Bi-Hamiltinian structure related to deformed $\mathfrak{so}(n)$

Alina Dobrogowska

Institute of Mathematics, University of Bialystok, Poland

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We define the map $\alpha : \mathcal{L}_+ \to \mathcal{L}_+$ as follows

$$\alpha(x_+) := \sum_{0 \le i < j} \alpha_{ij} x_{ij} \, |i\rangle \langle j| \, ,$$

where

$$x_+ = \sum_{0 \le i < j} x_{ij} \, |i\rangle \langle j| \in \mathcal{L}_+.$$

$$\begin{pmatrix} 0 & x_{01} & x_{02} & x_{03} & \dots \\ 0 & 0 & x_{12} & x_{13} & \dots \\ 0 & 0 & 0 & x_{23} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \longmapsto \begin{pmatrix} 0 & \alpha_{01} x_{01} & \alpha_{02} x_{02} & \alpha_{03} x_{03} & \dots \\ 0 & 0 & \alpha_{12} x_{12} & \alpha_{13} x_{13} & \dots \\ 0 & 0 & 0 & \alpha_{23} x_{23} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Lemma

The map $\alpha : \mathcal{L}_+ \to \mathcal{L}_+$ *is an endomorphism of the associative algebra* $\mathcal{L}_+$

$$\alpha(x_+ y_+) = \alpha(x_+)\alpha(y_+),$$

where $x_+, y_+ \in \mathcal{L}_+$*, if and only if*

$$\alpha_{ij}\alpha_{jk} = \alpha_{ik}$$

for $0 \leq i < j < k$.

We assume that

$$\alpha_{ij} = a_i a_{i+1} \dots a_{j-1} \qquad \text{for} \qquad i < j.$$

$$\begin{pmatrix} 0 & x_{01} & x_{02} & x_{03} & \dots \\ 0 & 0 & x_{12} & x_{13} & \dots \\ 0 & 0 & 0 & x_{23} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \stackrel{\alpha}{\longmapsto} \begin{pmatrix} 0 & a_0 x_{01} & a_0 a_1 x_{02} & a_0 a_1 a_2 x_{03} & \dots \\ 0 & 0 & a_1 x_{12} & a_1 a_2 x_{13} & \dots \\ 0 & 0 & 0 & a_2 x_{23} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

We define

$$\mathcal{A}_\alpha := \{x_+^\top - \alpha(x_+) : x_+ \in \mathcal{L}_+\}$$

$$x = \begin{pmatrix} 0 & -a_0 x_{01} & -a_0 a_1 x_{02} & -a_0 a_1 a_2 x_{03} & \dots \\ x_{01} & 0 & -a_1 x_{12} & -a_1 a_2 x_{13} & \dots \\ x_{02} & x_{12} & 0 & -a_2 x_{23} & \dots \\ x_{03} & x_{13} & x_{23} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$\mathcal{A}_\alpha$ is Lie algebra.

The commutator in $\mathcal{A}_\alpha$ is given by

$$[e_{ij}, e_{nm}] = \delta_{mi} e_{nj} - \delta_{jn} e_{im} + \delta_{jm} \begin{cases} -\alpha_{ij} e_{ni} & \text{for} \quad n < i \\ \alpha_{nm} e_{in} & \text{for} \quad i < n \end{cases} + \\ + \delta_{in} \begin{cases} -\alpha_{nm} e_{mj} & \text{for} \quad m < j \\ \alpha_{ij} e_{jm} & \text{for} \quad j < m \end{cases} ,$$

where

$$e_{ij} := |j\rangle \langle i| - \alpha_{ij} |i\rangle \langle j| ,$$

$0 \leq i < j$.

Lie–Poisson space $(\mathcal{L}_+, \{\cdot,\cdot\}_\alpha)$

There is the duality

$$(\mathcal{A}_\alpha)^* \simeq \mathcal{L}_+$$

defined by

$$\langle x, \rho \rangle := Tr(x\rho),$$

for $x \in \mathcal{A}_\alpha$, $\rho \in \mathcal{L}_+$.
One has the Lie–Poisson brackets on $C^\infty(\mathcal{L}_+)$ given by

$$\{f,g\}_\alpha(\rho) = Tr\left\{\rho\left[(Df(\rho))^\top - \alpha\,(Df(\rho))\,, \right.\right.$$
$$\left.\left.(Dg(\rho))^\top - \alpha\,(Dg(\rho))\right]\right\},$$

for $f, g \in C^\infty(\mathcal{L}_+)$.

Lie group for this Lie algebra is $G\mathcal{A}_\alpha = \exp \mathcal{A}_\alpha$.

Example

If the infinite product $a_i a_{i+1} \dots$ converges to a non-zero number $\alpha_{i\infty}$ for $i \in \mathbb{N} \cup \{0\}$, then

$$G\mathcal{A}_\alpha = \left\{ g : g \eta_\alpha g^\top = \eta_\alpha \right\},$$

where

$$\eta_\alpha := \sum_{i=0}^{\infty} \alpha_{i\infty} |i\rangle\langle i| = \begin{pmatrix} a_0 a_1 a_2 \dots & 0 & 0 & \dots \\ 0 & a_1 a_2 \dots & 0 & \dots \\ 0 & 0 & a_2 \dots & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Bi-Hamiltonian structure

We have the bi-Hamiltonian structure on the manifold M if

1. $(M, \{\cdot,\cdot\}_1)$ and $(M, \{\cdot,\cdot\}_2)$ are Poisson spaces;
2. a pencil of Poisson brackets

$$\{\cdot,\cdot\}_\epsilon := \{\cdot,\cdot\}_1 - \epsilon\{\cdot,\cdot\}_2$$

   is also a Poisson bracket.

Lemma

Let $\alpha : \mathcal{L}_+ \rightarrow \mathcal{L}_+$, $\beta : \mathcal{L}_+ \rightarrow \mathcal{L}_+$ be algebra endomorphisms. Then the following conditions are equivalent:

(i)

$$p\{.,.\}_\alpha + (1-p)\{.,.\}_\beta = \{.,.\}_{p\alpha+(1-p)\beta}$$

for $p \in [0,1]$;

(ii)

$$(\alpha_{ij} - \beta_{ij})\,(\alpha_{jn} - \beta_{jn}) = 0$$

for $0 \le i < j < n$;

(iii)

$$(a_i \dots a_{j-1} - b_i \dots b_{j-1})\,(a_j - b_j) = 0$$

for $0 \le i < j$.

