XV International conference on
Geometry Integrability and Quantization
7-12 June 2013, Varna

Soliton Equations and Lax operators. Effects of boundary conditions and reductions.

V. S. Gerdjikov
Institute for Nuclear Research and Nuclear Energy,
Bulgarian Academy of Sciences,
72 Tsarigradsko chaussee, 1784 Sofia, Bulgaria

Based on:

- V. S. Gerdjikov, N. A. Kostov, T. I. Valchev. *On multicomponent NLS Equations with Constant Boundary Conditions*. Theor. Math. Phys. **159**, 786-794 (2009).

- V. S. Gerdjikov. On Reductions of Soliton Solutions of multi-component NLS models and Spinor Bose-Einstein condensates. AIP **CP 1186**, 15-27 (2009). **arXiv: 1001.0166 [nlin.SI]**

- V. S. Gerdjikov, N. A. Kostov and T. I. Valchev. *Bose-Einstein condensates with $F = 1$ and $F = 2$. Reductions and soliton interactions of multi-component NLS models.* Proceedings of SPIE Volume: 7501, 7501W (2009). **arXiv: 1001.0168 [nlin.SI]**

- V. S. Gerdjikov. *Bose-Einstein Condensates and spectral properties of multicomponent nonlinear Schrödinger equations.* Discrete and Continuous Dynamical Systems B (In press) **arXiv: 1001.0164 [nlin.SI]**

- V. S. Gerdjikov, G. G. Grahovski. Multi-component NLS Models on Symmetric Spaces: Spectral Properties versus Representations Theory. Submitted to SIGMA, January 2010.
- V. S. Gerdjikov. Riemann-Hilbert Problems with canonical normalization and families of commuting operators. Pliska Stud. Math. Bulgar. **21**, 201–216 (2012). **arXiv:1204.2928v1 [nlin.SI]**.
- V. S. Gerdjikov. On new types of integrable 4-wave interactions. AIP Conf. proc. **1487** pp. 272-279; (2012). **arXiv:1302.1116.**
- V. S. Gerdjikov. Derivative Nonlinear Schrödinger Equations with $\mathbb{Z}_N$ and $\mathbb{D}_N$–Reductions. Romanian Journal of Physics, **58**, Nos. 5-6, (2013) (In press).

Plan

- Spectral properties of L may change when going from one representation of $\mathfrak{g}$ to another ($\lim_{x\to\pm\infty} Q(x) = 0$).
- Spectral properties of L for potentials with constant boundary conditions, i.e. $\lim_{x\to\pm\infty} Q(x) = Q_{\pm}$.
- Spectral properties of L possessing $\mathbb{Z}_h$ as reduction groups.

Multi-component (matrix) NLS equations and the homogeneous and symmetric spaces – Fordy, Kulish (1983)

Lax operator:

$$L\psi(x,t,\lambda) \equiv i\frac{d\psi}{dx} + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0, \qquad (1)$$

where $J \in \mathfrak{h} \subset \mathfrak{g}$ and $Q(x,t) \equiv [J, \widetilde{Q}(x,t)] \in \mathfrak{g}/\mathfrak{h}$.

$Q(x,t)$ belongs to the co-adjoint orbit $\mathcal{M}_J$ of $\mathfrak{g}$ passing through J.

MNLS type models, related to **BD.I** symmetric spaces:

$$\begin{aligned} L\psi(x,t,\lambda) &\equiv i\partial_x\psi + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \\ M\psi(x,t,\lambda) &\equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \\ V_1(x,t) &= Q(x,t), \qquad V_0(x,t) = i\,\mathrm{ad}_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}_J^{-1}Q, Q(x,t)\right]. \end{aligned}$$

In the typical representation of $\mathfrak{g} \simeq so(n+2)$:

$$Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \qquad J = \mathrm{diag}(1, 0, \ldots 0, -1). \qquad (2)$$

For $n=2r-1$

$$\vec{q}=(q_1,\dots,q_r,q_0,q_{\bar{r}},\dots,q_{\bar{1}})^T, \quad \vec{p}=(p_1,\dots,p_r,p_0,p_{\bar{r}},\dots,p_{\bar{1}})^T),$$

while the matrix $s_0=S_0^{(n)}$ enters in the definition of $so(n)$: $X\in so(n)$ if $X+S_0^{(n)}X^TS_0^{(n)}=0$

$$S_0^{(n)}=\sum_{s=1}^{n+1}(-1)^{s+1}E_{s,n+1-s}^{(n)}. \tag{3}$$

J is dual to $e_1\in\mathbb{E}^r$ and allows us to introduce a grading: $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$

$$[X_1,X_2]\in\mathfrak{g}_0, \qquad [X_1,Y_1]\in\mathfrak{g}_1, \qquad [Y_1,Y_2]\in\mathfrak{g}_0, \tag{4}$$

for any $X_1,X_2\in\mathfrak{g}_0$ and $Y_1,Y_2\in\mathfrak{g}_1$.

