparing Yang–Mills fields with harmonic maps, we observe the following evident analogy between:

$$\{\text{(anti)holomorphic maps}\} \longleftrightarrow \{\text{(anti)instantons}\}$$

and

$$\{\text{harmonic maps}\} \longleftrightarrow \{\text{Yang–Mills fields}\}\,.$$

As we shall see from the Atiyah theorem and harmonic spheres conjecture, this formal analogy has, in fact, a much deeper meaning.

TWISTOR INTERPRETATION OF INSTANTONS

Basic twistor bundle over S^4. We shall identify the 4-sphere S^4 with the quaternion projective line by analogy with the identification of the 2-sphere S^2 with the complex projective line $\mathbb{CP}^1$.

Recall that the space of quaternions $\mathbb{H}$ consists of elements

$$q = x_1 + ix_2 + jx_3 + kx_4$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$, $i^2 = j^2 = k^2 = -1$ and the multiplication law is defined by the relation

$$ij = -ji = k.$$

The space $\mathbb{H}$ is a non-commutative field isomorphic, as a vector space, to $\mathbb{R}^4$. As a complex vector space $\mathbb{H}$ can be identified with $\mathbb{C}^2$ by writing quaternions in the form

$$q = z_1 + z_2 j$$

where $z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4 \in \mathbb{C}$.

Quaternion projective line $\mathbb{HP}^1$ consists of pairs $[q, q']$ of quaternions (not equal to zero simultaneously) which are defined up to multiplication (from the right) by a nonzero quaternion. We identify the Euclidean sphere $S^4 = \mathbb{R}^4 \cup \{\infty\}$ with the quaternion projective line $\mathbb{HP}^1$ and define the *basic twistor bundle* over S^4:

$$\pi\colon \mathbb{CP}^3 \xrightarrow{\mathbb{CP}^1} \mathbb{HP}^1$$

by the tautological formula

$$\boxed{[z_1, z_2, z_3, z_4] \longmapsto [z_1 + z_2 j, z_3 + z_4 j]}$$

where the 4-tuple $[z_1, z_2, z_3, z_4] \in \mathbb{CP}^3$ is defined up to multiplication by a nonzero complex number while the pair $[z_1 + z_2 j, z_3 + z_4 j] \in \mathbb{HP}^1$ is defined up to multiplication by a nonzero quaternion. The fibre of π coincides with the complex projective line $\mathbb{CP}^1$, invariant under multiplication from the right by j, i. e. under the map

$$j\colon [z_1, z_2, z_3, z_4] \longmapsto [-z_2, z_1, -z_4, z_3].$$

The constructed bundle $\pi: \mathbb{CP}^3 \to S^4$ has a nice interpretation in terms of complex structures on $\mathbb{R}^4$ due to Atiyah.

To describe it, consider the restriction of π to the Euclidean space $\mathbb{R}^4 \cong \mathbb{H}$:

$$\pi\colon \ \mathbb{CP}^3 \setminus \mathbb{CP}^1_\infty \longrightarrow \mathbb{R}^4$$

where the omitted complex projective line $\mathbb{CP}^1_\infty$ is identified with the fibre $\pi^{-1}(\infty)$ of the twistor bundle at $\infty \in S^4$.

The space $\mathbb{CP}^3 \setminus \mathbb{CP}^1_\infty$ is foliated by parallel complex projective planes $\mathbb{CP}^2$. These planes intersect in $\mathbb{CP}^3$ on the projective line $\mathbb{CP}^1_\infty$ so that each point p of $\mathbb{CP}^1_\infty$ defines one family of parallel planes. The tangent map π_* provides the tangent space $T_q\mathbb{R}^4$ at a point $q \in \mathbb{R}^4$ with the complex structure, induced from these parallel planes. Different families, determined by points $p \in \mathbb{CP}^1_\infty$, define different complex structures on $T_q\mathbb{R}^4$ so that the space of all complex structures on $T_q\mathbb{R}^4$, compatible with metric, can be identified with $\mathbb{CP}^1_\infty$.

Summing up, we can consider the twistor bundle $\pi\colon \ \mathbb{CP}^3 \setminus \mathbb{CP}^1_\infty \longrightarrow \mathbb{R}^4$ as a bundle of complex structures on $\mathbb{R}^4$, compatible with metric. The fibre of this bundle at a point $q \in \mathbb{R}^4$ consists of complex structures on the tangent space $T_q\mathbb{R}^4$, compatible with metric, and can be identified, as above, with $\mathbb{CP}^1_\infty$.

Atiyah–Hitchin–Singer construction and Penrose twistor program. We use the interpretation of basic twistor bundle as a bundle of complex structures for the extension of the twistor bundle construction to general Riemannian manifolds.