Example

$$\alpha \rightsquigarrow \{a_1, a_2, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots\}$$
$$\beta \rightsquigarrow \{a_1, a_2, \ldots, a_{k-1}, b_k, a_{k+1}, \ldots\}$$

Example

$$\alpha \rightsquigarrow \{a_1, a_2, a_3, 0, a_5, a_6, a_7, 0, \ldots\}$$
$$\beta \rightsquigarrow \{b_1, a_2, a_3, 0, b_5, a_6, a_7, 0, \ldots\}$$

Magri chain on $\mathcal{L}_+$

Lemma

Let h_0, h_1, ..., be a sequence of functions on M satisfying recursion relation

$$\{\cdot\,, h_{p+1}\}_1 = \{\cdot\,, h_p\}_2, \quad p = 0, 1, \ldots$$

then

$$\{h_p, h_q\}_1 = \{h_p, h_q\}_2 = 0 \quad p, q = 0, 1, \ldots$$

Let I_ϵ^k be a family of Casimirs for Poisson bracket $\{\cdot,\cdot\}_\epsilon$ indexed by $k \in \mathbb{N}$

$$\{\ \cdot, I_\epsilon^k\}_\epsilon = \{\ \cdot, I_\epsilon^k\}_1 - \epsilon\{\ \cdot, I_\epsilon^k\}_2 = 0.$$

Expanding I_ϵ^k as a series with respect to the parameter ϵ

$$I_\epsilon^k = \sum_{n=0}^{\infty} h_{k,n}\epsilon^n$$

one obtains

$$\{\ \cdot, h_{k,0}\}_1 = 0$$

and

$$\{\cdot\ , h_{k,l+1}\}_1 = \{\cdot\ , h_{k,l}\}_2.$$

Thus the functions $h_{k,l}$ are in involution

$$\{h_{k,n}, h_{l,m}\}_1 = 0 = \{h_{k,n}, h_{l,m}\}_2.$$

The sequence $\{h_{k,l}\}_{l\in\mathbb{N}\cup\{0\}}$ is called a Magri chain.

In order to obtain a system of integrals in involution by Magri's method we need

(i) pencil of Poisson brackets

$$\{.,.\}_{\alpha+\epsilon\beta} = \{.,.\}_{\alpha} + \epsilon\{.,.\}_{\beta},$$

(ii) the Casimir I_ϵ

$$\{I_\epsilon,\ \cdot\ \}_{\alpha+\epsilon\beta} = 0$$

Casimirs for $(\mathcal{L}_+, \{.,.\}_\alpha)$

The functions

$$I^k_\alpha(\rho_+) = Tr\left(\alpha_{0\infty}\rho_+^2 - \rho_+\eta_\alpha\rho_+^\top\delta_\alpha - \eta_\alpha\rho_+^\top\delta_\alpha\rho_+ + \eta_\alpha\left(\rho_+^\top\right)^2\delta_\alpha\right)^k$$

are Casimirs for $(\mathcal{L}_+, \{.,.\}_\alpha)$.

$$\eta_\alpha = \sum_{i=0}^{\infty}\alpha_{i\infty}|i\rangle\langle i| = \begin{pmatrix} a_0a_1a_2\ldots & 0 & 0 & \ldots \\ 0 & a_1a_2\ldots & 0 & \ldots \\ 0 & 0 & a_2\ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\delta_\alpha = \sum_{i=0}^{\infty}\alpha_{0i}|i\rangle\langle i| = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & a_0 & 0 & \ldots \\ 0 & 0 & a_0a_1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example

If $\alpha_{0\infty} \neq 0$, then the endomophism $\alpha : \mathcal{L}_+ \longrightarrow \mathcal{L}_+$ is given by

$$\alpha(x_+) = \eta_\alpha x_+ \eta_\alpha^{-1} \qquad \text{for} \qquad x_+ \in \mathcal{A}_+.$$

where $\eta_\alpha = \sum_{i=0}^{\infty} \alpha_{i\infty} |i\rangle\langle i|$. The functions

$$I_\alpha^k(\rho_+) = Tr\left(\rho_+ - \eta_\alpha \rho_+^\top \eta_\alpha^{-1}\right)^{2k}, \qquad k \in \mathbb{N},$$

are Casimirs for $(\mathcal{L}_+, \{.,.\}_\alpha)$.

Casimirs for $(\mathcal{L}_+, \{.,.\}_{\alpha+\epsilon\beta})$

Substituting $\alpha + \epsilon\beta$ in place of α we obtain Casimirs for $(\mathcal{L}_+, \{.,.\}_{\alpha+\epsilon\beta})$:

$$\begin{aligned} I^k_\epsilon(\rho_+) = & Tr\Big((1+\epsilon)\left(\alpha_{0\infty} + \epsilon\beta_{0\infty}\right)\rho_+^2 - \\ & - \rho_+\left(\eta_\alpha + \epsilon\eta_\beta\right)\rho_+^\top\left(\delta_\alpha + \epsilon\delta_\beta\right) - \\ & - \left(\eta_\alpha + \epsilon\eta_\beta\right)\rho_+^\top\left(\delta_\alpha + \epsilon\delta_\beta\right)\rho_+ + \\ & + \left(\eta_\alpha + \epsilon\eta_\beta\right)\left(\rho_+^\top\right)^2\left(\delta_\alpha + \epsilon\delta_\beta\right)\Big)^k . \end{aligned}$$

Expanding $I_\epsilon^k(\rho_+)$ as a series with respect to the parameter ϵ

$$I_\epsilon^k(\rho_+) = \sum_{n=0}^{2k} h_{k,n}(\rho_+)\epsilon^n$$

we obtain the system of integrals in involution

$$\{h_{k,n}, h_{l,m}\}_\alpha = 0 = \{h_{k,n}, h_{l,m}\}_\beta \quad k, l, n, m \in \mathbb{N} \cup \{0\}.$$