The grading splits $\Delta^+=\Delta_0^+\cup\Delta_1^+$
$(\alpha,e_1)=0$; the roots in $\beta\in\Delta_1^+$ satisfy $(\beta,e_1)=1$.

The Lax pair can be considered in any representation of $so(n)$:

$$Q(x,t)=\sum_{\alpha\in\Delta_1^+}\left(q_\alpha(x,t)E_\alpha+p_\alpha(x,t)E_{-\alpha}\right). \tag{5}$$

The generic MNLS type equations related to **BD.I**. acquire the form

$$\begin{aligned} i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q},\vec{p})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{p} = 0, \\ i\vec{p}_t - \vec{p}_{xx} - 2(\vec{q},\vec{p})\vec{p} + (\vec{p}, s_0\vec{p})s_0\vec{q} = 0, \end{aligned} \tag{6}$$

Canonical reduction: $\vec{p} = \epsilon\vec{q}^*$, $\epsilon = \pm 1$ and Hamiltonian:

$$H_{\rm MNLS} = \int_{-\infty}^{\infty} dx \left((\partial_x\vec{q}, \partial_x\vec{q^*}) - \epsilon(\vec{q}, \vec{q^*})^2 + \epsilon(\vec{q}, s_0\vec{q})(\vec{q^*}, s_0\vec{q^*}) \right), \tag{7}$$

0.1 Direct Scattering Problem for L

Jost solutions:

$$\lim_{x\to-\infty} \phi(x,t,\lambda)e^{i\lambda Jx} = \leavevmode\hbox{1\kern-0.3em 1}, \qquad \lim_{x\to\infty} \psi(x,t,\lambda)e^{i\lambda Jx} = \leavevmode\hbox{1\kern-0.3em 1} \tag{8}$$

The scattering matrix

$$T(\lambda,t) \equiv \psi^{-1}\phi(x,t,\lambda) \in SO(n+2).$$

has the following block-matrix structure

$$T(\lambda,t)=\begin{pmatrix} m_1^+ & -\vec{b}^{-T} & c_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{B}^- \\ c_1^+ & \vec{B}^{+T}s_0 & m_1^- \end{pmatrix}, \qquad \hat{T}(\lambda,t)=\begin{pmatrix} m_1^- & \vec{b}^{-T} & c_1^- \\ -\vec{B}^+ & \hat{\mathbf{T}}_{22} & s_0\vec{b}^- \\ c_1^+ & -\vec{b}^{+T}s_0 & m_1^+ \end{pmatrix}, \tag{9}$$

Here $\vec{b}^{\pm}(\lambda,t)$ and $\vec{B}^{\pm}(\lambda,t)$ are n-component vectors, $\mathbf{T}_{22}(\lambda)$ and $\boldsymbol{m}^{\pm}(\lambda)$ are $n\times n$ block matrices, and $m_1^{\pm}(\lambda)$, $c_1^{\pm}(\lambda)$ are scalar functions. Such parametrization is compatible with the generalized Gauss decompositions of $T(\lambda)$.

Generalized Gauss factors of $T(\lambda)$ as follows:

$$T(\lambda,t)=T_J^-D_J^+\hat{S}_J^+=T_J^+D_J^-\hat{S}_J^-, \tag{10}$$

$$T_J^-=e^{(\vec{\rho}^+,\vec{E}^-)}=\begin{pmatrix} 1 & 0 & 0 \\ \vec{\rho}^+ & \mathbb{1} & 0 \\ c_1^{'',+} & \vec{\rho}^{+,T}s_0 & 1 \end{pmatrix}, \quad T_J^+=e^{(-\vec{\rho}^-,\vec{E}^+)}=\begin{pmatrix} 1 & -\vec{\rho}^{-,T} & c_1^{'',-} \\ 0 & \mathbb{1} & -s_0\vec{\rho}^- \\ 0 & 0 & 1 \end{pmatrix},$$

