Star products
-formal and nonformal-

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In this talk, we give a brief review on star products.

1. We introduce a star product on complex polynomials.
2. We can extend the product to functions by two different ways.
    - One is to extend by means of formal power series,
    - Another is nonformal extension.
3. Also we give some resent results in nonformal star products.

Based on the joint works with H. Omori, Y. Maeda, N. Miyazaki.

1. Star product on polynomials

- Star product is regarded as an idea to introduce an associative product on polynomials.
- In this talk, we mainly consider functions of two variables $(u, v) = (u_1, u_2)$ for simplicity. Generalization to an arbitrary number of variables is direct.

- The Moyal product is a typical example of star product, which is attached to the canonical coordinate:

$$f *_0 g = f \exp\left(\frac{i\hbar}{2}\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)g$$
$$= fg + \frac{i\hbar}{2} f\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)g + \cdots + \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n f\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)^n g + \cdots \qquad (1)$$

where $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\overleftarrow{\partial}\Lambda\overrightarrow{\partial} = (\overleftarrow{\partial_{u_1}}, \overleftarrow{\partial_{u_2}})\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \overrightarrow{\partial_{u_1}} \\ \overrightarrow{\partial_{u_2}} \end{pmatrix}$ is a biderivation given by

$$f\overleftarrow{\partial}\Lambda\overrightarrow{\partial}g = f\left(\overleftarrow{\partial_{u_2}}\overrightarrow{\partial_{u_1}} - \overleftarrow{\partial_{u_1}}\overrightarrow{\partial_{u_2}}\right)g = \partial_{u_2} f \partial_{u_1} g - \partial_{u_1} f \partial_{u_2} g \qquad (2)$$

- The Moyal product is well-defined on complex polynomials.

- By replacing the matrix Λ with an arbitrary complex matrix, we define a product on complex polynomials such that

$$f *_{\Lambda} g = f \exp\left(\frac{i\hbar}{2} \overleftarrow{\partial} \Lambda \overrightarrow{\partial}\right) g \tag{3}$$

- It is easy to see

Proposition

For an arbitrary Λ, the product $*_{\Lambda}$ is associative.

- We call $*_{\Lambda}$ a star product given by Λ.

We remark here that

Remark

1. *when* $\Lambda = 0$, $*_\Lambda$ *is a usual multiplication of polynomials*
2. *when* Λ *is symmetric,* $*_\Lambda$ *is commutative.*

$$f *_\Lambda g = f \exp\left(\frac{i\hbar}{2}\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)g$$
$$= fg + \frac{i\hbar}{2} f\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)g + \cdots + \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n f\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)^n g + \cdots$$

- We consider matrices Λ and Λ' with the common skew symmetric part. We put the difference of these as $K = \Lambda' - \Lambda$ and we define a linear isomorphism of polynomials

$$T_K f = \exp\left(\frac{i\hbar}{4}\partial K \partial\right) f = \sum_{n\geq 0} \frac{1}{n!}\left(\frac{i\hbar}{4}\right)^n (\partial K \partial)^n f, \ \partial K \partial = \sum_{ij} K_{ij}\partial_{u_i}\partial_{u_j}$$

Proposition

T_K is an intertwiner between the products $*_\Lambda$ and $*_{\Lambda'}$;

$$(T_K f *_{\Lambda'} T_K g) = T_K(f *_\Lambda g).$$

- Then the algebraic structure of $*_\Lambda$ on polynomials depends only on the skew symmetric part of Λ. $\Lambda = K + J \Longrightarrow K = 0$.

2. Formal extension

We consider to extend the star product to some space of functions.

We have two directions.

1. One is formal star product– star product on the space of all formal power series of $\hbar$ with coefficients in smooth functions
2. another is nonformal deformation.

We extend the star product $*_\Lambda$ to the space of all formal power series with coefficients in smooth functions on $\mathbb{R}^2$.

2. Formal extension

2.1. Extended product

Let us consider the space of all formal power series

$$\mathcal{A}_\hbar = C^\infty(\mathbb{R}^2)[[\hbar]] \tag{4}$$

Then we have

Proposition

The star product $*_\Lambda$ is well-defined on $\mathcal{A}_\hbar$ such that

$$f *_\Lambda g = fg + \frac{i\hbar}{2} C_1(f,g) + \cdots + \left(\frac{i\hbar}{2}\right)^n C_n(f,g) + \cdots \tag{5}$$

where $C_n = \frac{1}{n!}(\overleftarrow{\partial}\Lambda\overrightarrow{\partial})^n$ is a bidifferential operator. And we have an associative algebra $(\mathcal{A}_\hbar, *_\Lambda)$.