We obtain the following hierarchy of Hamilton equations

$$\frac{\partial \rho_{ij}}{\partial t_{k,m}} = \sum_{n=i+1}^{j-1} \left(\alpha_{nj} \rho_{in} \frac{\partial h_{k,m}}{\partial \rho_{nj}} - \alpha_{in} \frac{\partial h_{k,m}}{\partial \rho_{in}} \rho_{nj} \right) + \\ + \sum_{n=0}^{i-1} \left(\rho_{nj} \frac{\partial h_{k,m}}{\partial \rho_{ni}} - \alpha_{in} \frac{\partial h_{k,m}}{\partial \rho_{nj}} \rho_{ni} \right) + \\ + \sum_{n=j+1}^{\infty} \left(\alpha_{ij} \rho_{jn} \frac{\partial h_{k,m}}{\partial \rho_{in}} - \frac{\partial h_{k,m}}{\partial \rho_{jn}} \rho_{in} \right),$$

where $\rho_+ = \sum_{0 \leq i < j} \rho_{ij} |i\rangle\langle j|$, $m \leq 2k$, $k \in \mathbb{N}$.

References:

- A. Odzijewicz, A. Dobrogowska, *Integrable Hamiltonian systems related to the Hilbert-Schmidt ideal*, J. Geom. Phys., **61**, (2011), no. 8, 1426-1445.
- A. Dobrogowska, A. Odzijewicz, *Integrable relativistic systems given by Hamiltonians with momentum-spin-orbit coupling*, Regul. Chaotic Dyn., **17**, (2012), no. 6, 492-505

Example (5×5)– bi–Hamiltonian structure on $\mathcal{L}_+$ and the related integrable systems

In this case:

$$a_0 = 0,$$
$$a_1 = a,$$
$$a_2 = a_3 = 1,$$

and

$$\eta_\alpha = \left(\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \eta^a \end{array}\right),$$

The Lie algebra $\mathcal{A}_\alpha = \mathcal{E}_a(1,3)$

$$\left(\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline x_{01} & 0 & -ax_{12} & -ax_{13} & -ax_{14} \\ x_{02} & x_{12} & 0 & -x_{23} & -x_{24} \\ x_{03} & x_{13} & x_{23} & 0 & -x_{34} \\ x_{04} & x_{14} & x_{24} & x_{34} & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline y & X \end{array}\right),$$

where $y \in \mathbb{R}^4$ and $X \in Mat_{4\times 4}(\mathbb{R})$ satisfies

$$X\eta^a + \eta^a X^\intercal = 0.$$

We obtain for:
$a = 1$ – Euclidean algebra,
$a = -1$ – Poincaré algebra,
$a = 0$ – Galilean algebra.

Lie group $g \in E_a(1,3)$

$$g = \left(\begin{array}{c|c} 1 & 0 \\ \hline \tau & \Lambda \end{array} \right),$$

where $\tau \in \mathbb{R}^4$ and $\Lambda \in Mat_{4\times 4}(\mathbb{C})$ satisfies

$$\Lambda \eta^a \Lambda^\intercal = \eta^a.$$

The pencil of metric tensors

$$ds_a^2 = \eta^a_{\mu\nu} dx^\mu dx^\nu := a(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

on the four-dimensional affine space $\mathbb{E}_a^{1,3}$ with coordinates x^μ, $\mu = 0, 1, 2, 3$.

We have $\rho \in \mathcal{L}_+$

$$\rho = \begin{pmatrix} 0 & P_0 & P_1 & P_2 & P_3 \\ 0 & 0 & L_1 & L_2 & L_3 \\ 0 & 0 & 0 & J_3 & -J_2 \\ 0 & 0 & 0 & 0 & J_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $P = (P_\mu)$ the four-momentum and $M = (M_{\mu\nu})$ relativistic angular momentum

$$M_{0,k} = -M_{k,0} := L_k, \qquad M_{kl} := \epsilon_{kln} J_n.$$

We find that Lie–Poisson bracket $\{\cdot,\cdot\}_\alpha$ of $f,g \in C^\infty(\mathcal{L}_+)$ is expressed as follows

$$\begin{gathered}
\{f,g\}_\alpha(\rho) := Tr\left(\rho[Df(\rho), Dg(\rho)]\right) = \\
= aP_0\left(\frac{\partial f}{\partial \vec{P}}\cdot\frac{\partial g}{\partial \vec{L}} - \frac{\partial f}{\partial \vec{L}}\cdot\frac{\partial g}{\partial \vec{P}}\right)+ \\
+\vec{J}\cdot\left(a\left(\frac{\partial f}{\partial \vec{L}}\times\frac{\partial g}{\partial \vec{L}}\right)+\frac{\partial f}{\partial \vec{J}}\times\frac{\partial g}{\partial \vec{J}}\right)+ \\
+\frac{\partial g}{\partial P_0}\vec{P}\cdot\frac{\partial f}{\partial \vec{L}} - \frac{\partial f}{\partial P_0}\vec{P}\cdot\frac{\partial g}{\partial \vec{L}}+ \\
+\vec{P}\cdot\left(\frac{\partial f}{\partial \vec{P}}\times\frac{\partial g}{\partial \vec{J}}+\frac{\partial f}{\partial \vec{J}}\times\frac{\partial g}{\partial \vec{P}}\right)+ \\
+\vec{L}\cdot\left(\frac{\partial f}{\partial \vec{L}}\times\frac{\partial g}{\partial \vec{J}}+\frac{\partial f}{\partial \vec{J}}\times\frac{\partial g}{\partial \vec{L}}\right).
\end{gathered}$$