Let N be an even-dimensional oriented Riemannian manifold of dimension $2n$. Consider the bundle $\pi\colon \mathcal{J}(N) \to N$ of complex structures on N, compatible with Riemannian metric. The fibre of this bundle at $q \in N$ coincides with the space $\mathcal{J}(T_qN)$ of complex structures J_q on the tangent space T_qN, compatible with the metric.

The bundle $\pi\colon \mathcal{J}(N) \to N$ is associated with the principal bundle $\mathrm{O}(N) \to N$ of orthonormal frames on N and its fibre $\pi^{-1}(q)$ can be identified with the complex homogeneous space $\mathrm{O}(2n)/\mathrm{U}(n)$.

The bundle $\pi : \mathcal{J}(N) \to N$ can be provided with a natural almost complex structure, introduced by Atiyah–Hitchin–Singer.

Namely, the Levi-Civita connection ${}^N\nabla$ on N generates a natural connection on $\mathrm{O}(N)$, hence on $\mathcal{J}(N)$. This connection determines a vertical-horizontal decomposition

$$T\mathcal{J}(N) = V \oplus H.$$

In terms of this decomposition, we define an almost complex structure $\mathcal{J}^1$ on $\mathcal{J}(N)$ by setting

$$\mathcal{J}^1 = \mathcal{J}^v \oplus \mathcal{J}^h$$

where $\mathcal{J}^v_z$ at $z \in \mathcal{J}(N)$ coincides with the canonical complex structure on the vertical space V_z, identified with $\mathrm{O}(2n)/\mathrm{U}(n)$.

The horizontal component $\mathcal{J}^h_z$ at z coincides with the complex structure $J(z)$ on the horizontal space H_z, given by the point z of the twistor bundle with H_z, identified with the tangent space $T_{\pi(z)}N$ by π_*. We recall that the fibre $\pi^{-1}(q)$ of $\pi : \mathcal{J}(N) \to N$ at $q = \pi(z) \in N$ consists of complex structures on T_qN and we denote by $J(z)$ the complex structure on T_qN, corresponding to the point $z \in \pi^{-1}(q)$. This construction provides $(\mathcal{J}(N), \mathcal{J}^1)$ with the structure of an almost complex manifold.

We formulate now an heuristic *Penrose twistor program:*

Construct for a given Riemannian manifold N a twistor bundle $\pi : Z \to N$, where the twistor space Z is an almost complex manifold, with the following characterictic property: there should be a 1–1 correspondence between

$$\left\{ \begin{matrix} \text{objects of Riemannian} \\ \text{geometry on } N \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{objects of holomorphic} \\ \text{geometry on } Z \end{matrix} \right\}.$$

Such a correspondence, being established, yields a method of studying the real geometry of the Riemannian manifold N via the complex geometry of its twistor space Z.

The above Atiyah–Hitchin–Singer construction gives an example of such a twistor bundle $\mathcal{J}(N) \to N$ with the twistor space $Z = \mathcal{J}(N)$ provided with the almost complex structure $\mathcal{J}^1$.

Atiyah–Ward and Donaldson theorems. Since from now on we deal only with the complex projective spaces $\mathbb{CP}^1$ and $\mathbb{CP}^3$, we shorten their notation to $\mathbb{P}^1$ and $\mathbb{P}^3$.

Return to the problem of description of

$$\left\{ \begin{array}{c} \text{moduli space of} \\ G\text{-instantons on } \mathbb{R}^4 \end{array} \right\} = \frac{\{G\text{-instantons on } \mathbb{R}^4\}}{\{\text{gauge transforms}\}}.$$

Using the basic twistor bundle $\pi\colon \mathbb{P}^3 \setminus \mathbb{P}^1 \to \mathbb{R}^4$, Atiyah and Ward have reduced this problem to a problem of description of certain holomorphic bundles over the 3-dimensional complex projective space $\mathbb{P}^3$. Namely, according to Atiyah–Ward theorem, there is a 1–1 correspondence between:

$$\left\{ \begin{array}{c} \text{moduli space of} \\ G\text{-instatons on } \mathbb{R}^4 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{based equivalence classes of holomorphic} \\ G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^3\text{, trivial on } \pi\text{-fibers} \end{array} \right\}.$$

Here, $G^{\mathbb{C}}$ is the complexification of the group G and the term "based" means that the equivalence of $G^{\mathbb{C}}$-bundles over $\mathbb{P}^3$ is defined "modulo" $\mathbb{P}^1_\infty$, i. e. all mappings, defining the equivalence of the bundles, should be equal to identity on $\mathbb{P}^1_\infty$.