Note that $\{f,g\} = \frac{1}{2}(C_1(f,g) - C_1(g,f))$ is the Poisson bracket given by the skew symmetric part of Λ.

The concept of formal star product leads to deformation quantization on Poisson manifolds.

Let us consider a Poisson manifold $(M, \{\ ,\ \})$, and we put $\mathcal{A}_\hbar(M) = C^\infty(M)[[\hbar]]$.

Definition

An associative product $*$ *on* $\mathcal{A}_\hbar(M)$ *is called a deformation quantization on* $(M, \{\ ,\ \})$ *when it has an expansion*

$$f * g = fg + \frac{i\hbar}{2} C_1(f,g) + \cdots + (\frac{i\hbar}{2})^n C_n(f,g) + \cdots \tag{6}$$

for any $f, g \in \mathcal{A}_\hbar(M)$*, where* C_n *is a bidifferential operator on* M *and*

$$\frac{1}{2}\left(C_1(f,g) - C_1(g,f)\right) = \{f, g\}\,. \tag{7}$$

Remark that $*$ is localized to an arbtrary domain $U \subset M$, that is, we have a star product $f * g$ for any $f, g \in \mathcal{A}_\hbar(U)$.

When $(M, \{\ ,\ \})$ is symplectic, the deformation quantization $*$ has a nice property.

On a Darboux chart $(U, (u_1, \cdots, u_n, v_1, \cdots, v_n))$, the Poisson bracket is expressed in the form

$$\{f, g\} = \sum_i \frac{\partial f}{\partial_{u_i}} \frac{\partial g}{\partial_{v_i}} - \frac{\partial f}{\partial_{v_i}} \frac{\partial g}{\partial_{u_i}} = f \overleftarrow{\partial} \Lambda \overrightarrow{\partial} g, \qquad (8)$$

where $\Lambda = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} = J_0$.

On this U, we have the Moyal star product $*_{J_0}$ on $\mathcal{A}_\hbar(U) = C^\infty(U)[[\hbar]]$.

Further, we have

Proposition (Quantized Darboux theorem)

For any deformation quantization $*$ on a symplectic manifold $(M, \{\ ,\ \})$, locally the product $*$ is isomorphic to the Moyal product $*_{J_0}$ on $C^\infty(U)[[\hbar]]$.

We call the local Moyal procut algebra $(C^\infty(U)[[\hbar]], *_{J_0})$ a quantized Darboux chart.

On the other hand, by gluing local Moyal algebras we obtain a deformation quantizaion on $(M, \{\ ,\ \})$.

Theorem (DeWilde-Lecomte, Fedosov, OMY)

For any symplectic manifold $(M, \{\ ,\ \})$*, there exists a deformation quantization which has quantized Darboux charts.*

Further, we have an existence of a deformation quantization on an arbitrary Poisson manifolds.

Theorem (Kontsevich)

For a Poisson manifold, there exists a deformation quantization.

3. Nonformal extension

Now we consider nonformal extension of star product.

The situation is quite different from formal extension. For instance,

- The expansion

$$
\begin{aligned}
f *_\Lambda g &= f \exp\left(\frac{i\hbar}{2} \overleftarrow{\partial} \Lambda \overrightarrow{\partial}\right) g \\
&= fg + \frac{i\hbar}{2} f \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial}\right) g + \cdots + \frac{1}{n!} \left(\frac{i\hbar}{2}\right)^n f \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial}\right)^n g + \cdots
\end{aligned}
$$

  is not convergent for functions f, g in general.
- Gluing of local star product algebra is not convergent in general. So, we cannot consider a nonformal star product on a general Poisson manifold.

Instead of considering on a manifold, we consider star products on holomorphic functions on $\mathbb{C}^2$. For every positive number p we put

Definition

$$\mathcal{E}_p = \{ f \in Hol(\mathbb{C}^2) \,|\, |f|_{p,s} < \infty, \quad \forall s > 0 \}$$

where $|f|_{p,s}$ *is a semi-norm give by*

$$|f|_{p,s} = \sup_{z \in \mathbb{C}^2} |f(z)| \exp\left(-s|z|^p\right) \tag{9}$$

The space $\mathcal{E}_p$ is a commuative Frechét algebra under usual multiplication of functions, with $\mathcal{E}_p \subset \mathcal{E}_{p'}$, for $p < p'$.

