Constant Curvature solutions of Grassmannian sigma models

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Two-dimensional Grassmannian sigma model

We are interested in the set of maps Z from the two-sphere S^2 into the Grassmann target manifold $G(m,n)$ which are stationary points of an energy functional.
The Grassmann manifold $G(m,n)$ is the collection of m-dimensional linear subspaces in $\mathbb{C}^n$, it can be viewed as the quotient

$$G(m,n) = \frac{U(n)}{U(m) \times U(n-m)}, \tag{1}$$

where $U(n)$ is the unitary group of $n \times n$ complex matrix. The maps Z can thus be parametrized by m column vectors in $\mathbb{C}^n$ put in an array satisfying $Z^\dagger Z = \mathbb{I}_m$.

The Grassmannian sigma model is thus the collection of maps

$$Z = (z_1, z_2, \cdots, z_m), \quad z_i \in \mathbb{C}^n, \quad z_i^\dagger z_j = \delta_{ij}, \tag{2}$$

which are stationary points of the energy functional

$$\mathcal{E} = \int_\Omega \mathrm{Tr}(D_\mu Z)^\dagger D_\mu Z \, dx_1 dx_2, \tag{3}$$

where D_μ are the covariant derivatives defined as

$$D_\mu \Lambda = \partial_\mu \Lambda - \Lambda(Z^\dagger \partial_\mu Z). \tag{4}$$

Here Ω is an open and connected subset of the two-dimensional Euclidean space $\mathbb{R}^2$ and x_1 and x_2 are local coordinates on Ω. The energy functional (3) is invariant under global $U(n)$ and local $U(m)$ gauge transformations, *i.e.* the transformation $Z \to UZV$ leaves the energy functional invariant for $U \in U(n)$ and $V(x_1, x_2) \in U(m)$. The Euler-Lagrange equations are obtained by the least principle action and are given by

$$D_\mu D_\mu Z + Z(D_\mu Z)^\dagger D_\mu Z = 0, \quad Z^\dagger Z = \mathbb{I}_m. \tag{5}$$

The two-dimensional Grassmannian sigma model may also be formulated in an gauge invariant way using orthogonal projectors. An orthogonal projector $\mathbb{P}$ of rank m is defined as

$$\mathbb{P}^\dagger = \mathbb{P}, \quad \mathbb{P}^2 = \mathbb{P}, \quad \mathrm{Tr}\,\mathbb{P} = m. \tag{6}$$

Thus in the $G(m,n)$ model, we see that $\mathbb{P}$ defined as

$$\mathbb{P} = ZZ^\dagger \tag{7}$$

is an orthogonal projector.

In this formulation, the energy functional (3) and Euler-Lagrange equations (5) may be expressed, respectively, as

$$\mathcal{E} = \int_\Omega \mathrm{Tr}(\partial_+ \mathbb{P} \partial_- \mathbb{P}) dx_+ dx_- \tag{8}$$

and

$$[\partial_+ \partial_- \mathbb{P}, \mathbb{P}] = 0, \quad \mathbb{P}^2 = \mathbb{P}. \tag{9}$$

Here the local coordinates $x_\pm$ are defined as $x_\pm = x_1 \pm i x_2$ and $\partial_\pm = \frac{1}{2}(\partial_{x_1} \mp i \partial_{x_2})$.

Special solutions of the Euler-Lagrange equations

In the $G(1, n) \cong \mathbb{C}P^{n-1}$, Zakrzewski and Din have constructed all finite action solutions of the Euler-Lagrange equations. In that construction, we get three classes of solutions: holomorphics, antiholomorphics and mixed. The mixed and antiholomorphics solutions can be determined from the holomorphic ones by the following procedure:
An holomorphic solution of the Euler-Lagrange equation has the form

$$Z = \frac{f}{|f|}, \quad f = f(x_+). \tag{10}$$

One then introduce an orthogonalizing operator P_+ defined as

$$P_+ g = \partial_+ g - \frac{g^\dagger \partial_+ g}{|g|^2} g \tag{11}$$

and construct from $f = f(x_+) \in \mathbb{C}^n$ the orthogonalized set $\{f, P_+ f, P_+^2 f, \cdots, P_+^{n-1} f\}$ where $P_+^n f = 0$ and $P_+^i f = P_+(P_+^{i-1} f)$. In this set, f is holomorphic, P_+^{n-1} is antiholomorphic and $P_+^j f$ are mixed for $1 \leq j \leq n-2$. Thus, we have that all finite action solutions of the $\mathbb{C}P^{n-1}$ sigma model are given as