$$S_J^+=e^{(\vec{\tau}^+,\vec{E}^+)}=\begin{pmatrix} 1 & \vec{\tau}^{+,T} & c_1^{',-} \\ 0 & \mathbb{1} & s_0\vec{\tau}^+ \\ 0 & 0 & 1 \end{pmatrix}, \quad S_J^-=e^{(-\vec{\tau}^-,\vec{E}^-)}=\begin{pmatrix} 1 & 0 & 0 \\ -\vec{\tau}^- & \mathbb{1} & 0 \\ c_1^{',+} & -\vec{\tau}^{-,T}s_0 & 1 \end{pmatrix},$$

$$D_J^+ = \begin{pmatrix} m_1^+ & 0 & 0 \\ 0 & \mathbf{m}_2^+ & 0 \\ 0 & 0 & 1/m_1^+ \end{pmatrix}, \qquad D_J^- = \begin{pmatrix} 1/m_1^- & 0 & 0 \\ 0 & \mathbf{m}_2^- & 0 \\ 0 & 0 & m_1^- \end{pmatrix}, \tag{11}$$

$$c_1^{'',\pm} = \frac{1}{2}(\vec{\rho}^{\pm,T} s_0 \vec{\rho}^{\pm}), \qquad c_1^{',\pm} = \frac{1}{2}(\vec{\tau}^{\mp,T} s_0 \vec{\tau}^{\mp}) \tag{12}$$

where

$$\vec{\rho}^- = \frac{\vec{B}^-}{m_1^-}, \qquad \vec{\tau}^- = \frac{\vec{B}^+}{m_1^-}, \qquad \vec{\rho}^+ = \frac{\vec{b}^+}{m_1^+}, \qquad \vec{\tau}^+ = \frac{\vec{b}^-}{m_1^+},$$

If $Q(x,t)$ evolves according to MNLS then the scattering matrix and its elements satisfy the following linear evolution equations

$$i\frac{d\vec{b}^\pm}{dt} \pm \lambda^2 \vec{b}^\pm(t,\lambda) = 0, \qquad i\frac{d\vec{B}^\pm}{dt} \pm \lambda^2 \vec{B}^\pm(t,\lambda) = 0, \qquad i\frac{dm_1^\pm}{dt} = 0, \qquad i\frac{d\mathbf{m}_2^\pm}{dt} = 0, \tag{13}$$

so $D^\pm(\lambda)$ are generating functionals of the integrals of motion.

0.2 Riemann-Hilbert Problem

The ISP reduces a Riemann-Hilbert problem (RHP) for the fundamental analytic solution (FAS)

$$\chi^{\pm}(x,t,\lambda) = \phi(x,t,\lambda)S_J^{\pm}(t,\lambda) = \psi(x,t,\lambda)T_J^{\mp}(t,\lambda)D_J^{\pm}(\lambda). \qquad (14)$$

i.e.

$$\xi^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda)e^{i\lambda Jx}$$

are analytic functions of λ for $\lambda \in \mathbb{C}_{\pm}$.

The FAS for real λ are linearly related

$$\chi^{+}(x,t,\lambda) = \chi^{-}(x,t,\lambda)G_{0,J}(\lambda,t), \qquad G_{0,J}(\lambda,t) = \hat{S}_J^{-}(\lambda,t)S_J^{+}(\lambda,t). \qquad (15)$$

Equivalently for the FAS $\xi^{\pm}(x,t,\lambda) = \chi^{\pm}(x,t,\lambda)e^{i\lambda Jx}$ which satisfy the equation:

$$i\frac{d\xi^{\pm}}{dx} + Q(x)\xi^{\pm}(x,\lambda) - \lambda[J,\xi^{\pm}(x,\lambda)] = 0, \qquad \lim_{\lambda\to\infty} \xi^{\pm}(x,t,\lambda) = \mathbb{1}. \qquad (16)$$

Then these FAS satisfy

$$\xi^{+}(x,t,\lambda) = \xi^{-}(x,t,\lambda) G_J(x,\lambda,t), \qquad G_J(x,\lambda,t) = e^{-i\lambda J x} G_{0,J}^{-}(\lambda,t) e^{i\lambda J x}. \tag{17}$$

Given the solutions $\xi^{\pm}(x,t,\lambda)$ one recovers $Q(x,t)$ via the formula

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda \left(J - \xi^{\pm} J \widehat{\xi}^{\pm}(x,t,\lambda) \right) = [J, \xi_1(x)], \tag{18}$$

By $\xi_1(x)$ above we have denoted $\xi_1(x) = \lim_{\lambda\to\infty} \lambda(\xi(x,\lambda) - \mathbb{1})$.