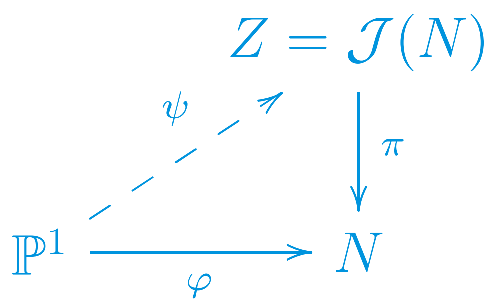
This result has the following 2-dimensional reduction to the space $\mathbb{P}^1 \times \mathbb{P}^1$, given by the Donaldson theorem:

$$\left\{ \begin{array}{c} \text{moduli space of} \\ G\text{-instantons on } \mathbb{R}^4 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{based equivalence classes of holomorphic} \\ G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^1 \times \mathbb{P}^1, \text{ trivial on } \mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty \end{array} \right\}$$

where $\mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty$ denotes the union of two complex projective lines "at infinity" of $\mathbb{P}^1 \times \mathbb{P}^1$.

TWISTOR INTERPRETATION OF HARMONIC SPHERES

Eells–Salamon theorem. Guided by the Penrose twistor program, we may suppose that the problem of construction of harmonic spheres $\varphi\colon \mathbb{P}^1 \to N$ in a given Riemannian manifold N should reformulate as a problem of construction of holomorphic spheres $\psi\colon \mathbb{P}^1 \to Z$ in its twistor space $(Z = \mathcal{J}(N), \mathcal{J}^1)$ such that $\varphi = \pi \circ \psi$:



And it is almost true. In fact, projections of holomorphic spheres $\psi\colon \mathbb{P}^1 \to Z$ to N do satisfy some partial differential equations of 2nd order on N. However, these equations are not harmonic but ultrahyperbolic, i.e. "harmonic with a wrong signature" (n, n) instead of the required signature $(2n, 0)$.

By this reason, we have to change the definition of the almost complex structure on Z if we want to construct harmonic spheres in N as projections of holomorphic spheres in Z.

Namely, we provide Z with a new almost complex structure $\mathcal{J}^2$ which is given in terms of the vertical-horizontal decomposition

$$T\mathcal{J}(N) = V \oplus H$$

by

$$\mathcal{J}^2 = (-\mathcal{J}^v) \oplus \mathcal{J}^h.$$

This particular almost complex structure, introduced by Eells and Salamon, is responsible for the twistor description of harmonic spheres.

Here is a formal definition of the twistor bundle which will be used in the sequel.

Definition. A smooth bundle $\pi : Z \to N$ of an almost complex manifold $(Z, \mathcal{J})$ over a Riemannian manifold N will be called the *twistor bundle* of N if the projection $\varphi := \pi \circ \psi$ of any holomorphic sphere $\psi\colon \mathbb{P}^1 \to Z$ to N is a harmonic sphere $\varphi\colon \mathbb{P}^1 \to N$.

Theorem (Eells–Salamon)**.** *The twistor bundle*

$$\pi\colon Z = \mathcal{J}(N) \longrightarrow N,$$

provided with the almost complex structure $\mathcal{J}^2$, *is the twistor bundle, i.e. projection* $\varphi := \pi \circ \psi$ *of any holomorphic sphere* $\psi\colon \mathbb{P}^1 \to Z$ *to* N *is a harmonic sphere* $\varphi\colon \mathbb{P}^1 \to N$.

Using this theorem, we can construct harmonic spheres in N from holomorphic spheres in its twistor space Z.

However, we note that the almost complex structure $\mathcal{J}^1$ on $\mathcal{J}(N)$ is *integrable* $\Leftrightarrow N$ *is conformally flat* while the almost complex structure $\mathcal{J}^2$ is *never integrable.*

Taking this into account, Eells–Salamon theorem may look not helpful as a method of construction of harmonic spheres in N. Indeed, it reduces the problem of construction of harmonic spheres in the Riemannian manifold N to the problem of construction of holomorphic spheres in the almost complex manifold $(Z, \mathcal{J}^2)$.

But the almost complex structure $\mathcal{J}^2$, being non-integrable, might be quite bizarre. For example, such a structure may have no non-constant holomorphic functions even locally. Our advantage is that we are dealing not with holomorphic functions, i.e. holomorphic maps $f\colon Z \to \mathbb{C}$, but with a dual object — holomorphic maps $\psi\colon \mathbb{C} \to Z$. Such a map is holomorphic with respect to the almost complex structure $\mathcal{J}^2$ on $Z \iff$ it satisfies the Cauchy–Riemann equation

$$\bar{\partial}_J \psi = 0$$

with respect to the pulled-back almost complex structure $J := \psi^*(\mathcal{J}^2)$ on $\mathbb{C}$. And this structure J is integrable as any almost complex structure in complex dimension 1.

In particular, the above Cauchy–Riemann equation has many local solutions.

Complex Grassmann manifolds and flag bundles. We apply the twistor approach to the description of harmonic spheres in the complex Grassmann manifolds $G_r(\mathbb{C}^d)$. In this case it is natural to choose as their twistor spaces the bundles of complex structures over $G_r(\mathbb{C}^d)$, invariant under the action of the unitary group $\mathrm{U}(d)$. Such bundles coincide with the flag bundles defined below.