$$Z = \frac{P_+^i f}{|P_+^i f|}, \quad 1 \leq i \leq n-1. \tag{12}$$

For the general $G(m,n)$ sigma model, we may construct special solutions using the ones of the $\mathbb{C}P^{n-1}$. Indeed, we have that

$$Z=\left(\frac{P_+^{i_1}f}{|P_+^{i_1}f|},\frac{P_+^{i_2}f}{|P_+^{i_2}f|},\cdots,\frac{P_+^{i_m}f}{|P_+^{i_m}f|}\right),\quad 0\leq i_1<i_2<\cdots<i_m\leq n-1, \tag{13}$$

solves the Euler-Lagrange equations. There is a lot more finite action solutions of the Euler-Lagrange equations, but in this conference we will mostly concentrate on those solutions. The holomorphic solution is such that $i_j=j-1$ for $j=1,2,\cdots,m$.

We refer the interested reader to the book of Zakrzewski for further discussion on the construction of solutions of the $G(m,n)$ sigma model.

Surfaces in $\mathbb{R}^{n^2-1}$ obtained from the $G(m,n)$ sigma model

The projector formalism , can be used, among other things, for the construction of surfaces in $\mathbb{R}^{n^2-1}$ obtained from the $G(m,n)$ sigma model. For this purpose, it is convenient to write the Euler-Lagrange equations as a conservation law as

$$\partial_+ K - \partial_- K^\dagger = 0, \quad K = [\partial_- \mathbb{P}, \mathbb{P}]. \tag{14}$$

To generate surfaces in $\mathbb{R}^{n^2-1}$, we follow the Weierstrass immersion formula and define X, the coordinates of the surface, by line integrals

$$X(x_-, x_+) = i \int_\gamma (K^\dagger dx'_+ + K dx'_-) \in su(n). \qquad (15)$$

We thus see that for solutions $\mathbb{P}$ of the Euler-Lagrange equations, the line integrals do not depend on the contour of integration γ , but only on its endpoints (of which one is taken at ∞ and the other at (x_+, x_-)).

In order to generate surfaces in $\mathbb{R}^{n^2-1}$ from the solutions of the Grassmannian sigma model and to deduce some of there properties, we identify the Euclidean space $\mathbb{R}^{n^2-1}$ with the Lie algebra $su(n)$ (remember $X \in su(n)$).
Having done such an identification, we consider a scalar product on $su(n)$ given by

$$(A,B) = -\frac{1}{2}\mathrm{Tr}(AB), \quad A,B \in su(n). \tag{16}$$

We may thus calculate different properties of these surfaces defined by line integrals such as the first fundamental form I and the Gaussian curvature $\mathcal{K}$. We have, for solutions of the Grassmannian sigma model,

$$I = (dX, dX) = \mathrm{Tr}(\partial_+ \mathbb{P} \partial_- \mathbb{P}) dx_+ dx_- = \mathcal{L} dx_+ dx_- \qquad (17)$$

and

$$\mathcal{K} = -\frac{1}{\mathcal{L}} \partial_+ \partial_- \ln \mathcal{L}. \qquad (18)$$

Constant curvature solutions of Grassmannian sigma model

In order to get constant Gaussian curvature solutions, the Lagrangian density must be such that

$$\mathcal{L} \propto (1+|x|^2)^{-2}. \tag{19}$$

In the $G(m,n)$ sigma model, we consider orthogonal projectors of the form

$$\mathbb{P} = ZZ^\dagger = \sum_{i=0}^{n-1} \alpha_i \frac{P_+^i f \otimes (P_+^i f)^\dagger}{|P_+^i f|^2}, \quad \alpha_i \in \{0,1\}, \tag{20}$$

where m of the α_i's are non-zero.

In this case, the Lagrangian density is explicitly given as

$$\mathcal{L} = \sum_{i=1}^{n-1} (\alpha_{i-1} - \alpha_i)^2 \frac{|P_+^i f|^2}{|P_+^{i-1} f|^2} = \partial_+ \partial_- \ln \prod_{i=1}^{n-1} M_i^{(\alpha_{i-1}-\alpha_i)^2}, \quad (21)$$

where the quantities M_i are defined as

$$M_i = \prod_{k=0}^{i-1} |P_+^k f|^2, \quad M_0 = 1. \quad (22)$$

This new convenient way of writing the Lagrangian density permits us to restate the criterion for constant curvature solutions. Indeed, in order to get constant curvature solutions, one as to impose the condition

$$\prod_{i=1}^{n-1} M_i^{(\alpha_{i-1}-\alpha_i)^2} \propto (1+|x|^2)^r, \quad r \in \mathbb{N} \tag{23}$$

and, in this case (actually it is in general), the Gaussian curvature $\mathcal{K}$ is constant and is given by

$$\mathcal{K} = \frac{4}{r}. \tag{24}$$

So our main goal is to investigate the possible values of the integer r and to give a complete classification of constant curvature solutions of the Grassmannian $G(m,n)$ sigma model.