1 Resolvent and spectral decompositions in the typical representation of $\mathfrak{g} \simeq B_r$

Theorem 1. *Let $Q(x)$ be a potential of L which falls off fast enough for $x \to \pm\infty$ and the corresponding RHP has a finite number of simple singularities at the points $\lambda_j^{\pm} \in \mathbb{C}_{\pm}$, i.e. $\chi^{\pm}(x,\lambda)$ have simple poles and zeroes at $\lambda_j^{\pm}$. Then*

1. *$R^{\pm}(x,y,\lambda)$ is an analytic function of λ for $\lambda \in \mathbb{C}_\pm$ having pole singularities at $\lambda_j^{\pm} \in \mathbb{C}_\pm$;*
2. *$R^{\pm}(x,y,\lambda)$ is a kernel of a bounded integral operator for $\operatorname{Im}\lambda \neq 0$;*
3. *$R(x,y,\lambda)$ is uniformly bounded function for $\lambda \in \mathbb{R}$ and provides a kernel of an unbounded integral operator;*
4. *$R^{\pm}(x,y,\lambda)$ satisfy the equation:*

$$L(\lambda)R^{\pm}(x,y,\lambda) = \Pi_1\delta(x-y), \qquad \Pi_1 = \operatorname{diag}(1,0,\ldots,0,1). \tag{19}$$

By definition,

- the continuous spectrum of L fills up the lines in the complex λ-plane for which $R(x,y,\lambda)$ a kernel of an unbounded integral operator;
- the discrete spectrum of L is located at the pole singularities of of $R(x,y,\lambda)$.

In our case J has n vanishing eigenvalues which makes the problem more difficult.

We can rewrite the Lax operator in the form:

$$
\begin{aligned}
i\frac{\partial \chi_1}{\partial x} + \vec{q}^T \vec{\chi}_0 = \lambda \chi_1, \\
i\frac{\partial \vec{\chi}_0}{\partial x} + \vec{q}^* \chi_1 + s_0 \vec{q} \chi_{-1} = 0, \\
i\frac{\partial \chi_{-1}}{\partial x} + \vec{q}^\dagger s_0 \vec{\chi}_0 = \lambda \chi_{-1},
\end{aligned} \tag{20}
$$

where we have split the eigenfunction $\chi(x,\lambda)$ of L into three according to the natural block-matrix structure compatible with J:

$$
\chi(x,\lambda) = \begin{pmatrix} \chi_1 \\ \vec{\chi}_0 \\ \chi_{-1} \end{pmatrix}.
$$

The equation for $\vec{\chi}_0$ can not be treated as eigenvalue equations; they can be formally integrated with:

$$
\vec{\chi}_0(x,\lambda) = \vec{\chi}_{0,\mathrm{as}} + i\int^x dy\, \left(\vec{q}^* \chi_1 + s_0 \vec{q} \chi_{-1}\right), \tag{21}
$$

which eventually casts the Lax operator into the following integro-differential system with non-degenerate λ dependence.

$$\begin{aligned} i\frac{\partial \chi_1}{\partial x} + i\vec{q}^T(x) \int^x dy\, \left(\vec{q}^{\,*}\chi_1 + s_0\vec{q}\chi_{-1}\right)(y,\lambda) = \lambda\chi_1, \\ i\frac{\partial \chi_{-1}}{\partial x} + i\vec{q}^{\,\dagger}(x)s_0 \int^x dy\, \left(\vec{q}^{\,*}\chi_1 + s_0\vec{q}\chi_{-1}\right)(y,\lambda) = -\lambda\chi_{-1}, \end{aligned} \tag{22}$$

Similarly we can treat the operator which is adjoint to L whose FAS $\hat{\chi}(x,\lambda)$ are the inverse to $\chi(x,\lambda)$, i.e. $\hat{\chi}(x,\lambda) = \chi^{-1}(x,\lambda)$. Splitting each of the rows of $\hat{\chi}(x,\lambda)$ into components as follows $\hat{\chi}(x,\lambda) = (\hat{\chi}_1, \hat{\vec{\chi}}_0, \hat{\chi}_{-1})$ we get:

$$\begin{aligned} i\frac{\partial \hat{\chi}_1}{\partial x} - (\hat{\vec{\chi}}_0, \vec{q}^{\,*}) - \lambda\hat{\chi}_1 = 0, \\ i\frac{\partial \hat{\vec{\chi}}_0}{\partial x} - \hat{\chi}_1\vec{q}^T - \hat{\chi}_{-1}\vec{q}^{\,\dagger}s_0 = 0, \\ i\frac{\partial \hat{\chi}_{-1}}{\partial x} - (\hat{\vec{\chi}}_0, s_0\vec{q}) - \lambda\hat{\chi}_{-1} = 0, \end{aligned} \tag{23}$$