Definition. The *flag manifold* $F_{\mathbf{r}}(\mathbb{C}^d)$ in $\mathbb{C}^d$ of *type* $\mathbf{r} = (r_1, \dots, r_n)$ with $d = r_1 + \dots + r_n$ consists of flags $\mathcal{W} = (W_1, \dots, W_n)$, i.e. nested sequences of complex subspaces

$$W_1 \subset \dots \subset W_n = \mathbb{C}^d,$$

such that the dimension of the subspace $V_1 := W_1$ is equal to r_1 and dimensions of the subspaces $V_i := W_i \ominus W_{i-1}$ are equal to r_i for $1 < i \leqslant n$.

The flag manifold $F_{\mathbf{r}}(\mathbb{C}^d)$ admits a description as a homogeneous space of the unitary group $\mathrm{U}(d)$:

$$F_{\mathbf{r}}(\mathbb{C}^d) = \mathrm{U}(d)/\,\mathrm{U}(r_1) \times \cdots \times \mathrm{U}(r_n).$$

It is a compact Kähler manifold which has an $\mathrm{U}(d)$-invariant complex structure, denoted by $\mathcal{J}^1$.

Definition. For the construction of a flag bundle over the Grassmann manifold $G_r(\mathbb{C}^d)$ we fix an ordered subset $\sigma \subset \{1, \ldots, n\}$, such that $\sum_{i\in\sigma} r_i = r$, and define the *flag bundle*

$$\pi_\sigma\colon F_{\mathbf{r}}(\mathbb{C}^d) \longrightarrow G_r(\mathbb{C}^d)$$

by

$$\pi_\sigma\colon \mathcal{W} = (W_1, \ldots, W_n) \longmapsto W := \bigoplus_{i\in\sigma} V_i.$$

Harmonic spheres in Grassmann manifolds: Burstall–Salamon theorem.

The flag bundle π_σ can be provided with an almost complex structure $\mathcal{J}^2_\sigma$ so that the following analogue of Eells–Salamon theorem holds.

Theorem (Burstall–Salamon). *The flag bundle*

$$\pi_\sigma\colon\ (F_{\mathbf{r}}(\mathbb{C}^d), \mathcal{J}^2_\sigma) \longrightarrow G_r(\mathbb{C}^d),$$

provided with the almost complex structure $\mathcal{J}^2_\sigma$*, is a twistor bundle, i.e. the projection* $\varphi = \pi_\sigma \circ \psi$ *of any holomorphic sphere* $\psi\colon \mathbb{P}^1 \to F_{\mathbf{r}}(\mathbb{C}^d)$ *to* $G_r(\mathbb{C}^d)$ *is a harmonic sphere* $\varphi\colon \mathbb{P}^1 \to G_r(\mathbb{C}^d)$ *in* $G_r(\mathbb{C}^d)$.

Moreover, the converse is also true: any harmonic sphere $\varphi\colon \mathbb{P}^1 \to G_r(\mathbb{C}^d)$ *in* $G_r(\mathbb{C}^d)$ *may be obtained in this way from some flag bundle* $\pi_\sigma\colon F_{\mathbf{r}}(\mathbb{C}^d) \to G_r(\mathbb{C}^d)$.

Using this twistor interpretation of harmonic spheres in $G_r(\mathbb{C}^d)$, we can reduce their description to the description of holomorphic spheres in flag manifolds $F_{\mathbf{r}}(\mathbb{C}^d)$.

The latter problem was solved by Wood. The idea of his construction is roughly the following. A map $\psi\colon \mathbb{P}^1 \to F_{\mathbf{r}}(\mathbb{C}^d)$ may be considered as a decomposition of the trivial bundle $\mathbb{P}^1 \times \mathbb{C}^d$ into the direct sum of subbundles

$$\mathbb{P}^1 \times \mathbb{C}^d = \psi_1 \oplus \cdots \oplus \psi_n$$

where $\psi_i := \psi^* T_i$ with T_i being the ith tautological bundle over $F_{\mathbf{r}}(\mathbb{C}^d)$.

A map $\psi\colon \mathbb{P}^1 \to F_{\mathbf{r}}(\mathbb{C}^d)$ is $\mathcal{J}^1$-holomorphic $\iff$ all subbundles $\psi_1, \ldots, \psi_n$ are holomorphic. Wood has proposed a procedure which allows to rebuild the above decomposition into a decomposition

$$\mathbb{P}^1 \times \mathbb{C}^d = \tilde{\psi}_1 \oplus \cdots \oplus \tilde{\psi}_m,$$

corresponding to a $\mathcal{J}^2$-holomorphic sphere, where subbundles $\tilde{\psi}_i$ are either holomorphic or anti-holomorphic.