The $\mathbb{C}P^{n-1}$ sigma model

Let us first consider the holomorphic solution corresponding to $\alpha_0 = 1$ and $\alpha_i = 0$ for $i \geq 1$. In this case, the Lagrangian density reduces to

$$\mathcal{L} = \partial_+ \partial_- \ln |f|^2, \tag{25}$$

where we recall that $f = f(x_+)$.

So it as been shown that up to gauge symmetry, the function f which leads to constant Gaussian curvature is the Veronese curve given explicitly as

$$f^T(x_+) = \left(1, \sqrt{\binom{n-1}{1}}x_+, \cdots, \sqrt{\binom{n-1}{r}}x_+^r, \cdots, x_+^{n-1}\right). \tag{26}$$

In this case, the Gaussian curvature is given by

$$\mathcal{K} = \frac{4}{n-1}. \tag{27}$$

For non-holomorphic solutions ($\alpha_j \neq 0$), the Lagrangian density is given as

$$\mathcal{L} = \partial_+\partial_- \ln M_j M_{j+1}. \tag{28}$$

Using the following property of the Veronese curve

$$|P_+^k f|^2 \propto (1+|x|^2)^{n-1-2k}, \tag{29}$$

we get, for $\alpha_j \neq 0$, a constant Gaussian curvature of

$$\mathcal{K} = \frac{4}{r_j(n)}, \quad r_j(n) = (n-1) + 2j(n-1-j). \tag{30}$$

From this result, we see that for $j = 0$ (holomorphic solution) and $j = n-1$ (antiholomorphic solution) we get the maximal value of the Gaussian curvature given as $\mathcal{K} = \frac{4}{n-1}$. For the $\mathbb{C}P^{n-1}$, we have given a complete classification of constant curvature solutions in terms of the Veronese curves.

The general Grassmannian sigma model

We have shown previously that

$$\mathbb{P} = \sum_{i=0}^{n-1} \alpha_i \frac{P_+^i f \otimes (P_+^i f)^\dagger}{|P_+^i f|^2}, \quad \alpha_i \in \{0,1\} \tag{31}$$

and $f = f(x_+)$ solves the Euler-Lagrange equations and leads to finite action solutions of our Grassmannian sigma model. If $f \in \mathbb{C}^n$ is choosen to be the Veronese curve, then $\mathbb{P}$ given above leads to constant Gaussian curvature solution given by

$$\mathcal{K} = \frac{4}{\sum_{i=1}^{n-1}(\alpha_i - \alpha_{i-1})^2 i(n-i)}. \tag{32}$$

This result generalises the result for the $\mathbb{C}P^{n-1}$ sigma model. This, however, is not a complete classification of constant curvature solutions for more general Grassmannians. We may miss out some solutions which are not based on the Veronese curves.

Conjectures in the Holomorphic Cases

For the holomorphic solutions of constant Gaussian curvature, we have establish, from our investigation and results known so far, two conjectures:

- **Conjecture 1**: The maximal value of r for which there exists a holomorphic solution of $G(m,n)$ of constant curvature is $r = m(n-m)$. Furthermore, this solution corresponds to the Veronese holomorphic curve.
- **Conjecture 2**: Holomorphic solutions of constant curvature may be constructed for all integers r such that $1 \leq r \leq m(n-m)$.

These two conjectures are true in the $G(1,n)$ case, using the simple embedding $G(1,j) \subset G(1,n)$ for $2 \leq j \leq n-1$ and the Veronese curve.
We have proved our conjectures in the $G(2,k)$ models for $k = 3,4,5,6$. A systematic way of proving our conjectures is still an open questions.

Future outlooks

- Give a proof of our conjectures using new methods from algebraic geometry such as the Plücker embedding and coordinates.
- Study the Supersymmetric version to better understand the classical model.
- Give a complete classification of constant Gaussian curvature solutions for general Grassmannians.

References

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