Again the equation for $\hat{\vec{\chi}}_0$ can be formally integrated with:

$$\hat{\vec{\chi}}_0(x,\lambda) = \hat{\vec{\chi}}_{0,\mathrm{as}} + i\int^x dy\, \left(\hat{\chi}_1(y,\lambda)\vec{q}^{\,T}(y) + \hat{\chi}_{-1}(y,\lambda)\vec{q}^{\,\dagger}(y)s_0\right), \qquad (24)$$

Now we get the following integro-differential system with non-degenerate λ dependence.

$$i\frac{\partial\hat{\chi}_1}{\partial x} - i\int^x dy\, \left(\hat{\chi}_1(y,\lambda)(\vec{q}^{\,T}(y),\vec{q}^{\,*}(x)) + \hat{\chi}_{-1}(y,\lambda)(\vec{q}^{\,\dagger}(y)s_0\vec{q}^{\,*}(x))\right) + \lambda\hat{\chi}_1 = 0,$$
$$i\frac{\partial\hat{\chi}_{-1}}{\partial x} - i\int^x dy\, \left(\hat{\chi}_1(y,\lambda)(\vec{q}^{\,T}(y)s_0\vec{q}(x)) + \hat{\chi}_{-1}(y,\lambda)(\vec{q}^{\,\dagger}(y),\vec{q}(x))\right) - \lambda\hat{\chi}_{-1} = 0, \qquad (25)$$

The kernel $R(x,y,\lambda)$ of the resolvent is given by:

$$R(x,y,\lambda) = \begin{cases} R^+(x,y,\lambda) & \text{for } \lambda\in\mathbb{C}^+, \\ R^-(x,y,\lambda) & \text{for } \lambda\in\mathbb{C}^-, \end{cases} \qquad (26)$$

where

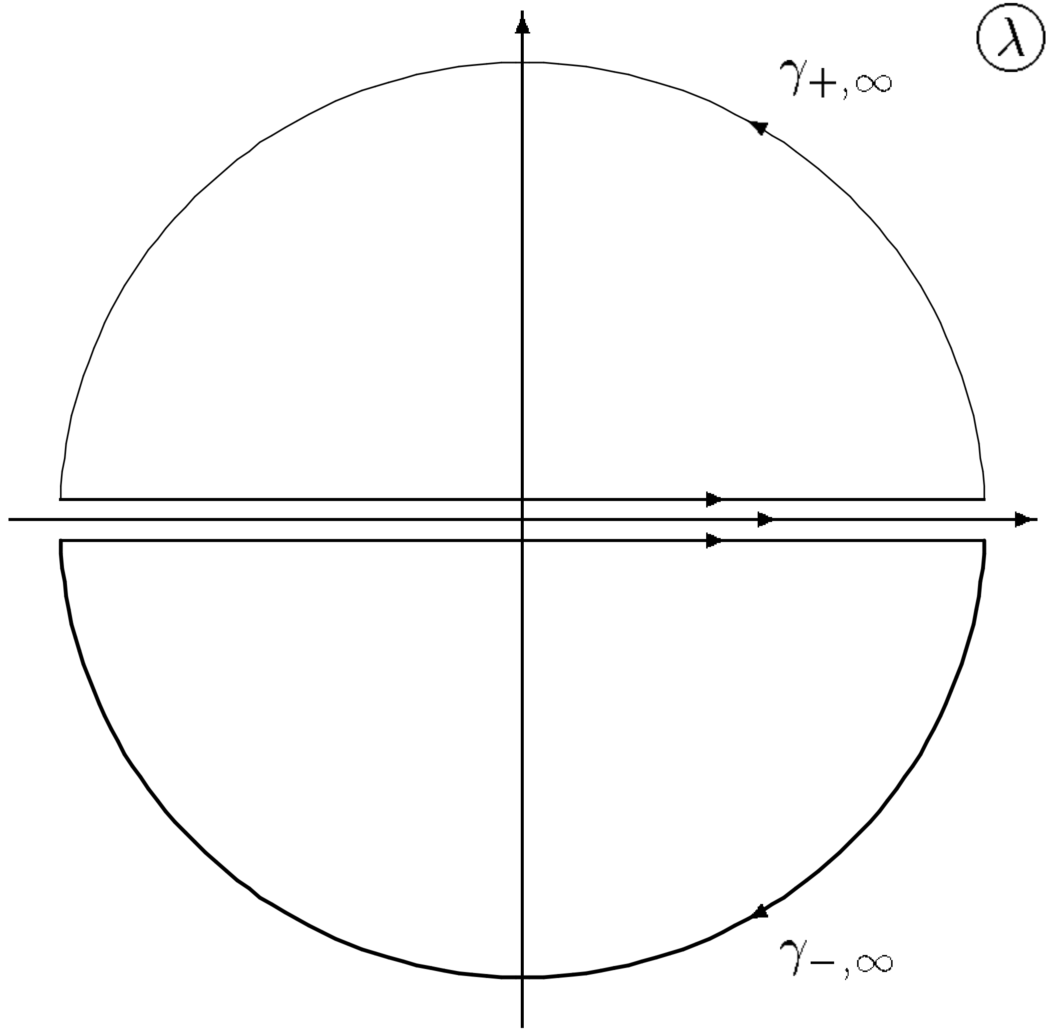
$$R^\pm(x,y,\lambda) = \pm i\chi^\pm(x,\lambda)\Theta^\pm(x-y)\hat{\chi}^\pm(y,\lambda), \qquad (27)$$