Loop spaces of compact Lie groups. We switch now to the infinite-dimensional target manifolds N, namely, we take for N the loop space ΩG of a compact Lie group G.

Definition. Let G be a compact Lie group. Then its *loop space* is $\boxed{\Omega G = LG/G}$ where $LG = C^\infty(S^1, G)$ is the *loop group* of G, i.e. the space of C^∞-smooth maps $S^1 \to G$, and G in the denominator is identified with the subgroup of constant maps $S^1 \to g_0 \in G$. Otherwise, ΩG can be thought of as the space of *based loops*, i.e. the maps $S^1 \to G$, sending $1 \in S^1 \mapsto e \in G$.

The space ΩG is an infinite-dimensional Kähler manifold. A complex structure on ΩG is induced from its representation as a homogeneous space of a complex Lie group

$$\Omega G = LG^{\mathbb{C}} / L_+G^{\mathbb{C}}$$

where $G^{\mathbb{C}}$ is the complexification of G, $LG^{\mathbb{C}} = C^\infty(S^1, G^{\mathbb{C}})$ is the complexified loop group of G, and $L_+G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$ is a subgroup of $LG^{\mathbb{C}}$, consisting of the maps which may be holomorphically extended to the unit disc Δ.

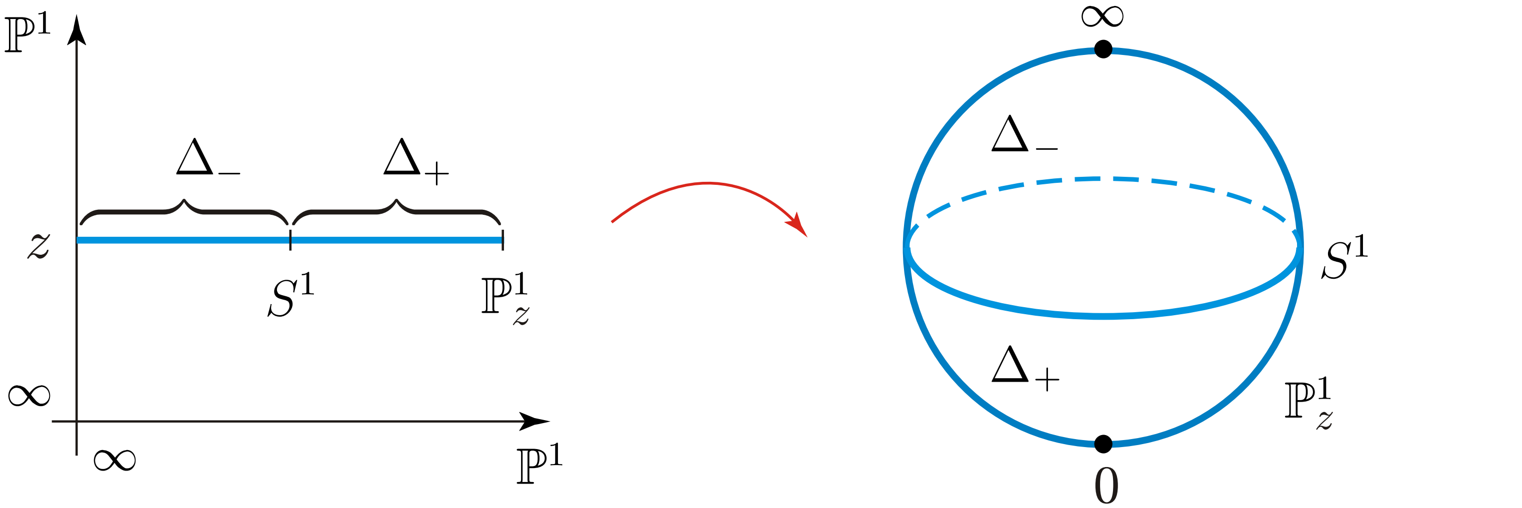
Holomorphic spheres in loop spaces: theorem of Atiyah. Recall that, according to Donaldson theorem:

$$\left\{\begin{array}{c}\text{moduli space of}\\ G\text{-instantons on }\mathbb{R}^4\end{array}\right\}\longleftrightarrow\left\{\begin{array}{c}\text{based equivalence classes of holo-}\\ \text{morphic }G^{\mathbb{C}}\text{-bundles over }\mathbb{P}^1\times\mathbb{P}^1,\\ \text{trivial on the union }\mathbb{P}^1_\infty\cup\mathbb{P}^1_\infty\end{array}\right\}.$$

Atiyah theorem asserts that the right hand side of this correspondence can be identified with the space of based holomorphic spheres in ΩG. In other words, there is a 1–1 correspondence:

$$\left\{\begin{array}{c}\text{based equivalence classes of holo-}\\ \text{morphic }G^{\mathbb{C}}\text{-bundles over }\mathbb{P}^1\times\mathbb{P}^1,\\ \text{trivial on the union }\mathbb{P}^1_\infty\cup\mathbb{P}^1_\infty\end{array}\right\}\longleftrightarrow\left\{\begin{array}{c}\text{based holomorphic spheres}\\ f:\mathbb{P}^1\to\Omega G,\text{ sending }\infty\\ \text{to the origin of }\Omega G\end{array}\right\}.$$

The proof of Atiyah theorem is based on the following construction.