Note that one has the following two invariants (Casimir functions) of the coadjoint representation

$$c_1 = \eta^a_{\mu\nu} P^\mu P^\nu = a P_0^2 + \vec{P} \cdot \vec{P},$$
$$c_2 = \eta^a_{\mu\nu} W^\mu W^\nu = a \left(\vec{P} \cdot \vec{J}\right)^2 + \left(a P_0 \vec{J} + \vec{L} \times \vec{P}\right)^2,$$

where the Pauli–Lubanski (spin) four–vector $W = (W^\mu)$

$$W^0 = -\vec{J} \cdot \vec{P},$$
$$\vec{W} = a P_0 \vec{J} + \vec{L} \times \vec{P},$$

while

$$\eta^a_{\mu\nu} P^\mu W^\nu = 0.$$

The coadjoint representation of $E_a(1,3)$ on the dual space of $\mathcal{L}_+$ has the form

$$Ad^*_g(P,M) = \Big((\eta^a)^{-1}\Lambda\eta^a P,$$

$$\Lambda\left(\pi_+\left(M\right) - \eta^a\pi_+\left(M^\top\right)(\eta^a)^{-1}\right)\Lambda^{-1} + \tau P^\top\Lambda^{-1} - \Lambda\eta^a P\tau^\top(\eta^a)^{-1}\Big),$$

$$Ad^*_g(W) = (\eta^a)^{-1}\Lambda\eta^a W,$$

where we represent $\rho \in \mathcal{L}_+$ by the four-momentum $P = (P_\mu)$ and the angular momentum.

$$M_{0,k} = -M_{k,0} := L_k, \qquad M_{kl} := \epsilon_{kln}J_n.$$

We note also that for a, b the Poisson brackets $\{\cdot,\cdot\}_a$ and $\{\cdot,\cdot\}_b$ define bi–Hamiltonian structure on $\mathcal{L}_+$, i.e. their linear combination $\{\cdot,\cdot\}_a + \epsilon\{\cdot,\cdot\}_b$, $\epsilon \in \mathbb{R}$, is also a Poisson bracket on $\mathcal{L}_+$.
Thus we obtain that Casimirs of $\{\cdot,\cdot\}_b$:

$$h_1 = bP_0^2 + \vec{P}\cdot\vec{P},$$
$$h_2 = b\left(\vec{P}\cdot\vec{J}\right)^2 + \left(bP_0\vec{J} + \vec{L}\times\vec{P}\right)^2$$

are the integrals of motion being in involution with respect to the Poisson bracket $\{\cdot,\cdot\}_a$.

Hamiltonian equations associated with the Hamiltonian

$$h=\frac{1}{2}\left(ch_1+dh_2\right)=$$

$$=\frac{c}{2}\left(bP_0^2+\vec{P}\cdot\vec{P}\right)+\frac{d}{2}\left(b\left(\vec{P}\cdot\vec{J}\right)^2+\left(bP_0\vec{J}+\vec{L}\times\vec{P}\right)^2\right)=$$

$$=\frac{1}{2}(b-a)\left(cP_0^2+d(b-a)P_0^2\vec{J}^2-\frac{d}{a}\vec{W}^2+2dP_0\vec{W}\cdot\vec{J}\right),$$

where $c,d\in\mathbb{R}$ are as follows

$$\frac{dP_0}{dt}=\{P_0,h\}_a=0,$$

$$\frac{d\vec{J}}{dt}=\{\vec{J},h\}_a=0,$$

$$\frac{d\vec{P}}{dt}=\{\vec{P},h\}_a=(b-a)dP_0\left(\vec{P}\times\left(\vec{P}\times\vec{L}\right)+bP_0\vec{J}\times\vec{P}\right),$$

$$\frac{d\vec{L}}{dt}=\{\vec{L},h\}_a=(b-a)\left(cP_0\vec{P}+bdP_0\vec{J}^2\vec{P}+dP_0\vec{L}\times\left(\vec{P}\times\vec{L}\right)+\right.$$

$$\left.+d\vec{P}^2\vec{J}\times\vec{L}-d\left(\vec{P}\cdot\vec{L}\right)\vec{J}\times\vec{P}+bdP_0^2\vec{J}\times\vec{L}\right).$$

In order to solve these equations it suffices to possess four functionally independent integrals of motion being in involution with respect to $\{\cdot,\cdot\}_a$. We choose h_1, h_2, $\vec{J}^2$ and J_3 as these integrals.

Using the variables $(\vec{P},\vec{W})$ we rewrite in the form

$$\frac{d\vec{P}}{dt} =(b-a)dP_0\left(-\vec{P}\times\vec{W}+(b-a)P_0\vec{J}\times\vec{P}\right),$$
$$\frac{d\vec{W}}{dt} =(b-a)d\left(\left(\vec{P}\cdot\vec{J}\right)\vec{P}\times\vec{W}+bP_0^2\vec{J}\times\vec{W}+aP_0\left(\vec{P}\cdot\vec{J}\right)\vec{J}\times\vec{P}\right).$$

Now let us introduce new variables

$$y := \vec{J} \cdot \vec{W},$$
$$z := \vec{J} \cdot \left(\vec{P} \times \vec{W} \right).$$

We find that these variables and $W_0 = -\vec{J} \cdot \vec{P}$ satisfy the following equations

$$\begin{aligned} \frac{dW_0}{dt} =& (b-a)dP_0 z, \\ \frac{dy}{dt} =& -(b-a)dW_0 z, \\ \frac{dz}{dt} =& -(b-a)dW_0 \left(c_2 P_0 + c_1 a P_0 \vec{J} \cdot \vec{J} - c_1 y \right), \end{aligned}$$

which can be integrated in quadratures:

$$t+t_0=\int \frac{dW_0}{(b-a)d\sqrt{\frac{-c_1}{4}W_0^4+\frac{c_1(h_2-c_2-(b^2-a^2)P_0^2\vec{J}^{\,2})}{2(b-a)}W_0^2+\beta}},$$

$$y(t)=-\frac{1}{2P_0}W_0^2(t)+\frac{h_2-c_2}{2P_0(b-a)}-\frac{b-a}{2}P_0\vec{J}^{\,2},$$

$$z(t)=\frac{1}{P_0}\sqrt{\frac{-c_1}{4}W_0^4(t)+\frac{c_1(h_2-c_2-(b^2-a^2)P_0^2\vec{J}^{\,2})}{2(b-a)}W_0^2(t)+\beta}.$$