The star product and the intertwiner are convergent for certain p. Namely, we have

Theorem

1. *For $0 < p \leq 2$, $(\mathcal{E}_p, *_\Lambda)$ is a Frechét algebra. Moreover, for any Λ' having the same skew symmetric part as Λ, $I_\Lambda^{\Lambda'} = \exp(\frac{i\hbar}{4}\partial K \partial)$ with $K = \Lambda' - \Lambda$ is well-defined intertwiner from $(\mathcal{E}_p, *_\Lambda)$ to $(\mathcal{E}_p, *_{\Lambda'})$.*
2. *For $p > 2$, the multiplication $*_\Lambda : \mathcal{E}_p \times \mathcal{E}_{p'} \to \mathcal{E}_p$ is well-defined for p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, and $(\mathcal{E}_p, *_\Lambda)$ is a $\mathcal{E}_{p'}$-bimodule.*

Since we have a complete topogical algebra, we can consider exponential element in the star product algebra $(\mathcal{E}_p, *_\Lambda)$.
For a polynomial H_* in $\mathcal{E}_p$, we want to define a star exponential $e_*^{t\frac{H_*}{i\hbar}}$. However, except special cases, the expansion $\sum_n \frac{t^n}{n!}\left(\frac{H_*}{i\hbar}\right)^n$ is not convergent, so we define a star exponential by means of the differential equation.

Definition

The star exponential $e_*^{t\frac{H_*}{i\hbar}}$ is given as a solution of the following differential equation

$$\frac{d}{dt}F_t = \frac{H_*}{i\hbar} *_\Lambda F_t, \quad F_0 = 1. \tag{10}$$

We are interested in the star exponentials of linear, and quadratic polynomials. For these, we can solve the differential equation and obtain explicit form.

- For simplicity, we consider matrices Λ having the skew symmetric part $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We write $\Lambda = K + J_0$ where K is a complex symmetric matrix.
- First we remark the following. For a linear polynomial $l = \sum_j a_j u_j$, we see directly

$$e^l \notin \mathcal{E}_1, \quad \in \mathcal{E}_{1+\epsilon}, \quad \forall \epsilon > 0. \tag{11}$$

- Then put the space

$$\mathcal{E}_{p+} = \cap_{q>p} \mathcal{E}_q \tag{12}$$

Proposition

For a linear function $l = \sum_j a_j u_j = \langle \boldsymbol{a}, \boldsymbol{u} \rangle$*, the star exponential is expressed as*

$$e_*^{t(l/i\hbar)} = e^{t^2 \boldsymbol{a} K \boldsymbol{a}/4i\hbar} e^{t(l/i\hbar)} \in \mathcal{E}_{1+}$$

Proposition

For $Q_* = \langle \boldsymbol{u}A, \boldsymbol{u}\rangle_*$ where A is a 2×2 complex symmetric matrix, the star exponential is expressed as

$$e_*^{t(Q_*/i\hbar)} = \frac{2^m}{\sqrt{\det(I - \kappa + e^{-2t\alpha}(I + \kappa))}} e^{\frac{1}{i\hbar}\langle \boldsymbol{u} \frac{1}{I - \kappa + e^{-2t\alpha}(I+\kappa)}(I - e^{-2t\alpha})J, \boldsymbol{u}\rangle}$$

where $\kappa = KJ_0$ and $\alpha = AJ_0$.

In what follows, we consider the star product for the simple case where

$$\Lambda = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$$

Then we see easily that the star product is commutative and explicitly given by $p_1 *_\Lambda p_2 = p_1 \exp\left(\frac{i\hbar\rho}{2}\overleftarrow{\partial_{u_1}}\overrightarrow{\partial_{u_1}}\right) p_2$. This means that the algebra is essentially reduced to space of functions of one varible u_1. Thus, we consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and we consider a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) e^{\frac{\tau}{2}\overleftarrow{\partial_w}\overrightarrow{\partial_w}} g(w)$$