Restrict a given holomorphic $G^{\mathbb{C}}$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ to the projective line $\mathbb{P}^1_z$, passing through a point $\mathbb{P}^1 \times \{z\}$ parallel to P^1_∞. This restricted bundle is determined by a transition function

$$\tilde{f}_z\colon\ S^1 \longrightarrow G^{\mathbb{C}}$$

which is holomorphic in a neighborhood of the equator S^1 in $\mathbb{P}^1_z$. Hence, $\tilde{f}_z \in LG^{\mathbb{C}}$ and we have a map

$$f\colon\ \mathbb{P}^1 \ni z \longmapsto \tilde{f}_z \in LG^{\mathbb{C}} \longmapsto f(z) \in \Omega G = LG^{\mathbb{C}}/L_+G^{\mathbb{C}}.$$

This map is holomorphic and based $\Longleftrightarrow$ the original $G^{\mathbb{C}}$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ is holomorphic and trivial on $\mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty$.

Harmonic spheres conjecture. Atiyah and Donaldson theorems imply that there is a 1–1 correspondence between:

$$\left\{\begin{array}{c}\text{moduli space of}\\ G\text{-instantons on } \mathbb{R}^4\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{based holomorphic spheres}\\ f\colon \mathbb{P}^1 \to \Omega G\end{array}\right\}.$$

So we have a correspondence between local minima of two functionals, namely:

$$\left\{\begin{array}{c}\text{Yang–Mills action on}\\ \text{gauge } G\text{-fields on } \mathbb{R}^4\end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{c}\text{energy of smooth}\\ \text{spheres in } \Omega G\end{array}\right\},$$

their local minima being given correspondingly by:

$$\left\{\begin{array}{c}\text{instantons and}\\ \text{anti-instantons}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{holomorphic and anti-}\\ \text{holomorphic spheres}\end{array}\right\}.$$

If we replace here the local minima by the critical points of the corresponding functionals, we shall arrive at the formulation of the *harmonic spheres conjecture*, asserting that it should exist a 1–1 correspondence between:

$$\left\{\begin{array}{c}\text{moduli space of Yang–}\\ \text{Mills } G\text{-fields on } \mathbb{R}^4\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{based harmonic spheres}\\ f\colon \mathbb{P}^1 \to \Omega G\end{array}\right\}.$$

We can consider the described transition from the local minima to the critical points of our functionals as a "realification" procedure. Indeed, if we replace smooth spheres in the right hand side of the above diagram by smooth functions $f\colon \mathbb{C} \to \mathbb{C}$ then the described transition will reduce to the replacement of holomorphic and anti-holomorphic functions by arbitrary harmonic functions (which are the sums of holomorphic and anti-holomorphic functions). In the case of smooth spheres in ΩG this transition from holomorphic and anti-holomorphic spheres to harmonic ones becomes non-trivial due to the non-linear character of Euler–Lagrange equations for the energy.

Unfortunately, a direct extension of Atiyah–Donaldson proof to the harmonic case is not possible. The reason is that the proof of Donaldson theorem, based on the monad method of construction of holomorphic vector bundles on complex projective spaces, is purely holomorphic. However, one can attempt to reduce the proof of the harmonic spheres conjecture to the holomorphic case by "pulling-up" both sides of the correspondence in this conjecture to their twistor spaces.

Hilbert–Schmidt Grassmannian. In order to construct a twistor bundle over the loop space ΩG we shall first embed ΩG into an infinite-dimensional Grassmannian and then construct its twistor bundle by analogy with the finite-dimensional case.

The role of the infinite-dimensional Grassmannian will be played by the Hilbert–Schmidt Grassmannian of a complex Hilbert space H, provided with a *polarization*. That is a complex Hilbert space H together with a decomposition

$$H = H_+ \oplus H_-$$

into the direct orthogonal sum of closed infinite-dimensional subspaces $H_\pm$. In the case of the space $H = L_0^2(S^1, \mathbb{C})$ of square integrable functions on S^1 with zero average such subspaces are given by

$$H_\pm = \left\{\gamma \in H : \gamma = \sum_{\pm k>0} \gamma_k e^{ik\theta}\right\}.$$

Definition. The *Hilbert–Schmidt Grassmannian* $\mathrm{Gr}_{\mathrm{HS}}(H)$ consists of closed subspaces $W \subset H$ such that the orthogonal projection $\pi_+ \colon W \to H_+$ is a Fredholm operator and orthogonal projection $\pi_- \colon W \to H_-$ is a Hilbert–Schmidt operator.