Without loss of generality we can assume $\vec{J} = (0, 0, J)$ and obtain

$$P_3 = -\frac{1}{J} W_0,$$

$$W_3 = \frac{1}{J}\left(-\frac{1}{2P_0} W_0^2 + \frac{h_2 - c_2}{2P_0(b-a)} - \frac{b-a}{2} P_0 J^2\right),$$

$$P_1^2 + P_2^2 = c_1 - aP_0^2 - \frac{1}{J^2} W_0^2,$$

$$W_1^2 + W_2^2 = c_2 - aW_0^2 - \frac{1}{J^2}\left(-\frac{1}{2P_0} W_0^2 + \frac{h_2 - c_2}{2P_0(b-a)} - \frac{b-a}{2} P_0 J^2\right)^2.$$

After passing to polar coordinates

$$P_1 = \sqrt{P_1^2 + P_2^2}\cos\varphi, \qquad P_2 = \sqrt{P_1^2 + P_2^2}\sin\varphi,$$
$$W_1 = \sqrt{W_1^2 + W_2^2}\cos\psi, \qquad W_2 = \sqrt{W_1^2 + W_2^2}\sin\psi$$

we get

$$\frac{d\varphi}{dt} = (b-a)d_2 P_0 \left(bP_0 J + \frac{y - aP_0 J^2}{J} - \frac{W_0^2 (y + aP_0 J^2)}{W_0^2 - c_1 J^2 + aJ^2 P_0^2} \right),$$
$$\frac{d\psi}{dt} = (b-a)dP_0 \left(bP_0 J + \frac{y^2 - a^2 P_0^2 J^4}{JP_0 (c_2 J^2 - aJ^2 W_0^2 - y^2)} - \frac{1}{P_0 J} W_0^2 \right).$$

We find that the solutions $\vec{W}(t)$, $\vec{P}(t)$ are expressed by first–coordinate of the spin four–vector $W_0(t)$ which is an elliptic function of t.

The case of relativistic particle with spin

The twistor space $\mathbb{T}$ is $\mathbb{C}^4$ equipped with in the Hermitian form Φ of the signature $(++--)$

$$\Phi = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The Grassmannian $G(2, \mathbb{T}) =: \mathbb{M}$ of the two–dimensional subspaces $z \subset \mathbb{T}$ of the twistor space $\mathbb{T} \longleftrightarrow$ the Mincowski space $\mathbb{M}^{1,3}$, which in our notation corresponds to $\mathbb{E}_a^{1,3}$, with $a = -1$.

One can enumerate the orbits $\mathbb{M}^{k,l}$ of the action of the conformal group $SU(2,2) = \left\{g \in GL(4, \mathbb{C}) : g^\dagger \Phi g = \Phi\right\}$ on $\mathbb{M}$ by signatures $sign\Phi \mid_z =: (k, l)$ of the restrictions $\Phi \mid_z$ of twistror forms Φ to subspace $z \in \mathbb{M}$. The orbit $\mathbb{M}^{00}$ is identified with $\overline{\mathbb{M}}^{1,3}$.

The case of massless particle with non-zero helicity

The phase space of such particle is the manifold the positive defined projective twistors

$$\mathbb{PT}^+ := \left\{ [v] \in \mathbb{CP}(3) : v^\dagger \Phi v > 0 \right\},$$

where $[v] := \mathbb{C}v$ is a one–dimensional complex subspace of $\mathbb{T}$ spanded by $0 \neq v \in \mathbb{T}$. The $SU(2,2)$– invariant symplectic form ω^+_α on $\mathbb{PT}^+$ is the Kähler form

$$\omega^+_\alpha := i\alpha\partial\overline{\partial} \log v^\dagger \Phi v.$$

In the subsequent we will use spinor coordinates $(\eta, \xi) \in \mathbb{C}^2 \times \mathbb{C}^2$ for the twistor $v = (\eta, \xi) \in \mathbb{T}$ defined by the decomposition $\mathbb{T} = \infty \oplus o$

$$\infty := \left\{ \begin{pmatrix} \eta \\ 0 \end{pmatrix} \in \mathbb{T} : \eta \in \mathbb{C}^2 \right\} \in \mathbb{M}^{00},$$

$$o := \left\{ \begin{pmatrix} 0 \\ \xi \end{pmatrix} \in \mathbb{T} : \xi \in \mathbb{C}^2 \right\} \in \mathbb{M}^{00}.$$

After passing to the homogeneous coordinates $\zeta_1 := \frac{\eta_1}{\xi_2}$, $\zeta_2 := \frac{\eta_2}{\xi_2}$, $\zeta := \frac{\xi_1}{\xi_2}$, where $\xi_2 \neq 0$, we obtain the coordinate representation for symplectic form:

$$\omega_\alpha^+ = \frac{-i\alpha}{\Delta^2}\left(d\overline{\zeta}_1 d\overline{\zeta}_2 d\overline{\zeta}\right) \wedge \begin{pmatrix} -\zeta\overline{\zeta} & -\zeta\overline{\zeta}\zeta_1 + \zeta_2 - \overline{\zeta}_2 \\ -\overline{\zeta} & -1 & \overline{\zeta}_1 \\ \zeta\overline{\zeta}_1 - \zeta_2 + \overline{\zeta}_2\,\zeta_1 & -\zeta_1\overline{\zeta}_1 \end{pmatrix} \begin{pmatrix} d\zeta_1 \\ d\zeta_2 \\ d\zeta \end{pmatrix},$$

where

$$\Delta = v^\dagger \Phi v = i\left(\overline{\zeta}\zeta_1 - \zeta\overline{\zeta}_1 + \zeta_2 - \overline{\zeta}_2\right).$$