$$\Theta^{\pm}(z) = \theta(\mp z)E_{11} - \theta(\pm z)(\mathbb{1} - E_{11}),$$

The completeness relation for the eigenfunctions of L is derived by contour integration method

$$\mathcal{J}'(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} d\lambda \Pi_1 R^+(x,y,\lambda) - \frac{1}{2\pi i}\oint_{\gamma_-} d\lambda \Pi_1 R^-(x,y,\lambda), \quad (28)$$

where $\Pi_1 = E_{11} + E_{n+2,n+2}$.



Фигура 1: The contours $\gamma_{\pm} = \mathbb{R} \cup \gamma_{\pm\infty}$.

Now the kernel of the resolvent has poles of second order at $\lambda = \lambda_k^\pm$; therefore

$$\Pi_1\delta(x-y)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \Pi_1 \left\{ |\chi^{[1]+}(x,\lambda)\rangle\langle\hat{\chi}^{[1]+}(y,\lambda)| - |\chi^{[n+2]-}(x,\lambda)\rangle\langle\hat{\chi}^{[n+2]-}(y,\lambda)| \right\}$$
$$+ 2i\sum_{j=1}^{N}\left\{ \operatorname*{Res}_{\lambda=\lambda_k^+} R^+(x,y,\lambda) + \operatorname*{Res}_{\lambda=\lambda_k^-} R^-(x,y,\lambda) \right\}.$$

where

$$\operatorname*{Res}_{\lambda=\lambda_k^\pm} R^\pm(x,y,\lambda) = \pm(\lambda_k^- - \lambda_k^+)\Pi_1\left(\chi^{+,(k)}(x)\dot{\hat{\chi}}^{+,(k)}(y) + \dot{\chi}^{+,(k)}(x)\hat{\chi}^{+,(k)}(y)\right) \tag{29}$$

- the continuous spectrum of L has multiplicity 2 and fills up the whole real axis $\mathbb{R}$ of the complex λ-plane;
- the resolvent kernel $R(x,y,\lambda)$ has second order poles at $\lambda = \lambda_k^\pm$.

2 Resolvent and spectral decompositions in the adjoint representation of $\mathfrak{g} \simeq B_r$

The simplest realization of L in the adjoint representation is to make use of the adjoint action of $Q(x) - \lambda J$ on $\mathfrak{g}$:

$$L_{\rm ad} e_{\rm ad} \equiv i \frac{\partial e_{\rm ad}}{\partial x} + [Q(x) - \lambda J_{\rm ad}, e_{\rm ad}(x,\lambda)] = 0. \qquad (30)$$

$e_{\rm ad}$ take values in the Lie algebra $\mathfrak{g}$; they are known also as the 'squared solutions' of L and appear in a natural way in the analysis of the transform from the potential $Q(x,t)$ to the scattering data of L.