Given a subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ we call the Fredholm index of the projection $\pi_+ \colon W \to H_+$ the *virtual dimension* of W.

Similar to the finite-dimensional case, the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ admits the following homogeneous representation

$$\mathrm{Gr}_{\mathrm{HS}}(H) = \frac{\mathrm{U}_{\mathrm{HS}}(H)}{\mathrm{U}(H_+) \times \mathrm{U}(H_-)}$$

where the *unitary Hilbert–Schmidt group* $\mathrm{U}_{\mathrm{HS}}(H)$ is defined by

$$\mathrm{U}_{\mathrm{HS}}(H) = \{A \in \mathrm{U}(H) : \pi_- \circ A \circ \pi_+ \text{ is Hilbert–Schmidt}\}.$$

The Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ is a Hilbert Kähler manifold, consisting of a countable number of connected components of a fixed virtual dimension:

$$\mathrm{Gr}_{\mathrm{HS}}(H) = \bigcup_d G_d(H) \qquad \text{where} \quad G_d(H) = \{W \in \mathrm{Gr}_{\mathrm{HS}}(H) : \text{virt.dim}\, W = d\}.$$

Virtual flag bundles and harmonic spheres in the Hilbert–Schmidt Grassmannian. The virtual flag manifold and flag bundles are defined by analogy with the finite-dimensional case.

Definition. The *virtual flag manifold* $F^d_{\mathbf{r}}(H)$ in H of *type* $\mathbf{r} = (r_1, \dots, r_n)$ with $d = r_1 + \dots + r_n$ consists of flags $\mathcal{W} = (W_1, \dots, W_n)$, i.e. nested sequences of complex subspaces

$$W_1 \subset \dots \subset W_n \subset H,$$

such that the virtual dimension of the subspace $V_1 := W_1$ is equal to r_1, and dimensions of subspaces $V_i := W_i \ominus W_{i-1}$ are equal to r_i for $1 < i \leqslant n$.

Definition. For the construction of a flag bundle over the Grassmann manifold $G_r(H)$ we fix an ordered subset $\sigma \subset \{1, \dots, n\}$, so that $\sum_{i\in\sigma} r_i = r$, and define the *virtual flag bundle*

$$\pi_\sigma\colon\ F^d_{\mathbf{r}}(H) \longrightarrow G_r(H),$$

by

$$\pi_\sigma\colon\ \mathcal{W} = (W_1, \dots, W_n) \longmapsto W := \bigoplus_{i\in\sigma} V_i.$$

As in the finite-dimensional case, we can provide the virtual flag bundle π_σ with an almost complex structure $\mathcal{J}_\sigma^2$ so that the following analogue of Burstall–Salamon theorem holds.

Theorem. *The virtual flag bundle*

$$\pi_\sigma \colon \left(F_{\mathbf{r}}^d(H), \mathcal{J}_\sigma^2\right) \longrightarrow G_r(H),$$

provided with the almost complex structure $\mathcal{J}_\sigma^2$, is a twistor bundle, i.e. the projection $\varphi = \pi_\sigma \circ \psi$ of any almost holomorphic sphere $\psi \colon \mathbb{P}^1 \to F_{\mathbf{r}}^d(H)$ to $G_r(H)$ is a harmonic sphere $\varphi \colon \mathbb{P}^1 \to G_r(H)$ in $G_r(H)$.

We think that the converse of this Theorem is also true, as in the finite-dimensional case.

Embedding of loop spaces into the Hilbert–Schmidt Grassmannian. Suppose that our compact Lie group G is realized as a subgroup of the unitary group $\mathrm{U}(N)$ and construct an embedding of ΩG into the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ where $H = L^2_0(S^1, \mathbb{C}^N)$.

Construct first an embedding of the loop group LG into the unitary Hilbert–Schmidt group $\mathrm{U}_{\mathrm{HS}}(H)$. For that we associate with a loop γ, belonging to the space $LG = C^\infty(S^1, G) \subset C^\infty(S^1, \mathrm{U}(N))$, the multiplication operator M_γ in the Hilbert space $H = L^2_0(S^1, \mathbb{C}^N)$, acting by the formula:

$$\boxed{h \in H = L^2_0(S^1, \mathbb{C}^N) \longmapsto M_\gamma h(z) := \gamma(z)h(z),} \qquad z \in S^1.$$

In other words, $M_\gamma h$ is a vector function from $H = L^2_0(S^1, \mathbb{C}^N)$, obtained by the pointwise application of the matrix function $\gamma \in C^\infty(S^1, \mathrm{U}(N))$ to the vector function $h \in H = L^2_0(S^1, \mathbb{C}^N)$. The operator M_γ belongs to the unitary group $\mathrm{U}_{\mathrm{HS}}(H)$ if $\gamma \in C^\infty(S^1, \mathrm{U}(N))$.