The momentum map $\mathcal{J}_\alpha^+ : (\mathbb{PT}^+, \omega_\alpha^+) \longrightarrow su(2,2)^*$ for the symplectic manifold $(\mathbb{PT}^+, \omega_\alpha^+)$ is the following one

$$\mathcal{J}_\alpha^+([v]) = i\alpha \left(\frac{1}{4}\mathbf{1} - \frac{vv^\dagger \Phi}{\Delta} \right),$$

and it leads to (in the $(\zeta_1, \zeta_2, \zeta)$ – coordinate representation) the formulas for four-momentum and relativistic angular momentum

$$P_0 = -\frac{i\alpha}{2\Delta}(\zeta\bar{\zeta} + 1),$$
$$P_1 = -\frac{i\alpha}{2\Delta}(\zeta + \bar{\zeta}),$$
$$P_2 = -\frac{\alpha}{2\Delta}(\bar{\zeta} - \zeta),$$
$$P_3 = -\frac{i\alpha}{2\Delta}(\zeta\bar{\zeta} - 1),$$

$$L_1 = \frac{\alpha}{2\Delta}(\zeta_2\overline{\zeta} + \overline{\zeta}_2\zeta + \zeta_1 + \overline{\zeta}_1),$$

$$L_2 = \frac{i\alpha}{2\Delta}(-\zeta_2\overline{\zeta} + \overline{\zeta}_2\zeta + \zeta_1 - \overline{\zeta}_1),$$

$$L_3 = \frac{\alpha}{2\Delta}(\zeta_1\overline{\zeta} + \overline{\zeta}_1\zeta - \zeta_2 - \overline{\zeta}_2),$$

$$J_1 = -\frac{i\alpha}{2\Delta}(\zeta_2\overline{\zeta} - \overline{\zeta}_2\zeta + \zeta_1 - \overline{\zeta}_1),$$

$$J_2 = -\frac{\alpha}{2\Delta}(\zeta_2\overline{\zeta} + \overline{\zeta}_2\zeta - \zeta_1 - \overline{\zeta}_1),$$

$$J_3 = -\frac{i\alpha}{2\Delta}(\zeta_1\overline{\zeta} - \overline{\zeta}_1\zeta - \zeta_2 + \overline{\zeta}_2),$$

$$W_0 = \frac{i\alpha^2}{4\Delta}(\zeta\overline{\zeta} + 1),$$

$$W_1 = -\frac{i\alpha^2}{4\Delta}(\zeta + \overline{\zeta}),$$

$$W_2 = -\frac{\alpha^2}{4\Delta}(\overline{\zeta} - \zeta), \qquad W_3 = -\frac{i\alpha^2}{4\Delta}(\zeta\overline{\zeta} - 1).$$

The momentum map $\mathcal{J}_\alpha^+$ is a Poisson map from the symplectic manifold $(\mathbb{PT}^+, \omega_\alpha^+)$ into Lie–Poisson space $(\mathcal{L}_+, \{\cdot,\cdot\}_\alpha)$, i.e. for $f, g \in C^\infty(\mathcal{L}_+)$ we have

$$\{f,g\}_a \circ \mathcal{J}_\alpha^+ = \{f \circ \mathcal{J}_\alpha^+, g \circ \mathcal{J}_\alpha^+\}_{\alpha,+},$$

where Poisson bracket $\{\cdot,\cdot\}_{\alpha,+}$ is defined by the symplectic form ω_α^+.

In the coordinates $(\zeta_1, \zeta_2, \zeta)$ it takes the form

$$\{F,G\}_{\alpha,+} = -\frac{\Delta}{\alpha}\left(\zeta_1\left(\frac{\partial F}{\partial\bar{\zeta}_2}\frac{\partial G}{\partial\zeta_1} - \frac{\partial F}{\partial\zeta_1}\frac{\partial G}{\partial\bar{\zeta}_2}\right) - \bar{\zeta}_1\left(\frac{\partial F}{\partial\bar{\zeta}_1}\frac{\partial G}{\partial\zeta_2} - \frac{\partial F}{\partial\zeta_2}\frac{\partial G}{\partial\bar{\zeta}_1}\right)\right.$$
$$+ (\zeta_2 - \bar{\zeta}_2)\left(\frac{\partial F}{\partial\bar{\zeta}_2}\frac{\partial G}{\partial\zeta_2} - \frac{\partial F}{\partial\zeta_2}\frac{\partial G}{\partial\bar{\zeta}_2}\right) + \frac{\partial F}{\partial\bar{\zeta}}\frac{\partial G}{\partial\zeta_1} - \frac{\partial F}{\partial\zeta_1}\frac{\partial G}{\partial\bar{\zeta}} +$$
$$+ \frac{\partial F}{\partial\zeta}\frac{\partial G}{\partial\bar{\zeta}_1} - \frac{\partial F}{\partial\bar{\zeta}_1}\frac{\partial G}{\partial\zeta} + \bar{\zeta}\left(\frac{\partial F}{\partial\zeta_2}\frac{\partial G}{\partial\bar{\zeta}} - \frac{\partial F}{\partial\bar{\zeta}}\frac{\partial G}{\partial\zeta_2}\right) +$$
$$\left. + \zeta\left(\frac{\partial F}{\partial\bar{\zeta}_2}\frac{\partial G}{\partial\zeta} - \frac{\partial F}{\partial\zeta}\frac{\partial G}{\partial\bar{\zeta}_2}\right)\right),$$

for $F, G \in C^\infty(\mathbb{PT}^+)$.

The four-momentum $P_\mu([v])$ and Pauli–Lubansky vector $W^\mu([v])$ satisfies the relationships

$$\eta^a_{\mu\nu} P^\mu([v]) P^\nu([v]) = 0 \quad \text{and} \quad W^\mu([v]) = \frac{\alpha}{2} P^\mu([v]).$$

So, one can pull back the solution $\left(\vec{P}(t), \vec{L}(t)\right)$ of Hamilton equations on the symplectic manifold $\left(\mathbb{PT}^+, \omega_\alpha^+\right)$ by the map

$$\zeta_1 = \frac{L_2 - J_1 + i(L_1 + J_2)}{2(P_3 - P_0)},$$

$$\zeta_2 = \frac{(P_1 + iP_2)(L_2 - J_1 + i(L_1 - J_2)) - (P_3 - P_0)(J_3 - iL_3)}{-2(P_3 - P_0)^2},$$

$$\zeta = \frac{P_1 - iP_2}{P_0 - P_3}.$$

So, if $P_0(t)$, $\vec{P}(t)$, $\vec{L}(t)$, $\vec{J}(t)$ satisfy equations then $\zeta_1(t),\zeta_2(t),\zeta(t))$ are the solution of Hamilton equations