Introduce:

$$e^{\pm}_{\alpha,{\rm ad}}(x,\lambda) = \chi^{\pm} E_\alpha \hat{\chi}^{\pm}(x,\lambda), \qquad e^{\pm}_{j,{\rm ad}}(x,\lambda) = \chi^{\pm} H_j \hat{\chi}^{\pm}(x,\lambda), \qquad (31)$$

where $\chi^{\pm}(x,\lambda)$ are the FAS of L and E_α, H_j form the Cartan-Weyl basis of $\mathfrak{g}$.

In the adjoint representation $J_{\rm ad} \cdot \equiv {\rm ad}_J \cdot \equiv [J, \cdot]$ has kernel. so we need the projector:

$$\pi_J X \equiv {\rm ad}_J^{-1} {\rm ad}_J X, \qquad (32)$$

In particular, the potential Q provides a generic element of the image of π_J, i.e. $\pi_J Q \equiv Q$.

From the Wronskian relations we are able to introduce two sets of squared solutions:

$$\mathbf{\Psi}_\alpha^\pm = \pi_J(\chi^\pm(x,\lambda)E_\alpha\hat{\chi}^\pm(x,\lambda)), \quad \mathbf{\Phi}_\alpha^\pm = \pi_J(\chi^\pm(x,\lambda)E_{-\alpha}\hat{\chi}^\pm(x,\lambda)), \quad \alpha \in \Delta_1^+.$$

We remind that the set Δ_1^+ contains all roots of $so(2r+1)$ for which $\alpha(J) > 0$.

Each of the above two sets are complete sets of functions in the space of allowed potentials. Apply again the contour integration method to the integral

$$\mathcal{J}_G(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} d\lambda G^+(x,y,\lambda) - \frac{1}{2\pi i}\oint_{\gamma_-} d\lambda G^-(x,y,\lambda), \qquad (33)$$

where the Green function is defined by:

$$G^{\pm}(x,y,\lambda) = G_1^{\pm}(x,y,\lambda)\theta(y-x) - G_2^{\pm}(x,y,\lambda)\theta(x-y),$$

$$G_1^{\pm}(x,y,\lambda) = \sum_{\alpha\in\Delta_1^+} \boldsymbol{\Psi}_{\pm\alpha}^{\pm}(x,\lambda) \otimes \boldsymbol{\Phi}_{\mp\alpha}^{\pm}(y,\lambda),$$

$$G_2^{\pm}(x,y,\lambda) = \sum_{\alpha\in\Delta_0\cup\Delta_1^-} \boldsymbol{\Phi}_{\pm\alpha}^{\pm}(x,\lambda) \otimes \boldsymbol{\Psi}_{\mp\alpha}^{\pm}(y,\lambda) + \sum_{j=1}^{r} \boldsymbol{h}_j^{\pm}(x,\lambda) \otimes \boldsymbol{h}_j^{\pm}(y,\lambda),$$

$$\boldsymbol{h}_j^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda) H_j \hat{\chi}^{\pm}(x,\lambda),$$

The result – VSG (1984) and after:

$$\begin{aligned}\delta(x-y)\Pi_{0J} &= \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda (G_1^+(x,y,\lambda) - G_1^-(x,y,\lambda)) \\ &\quad - 2i\sum_{j=1}^{N}(G_{1,j}^+(x,y) + G_{1,j}^-(x,y)),\end{aligned} \tag{34}$$

$$\Pi_{0J} = \sum_{\alpha\in\Delta_1^+} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha),$$

$$G^{\pm}_{1,j}(x,y) = \sum_{\alpha\in\Delta_1^+} (\dot{\boldsymbol{\Psi}}^{\pm}_{\pm\alpha;j}(x) \otimes \boldsymbol{\Phi}^{\pm}_{\mp\alpha;j}(y) + \boldsymbol{\Psi}^{\pm}_{\pm\alpha;j}(x) \otimes \dot{\boldsymbol{\Phi}}^{\pm}_{\mp\alpha;j}(y)).$$

- the continuous spectrum of $L_{\rm ad} \simeq \Lambda_{\pm}$ has multiplicity $2n$ and fills up the whole real axis $\mathbb{R}$ of the complex λ-plane;
- the Green function $G(x,y,\lambda)$ has second order poles at $\lambda = \lambda_k^{\pm}$;
- eq. (34) provides the spectral decomposition of $\Lambda_{\pm}$