The constructed embedding $LG \hookrightarrow \mathrm{U}_{\mathrm{HS}}(H)$ generates an isometric embedding

$$\Omega G \longrightarrow \mathrm{Gr}_{\mathrm{HS}}(H).$$

IDEA OF THE PROOF OF HARMONIC SPHERES CONJECTURE

Harmonic analogue of Atiyah theorem. Using the constructed isometric embedding $\Omega G \hookrightarrow \mathrm{Gr}_{\mathrm{HS}}(H)$, we can consider an arbitrary harmonic map $\varphi\colon \mathbb{P}^1 \to \Omega G$ as taking its values in the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$, hence, in one of its connected components $G_r(H)$ and use the twistor method.

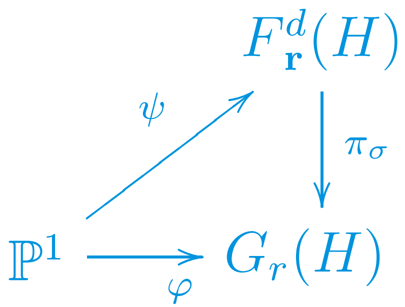
We start from a harmonic version of Atiyah theorem, relating based harmonic spheres $\varphi\colon \mathbb{P}^1 \to \Omega G$ to harmonic $G^{\mathbb{C}}$-bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. For a fixed $z \in \mathbb{P}^1$ we pull back the value $\varphi(z) \in \Omega G$ to $\tilde{\varphi}(z) \in LG^{\mathbb{C}}$ and consider $\tilde{\varphi}(z)$ as a transition function of a bundle over the projective line $\mathbb{P}^1_z$. By changing $z \in \mathbb{P}^1$, we obtain a $G^{\mathbb{C}}$-bundle E over $\mathbb{P}^1 \times \mathbb{P}^1$ which is harmonic and trivial over $\mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty$ iff the original map φ is based and harmonic.

If we consider the map $\varphi\colon \mathbb{P}^1 \to \Omega G$ as taking values in $\mathrm{Gr}_{\mathrm{HS}}(H)$ then the value $\varphi(z)$ at a fixed $z \in \mathbb{P}^1$ is interpreted in terms of $\mathrm{Gr}_{\mathrm{HS}}(H)$ as a subspace $W_z = M_{\tilde{\varphi}(z)} H_+$.

Twistor interpretation of the moduli space of Yang–Mills fields. Here is a twistor interpretation of the above construction.

A harmonic sphere $\varphi\colon\ \mathbb{P}^1 \to \Omega G$ may be considered as a harmonic sphere in a submanifold $G_r(H) \subset \mathrm{Gr}_{\mathrm{HS}}(H)$, consisting of subspaces $W \subset H$ of some fixed virtual dimension r.

In terms of the twistor flag bundle the harmonic sphere $\varphi\colon\ \mathbb{P}^1 \to G_r(H)$ is the projection of some $\mathcal{J}^2_\sigma$-holomorphic sphere $\psi\colon\ \mathbb{P}^1 \to F^d_{\mathbf{r}}(H)$ so that there is a commutative diagram



We pull back the value $\psi(z) = (\psi_1(z), \ldots, \psi_n(z))$ at a fixed $z \in \mathbb{P}^1$ to

$$\tilde{\psi}(z) = (\tilde{\psi}_1(z), \ldots, \tilde{\psi}_n(z))$$

where $\tilde{\psi}_i(z) \in LG^{\mathbb{C}}$.

In terms of $F^d_{\mathbf{r}}(H)$ the value $\psi(z) = (\psi_1(z), \ldots, \psi_n(z))$ is given by a virtual flag $\mathcal{W}(z) = (W_1(z), \ldots, W_n(z)$ where $W_i(z) = M_{\tilde{\psi}_i(z)} H_+$.

The functions $\tilde{\psi}_i(z) \in LG^{\mathbb{C}}$, being considered as transition functions, determine some bundles over $\mathbb{P}^1_z$. By changing $z \in \mathbb{P}^1$, we obtain for $i = 1, \ldots, n$ the $G^{\mathbb{C}}$-bundles E_i over $\mathbb{P}^1 \times \mathbb{P}^1$ trivial over $\mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty$. It follows from the definition of the almost complex structure $\mathcal{J}^2_\sigma$ that these bundles E_i should be either holomorphic or anti-holomorphic. So by Atiyah theorem they should correspond to instantons or anti-instantons on $\mathbb{R}^4$.

In this way we can associate with any Yang–Mills field on $\mathbb{R}^4$ a finite collection of instantons and anti-instantons on $\mathbb{R}^4$. This construction may be considered as a twistor interpretation of the moduli space of Yang–Mills fields on $\mathbb{R}^4$.