$$\frac{d}{dt}\zeta_1(t)=\{\zeta_1,h\circ\mathcal{J}^+_\alpha\}_{\alpha,+}=-\frac{\alpha}{4\Delta}(b-a)(\zeta\bar{\zeta}+1)\left(\left(c-\frac{d\alpha^2}{4}\right)\zeta-\right.$$
$$-\frac{\alpha^2}{4\Delta^2}d(b-a)\left(-6\zeta_2\bar{\zeta}_2\zeta^2\bar{\zeta}+4\zeta_1\zeta_2\zeta\bar{\zeta}-4\zeta_1\bar{\zeta}_1\zeta^2\bar{\zeta}+4\zeta_1\bar{\zeta}_2\zeta\bar{\zeta}+\right.$$
$$+3\zeta_1^2\zeta\bar{\zeta}^2+2\bar{\zeta}_1\bar{\zeta}_2\zeta^2+2\bar{\zeta}_1\zeta_2\zeta^2-4\zeta_2\bar{\zeta}_2\zeta+\bar{\zeta}_1^2\zeta^3+\zeta_2^2\zeta+\bar{\zeta}_2^2\zeta+$$
$$\left.\left.+2\zeta_1^2\bar{\zeta}+2\zeta_1\bar{\zeta}_2+2\zeta_1\zeta_2\zeta^2\bar{\zeta}^2+2\zeta_1\bar{\zeta}_1\zeta^2\bar{\zeta}\right)\right),$$
$$\frac{d}{dt}\zeta(t)=\{\zeta,h\circ\mathcal{J}^+_\alpha\}_{\alpha,+}=-\frac{\alpha^3}{8\Delta^3}d(b-a)^2(\zeta\bar{\zeta}+1)^3(\zeta_2\zeta-\zeta_1),$$

defined on $\left(\mathbb{PT}^+,\omega^+_\alpha\right)$ by the Hamiltonian $h\circ\mathcal{J}^+_\alpha$.

The case of a massive particle with the spin

In order to describe the phase space of a massive particle with the spin $s \neq 0$ let us consider twistor flag space

$$\mathbb{F} := \{([v], z) \in \mathbb{PT} \times \mathbb{M} : [v] \subset z\}.$$

Similarly to the case of Grassmannian $\mathbb{M}$ we will enumerate the orbits $\mathbb{F}^{k,lm}$ of the natural action of $SU(2,2)$ on $\mathbb{F}$ by the signatures $k = sign\ \Phi\ |_{[v]}$, $lm = sign\ \Phi\ |_z$ of the restrictions of twistor form to the flag $[v] \subset z$. We restrict our interest to the orbit $\mathbb{F}^{+,++}$ consisting of the positive flags.

One has the following double fibration

$$\begin{array}{ccc} & \mathbb{F}^{+,++} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{PT}^{+} & & \mathbb{M}^{++} \end{array}$$

of $\mathbb{F}^{+,++}$ over $\mathbb{PT}^{+}$ and $\mathbb{M}^{++}$. We show now that $\mathbb{M}^{++}$ is the phase space of massive spinless particle and $\mathbb{F}^{+,++}$ is the phase space of a massive particle with non-zero spin.

For these reasons let us pass to the coordinate description of $\mathbb{M}^{++}$ and $\mathbb{F}^{+,++}$ consistent with the decomposition $\mathbb{T} = \infty \oplus o$. We have

$$v = \begin{pmatrix} Z\xi \\ \xi \end{pmatrix} \quad \text{and} \quad z = \left\{ \begin{pmatrix} Z\xi \\ \xi \end{pmatrix} : \ \xi \in \mathbb{C}^2 \right\}$$

for $[v] \subset z$, where $Z \in Mat_{2\times 2}(\mathbb{C})$. The flag $[v] \subset z$ belongs to $\mathbb{F}^{+,++}$ iff the "imaginary" part of $Z = X + iY$, $X^\dagger = X$ and $Y^\dagger = Y$, is positive definite, i.e.

$$\det Y > 0 \quad \text{and} \quad \mathrm{Tr}\, Y > 0.$$

Let us take the product $\mathbb{PT}^+ \times \mathbb{PT}^+$ of the one–twistor phase spaces with the symplectic form

$$\omega^{12} = \pi_1^* \omega_{\alpha_1}^+ + \pi_2^* \omega_{\alpha_2}^+,$$

where $\pi_i : \mathbb{PT}^+ \times \mathbb{PT}^+ \longrightarrow \mathbb{PT}^+$ is the projection on the i–th component of the product. The symplectic form ω^{12} is invariant with respect to the natural action of $SU(2,2)$ on $\mathbb{PT}^+ \times \mathbb{PT}^+$ and the momentum map $\mathcal{J}^{1,2} : \mathbb{PT}^+ \times \mathbb{PT}^+ \longrightarrow su(2,2)^*$ for $(\mathbb{PT}^+ \times \mathbb{PT}^+, \omega^{12})$ is given by

$$\mathcal{J}^{1,2} = \mathcal{J}_\alpha^+ \circ \pi_1 + \mathcal{J}_\alpha^+ \circ \pi_2.$$

The function $s^2 : \mathbb{PT} \times \mathbb{PT} \longrightarrow \mathbb{R}$ defined by

$$s^2([v_1],[v_2]) := \left(\frac{\alpha_1 - \alpha_2}{4}\right)^2 + \frac{\alpha_1\alpha_2}{4}\frac{|v_1^\dagger \Phi v_2|^2}{v_1^\dagger \Phi v_1 v_2^\dagger \Phi v_2}$$

in an invariant of the conformal group $SU(2,2)$. The projective twistors are orthogonal $[v_1]\perp[v_2]$ with respect to the twistor form Φ iff

$$s^2([v_1],[v_2]) := \left(\frac{\alpha_1 - \alpha_2}{4}\right)^2 .$$

Any flag $[v] \subset z$ one can identify with the pair of twistors $([v_1], [v_2]) \in \mathbb{PT}^+ \times \mathbb{PT}^+$. Namely, one puts $z = span\{[v_1], [v_2]\} \in \mathbb{M}$ and $[v] = [v_1]$. Reducing symplectic form $\omega^{1,2}$ to the level submanifold of $\mathbb{PT}^+ \times \mathbb{PT}^+$ we obtain symplectic form (Kähler form) $\omega_{s,\delta}$ on $\mathbb{F}^{+,++}$ which in the coordinates $([\xi], Z) \in \mathbb{CP}(1) \times Mat_{2\times 2}(\mathbb{C})$ is given by

$$\omega_{s,\delta} = i\partial\overline{\partial}\log\left[\left(\det\left(Z - Z^\dagger\right)\right)^{s+2\delta}\left(\eta^\dagger\left(Z - Z^\dagger\right)\eta\right)^{4s}\right],$$

where $s := \frac{\alpha_1 - \alpha_2}{4}$ and $\delta := -\frac{\alpha_1 + \alpha_2}{4}$. The symplectic form rewrited in the variables P^μ, W^μ, X^μ is the Souriau symplectic form.