A direct calculation gives

$$\exp_{*_\tau} itw = \exp(itw - (\tau/4)t^2)$$

Hence for $\Re\tau > 0$, the star exponential $\exp_{*_\tau} niw = \exp(niw - (\tau/4)n^2)$ is rapidly decreasing with respect to integer n and then we can consider summations for τ satisfying $\Re\tau > 0$

$$\sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2niw = \sum_{n=-\infty}^{\infty} \exp\left(2niw - \tau n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niw}, \quad (q = e^{-\tau})$$

This is Jacobi's theta function $\theta_3(w, \tau)$

Then we have expression of theta functions as

$$\theta_{1*_\tau}(w) = \frac{1}{\mathrm{i}} \sum_{n=-\infty}^{\infty} (-1)^n \exp_{*_\tau}(2n+1)\mathrm{i}w, \quad \theta_{2*_\tau}(w) = \sum_{n=-\infty}^{\infty} \exp_{*_\tau}(2n+1)\mathrm{i}w$$

$$\theta_{3*_\tau}(w) = \sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2n\mathrm{i}w, \quad \theta_{4*_\tau}(w) = \sum_{n=-\infty}^{\infty} (-1)^n \exp_{*_\tau} 2n\mathrm{i}w$$

Remark that $\theta_{k*_\tau}(w)$ is the Jacobi's theta function $\theta_k(w,\tau)$, $k = 1,2,3,4$ respectively.

It is obvious by the exponential law

$$\exp_{*_\tau} 2\mathrm{i}w *_\tau \theta_{k*_\tau}(w) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$\exp_{*_\tau} 2\mathrm{i}w *_\tau \theta_{k*_\tau}(w) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

Then using $\exp_{*_\tau} 2\mathrm{i}w = \mathrm{e}^{-\tau}\mathrm{e}^{2\mathrm{i}w}$ and the product formula directly we have

$$\mathrm{e}^{2\mathrm{i}w-\tau}\theta_{k*_\tau}(w + \mathrm{i}\tau) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$\mathrm{e}^{2\mathrm{i}w-\tau}\theta_{k*_\tau}(w + \mathrm{i}\tau) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

As a simple example, we consider a Harmonic oscillator

$$H = \frac{1}{2}(p^2 + q^2) \tag{13}$$

We can obtain eigenvalues of the Schrödinger operator $\hat{H}$ by means of star product.

We consider the Moyal product. The star exponential is

$$e_*^{t(H/i\hbar)} = (\cos(t/2))^{-1}\, e^{\left((\tan(t/2))\frac{2H}{i\hbar}\right)} \tag{14}$$

We have a limit

$$\lim_{t \to -i\infty} e^{it/2} e_*^{t(H/i\hbar)} = f_0 = 2e^{-H/\hbar} \tag{15}$$

Then by direct calculation we have

$$H * f_0 = \frac{h}{2} f_0. \tag{16}$$

Further, by using f_0 we can construct functions f_n such that

$$H * f_n = \left(n + \frac{1}{2}\right)\hbar\, f_n, \quad n = 0, 1, 2, \ldots . \tag{17}$$

In this way, we can calculate eigenvalues of the Schrödinger operator. Kanazawa will give a talk here related star products and eigenvalue problems of MIC-Kepler problem.

Using this noncommutative exponential, we can construct several noncommutative special functions.

1. Deformation quantization

- F.Bayen, M,Flato, C.Fronsdal, A.Lichnerowicz, D.Sternheimer, *Deformation theory and quantization I, II*, Ann. Phys. 111, (1977), 61-151.
- M. De Wilde and P. Lecomte. Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds. *Lett. Math. Phys.*, 7:427–496, 1983.
- B. V. Fedosov. A simple geometrical construction of deformation quantization. *J. Diff. Geom.*, 40:213–238, 1994.
- H. Omori, Y. Maeda, and A. Yoshioka. Weyl manifolds and deformation quantization. *Advances in Math.*, 85:224–255, 1991.

- Kontsevich, Maxim Deformation quantization of Poisson manifolds. Lett. Math. Phys. 66 (2003), no. 3, 157–216.

2. Nonformal star products

- Quantum mechanics in phase space: An Overview With Selected Papers (World Scientific Series in 20th Century Physics), ed. Cosmas K. Zachos, David B. Fairlie, Thomas L. Curtright
- OMMY, Deformation Expression for Elements of Algebras, (I)–(VI) 2011, 2012, ArXiv. math-ph.