The reduced momentum map $\mathcal{J}_{s,\delta} : \mathbb{F}^{+,++} \longrightarrow su(2,2)^*$ is of the form

$$\mathcal{J}_{s,\delta}([\xi], Z) = \begin{pmatrix} ZP - i\delta\sigma_0 & -ZPZ^\dagger \\ P & -PZ^\dagger - i\delta\sigma_0 \end{pmatrix},$$

where

$$P = -\frac{i\alpha_1}{\xi^\dagger(Z - Z^\dagger)\xi}\xi\xi^\dagger - \frac{i\alpha_2}{det(Z - Z^\dagger)\xi^\dagger(Z - Z^\dagger)\xi}(\widetilde{Z} - \widetilde{Z}^\dagger)\widetilde{\xi\xi^\dagger}(\widetilde{Z} - \widetilde{Z}^\dagger).$$

For (2×2)–matrix calculus it is useful to introduce the following operation on $B \in Mat_{2\times 2}(\mathbb{C})$:

$$\widetilde{B} := \sigma_2 B^\intercal \sigma_2.$$

We define the Pauli–Lubansky four–vector $W = W^\mu \sigma_\mu$ in the following way

$$M\widetilde{P} =: R - iW,$$

where $R^\dagger = R$ and $W^\dagger = W$.

We obtain

$$M = ZP - \frac{1}{2}\mathrm{Tr}(ZP)\sigma_0.$$

and we also express M in the coordinates $([\xi], Z = X + iY)$:

$$M = -\frac{i\alpha_1}{\xi^\dagger(Z - Z^\dagger)\xi} Z\xi\xi^\dagger + \frac{i\alpha_1}{2\xi^\dagger(Z - Z^\dagger)\xi}\xi^\dagger Z\xi\sigma_0 -$$

$$-\frac{i\alpha_2}{det(Z - Z^\dagger)\xi^\dagger(Z - Z^\dagger)\xi}(\widetilde{Z} - \widetilde{Z}^\dagger)\widetilde{\xi\xi^\dagger}(\widetilde{Z} - \widetilde{Z}^\dagger) +$$

$$+\frac{i\alpha_2}{2det(Z - Z^\dagger)\xi^\dagger(Z - Z^\dagger)\xi}\overline{\xi^\dagger(Z - Z^\dagger)\sigma_2\overline{Z}\sigma_2(Z - Z^\dagger)\xi}\sigma_0.$$

We find that

$$W = -\frac{i\delta\alpha_1}{\xi^\dagger(Z-Z^\dagger)\xi}\widetilde{\xi\xi^\dagger} - \frac{i\alpha_1\alpha_2}{2det(Z-Z^\dagger)}(Z-Z^\dagger) -$$

$$-\frac{i\delta\alpha_2}{det(Z-Z^\dagger)\xi^\dagger(Z-Z^\dagger)\xi}(Z-Z^\dagger)\xi\xi^\dagger(Z-Z^\dagger).$$

and

$$\begin{aligned} \mathrm{Tr}\ PW &= 0, \\ \det W &= -s^2 \det P, \\ \mathrm{Tr}\ PY &= 2\delta. \end{aligned}$$

Using vector notation for M we find follows

$$\vec{L} = X_0\vec{P} + P_0\vec{X} - \vec{Y} \times \vec{P},$$
$$\vec{J} = Y_0\vec{P} + P_0\vec{Y} + \vec{X} \times \vec{P},$$

$$Y_0 = -\frac{1}{\det P}\left(W_0 - \delta P_0\right),$$
$$\vec{Y} = -\frac{1}{\det P}\left(\vec{W} + \delta\vec{P}\right),$$
$$\vec{X} = \frac{1}{\det P}\vec{J} \times \vec{P} + \frac{P_0}{\det P}\vec{L} - \frac{1}{P_0 \det P}\left(\left(\vec{P} \cdot \vec{L}\right) + \det P X_0\right)\vec{P}.$$

The formula given above allows us to obtain the time evolution $Y_0 = Y_0(t)$, $\vec{Y} = \vec{Y}(t)$ and $\vec{X} = \vec{X}(t)$ described by the Hamiltonian. For this reason we only need to assume that the evolution parameter t appearing in the Hamilton equations is the time related to the space–time coordinate X_0 by $X_0 = ct$, where c is the light velocity. We have

$$\vec{X} = -\frac{1}{(mc)^2}\vec{J} \times \left(\vec{P}(t) - \frac{P_0}{W_0(t)}\vec{W}(t)\right) +$$

$$+\left(ct + \frac{1}{(a-b)dP_0}\frac{d}{dt}\ln W_0(t) + (mc)^2\xi(t)\right)\frac{\vec{P}(t)}{P_0},$$

where m is the relativistic particle mass defined by $-(mc)^2 = c_1$, cP_0 and $\vec{J}$ are its energy and angular momentum, being integral of motions in the case under consideration.

References:

- A. Odzijewicz, A. Dobrogowska, *Integrable Hamiltonian systems related to the Hilbert-Schmidt ideal*, J. Geom. Phys., **61**, (2011), no. 8, 1426-1445.
- A. Dobrogowska, A. Odzijewicz, *Integrable relativistic systems given by Hamiltonians with momentum-spin-orbit coupling*, Regul. Chaotic Dyn., **17**, (2012), no. 6, 492-505