2.1 Expansion over the 'squared solutions'

The expansion of $Q(x)$

$$Q(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\tau_\alpha^+(\lambda)\boldsymbol{\Phi}_\alpha^+(x,\lambda) - \tau_\alpha^-(\lambda)\boldsymbol{\Phi}_{-\alpha}^-(x,\lambda)\right)$$

$$+ 2\sum_{k=1}^{N} \sum_{\alpha\in\Delta_1^+} \left(\tau_{\alpha;j}^+ \boldsymbol{\Phi}_{\alpha;j}^+(x) + \tau_{\alpha;j}^- \boldsymbol{\Phi}_{-\alpha;j}^-(x)\right),$$

$$Q(x) = -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\rho_\alpha^+(\lambda)\boldsymbol{\Psi}_{-\alpha}^+(x,\lambda) - \rho_\alpha^-(\lambda)\boldsymbol{\Psi}_\alpha^-(x,\lambda)\right)$$

$$- 2\sum_{k=1}^{N}\sum_{\alpha\in\Delta_1^+} \left(\rho_{\alpha;j}^+\boldsymbol{\Psi}_{-\alpha;j}^+(x) + \rho_{\alpha;j}^-\boldsymbol{\Psi}_{\alpha;j}^-(x)\right),$$

The next expansion is of $\mathrm{ad}_J^{-1}\delta Q(x)$:

$$\mathrm{ad}_J^{-1}\delta Q(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\delta\tau_\alpha^+(\lambda)\boldsymbol{\Phi}_\alpha^+(x,\lambda) + \delta\tau_\alpha^-(\lambda)\boldsymbol{\Phi}_{-\alpha}^-(x,\lambda)\right)$$

$$+\sum_{k=1}^{N}\sum_{\alpha\in\Delta_1^+} \left(\delta W_{\alpha;j}^+(x) - \delta' W_{-\alpha;j}^-(x)\right),$$

$$
\mathrm{ad}_J^{-1}\delta Q(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\delta\rho_\alpha^+(\lambda)\boldsymbol{\Psi}_{-\alpha}^+(x,\lambda) + \delta\rho_\alpha^-(\lambda)\boldsymbol{\Psi}_\alpha^-(x,\lambda)\right)
$$

$$
+\sum_{k=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\delta\tilde{W}_{-\alpha;j}^+(x) - \delta\tilde{W}_{\alpha;j}^-(x)\right),
$$

where

$$
\delta W_{\pm\alpha;j}^{\pm}(x) = \delta\lambda_j^{\pm}\tau_{\alpha;j}^{\pm}\dot{\boldsymbol{\Phi}}_{\pm\alpha;j}^{\pm}(x) + \delta\tau_{\alpha;j}^{\pm}\boldsymbol{\Phi}_{\pm\alpha;j}^{\pm}(x),
$$

$$
\delta\tilde{W}_{\mp\alpha;j}^{\pm}(x) = \delta\lambda_j^{\pm}\rho_{\alpha;j}^{\pm}\dot{\boldsymbol{\Psi}}_{\mp\alpha;j}^{\pm}(x) + \delta\rho_{\alpha;j}^{\pm}\boldsymbol{\Psi}_{\mp\alpha;j}^{\pm}(x),
$$

$$
\boldsymbol{\Phi}_{\pm\alpha;j}^{\pm}(x) = \boldsymbol{\Phi}_{\pm\alpha}^{\pm}(x,\lambda_j^{\pm}), \quad \dot{\boldsymbol{\Phi}}_{\pm\alpha;j}^{\pm}(x) = \partial_\lambda\boldsymbol{\Phi}_{\pm\alpha}^{\pm}(x,\lambda)|_{\lambda=\lambda_j^{\pm}}.
$$

Consider the class of variations of $Q(x,t)$ due to the evolution in t:

$$
\delta Q(x,t) \equiv Q(x,t+\delta t) - Q(x,t) = \frac{\partial Q}{\partial t}\delta t + (O)((\delta t)^2). \qquad (35)
$$

Assuming that δt is small and keeping only the first order terms in δt we get the expansions for $\mathrm{ad}_J^{-1}Q_t$. They are obtained from the above by replacing $\delta\rho_\alpha^{\pm}(\lambda)$ and $\delta\tau_\alpha^{\pm}(\lambda)$ by $\partial_t\rho_\alpha^{\pm}(\lambda)$ and $\partial_t\rho_\alpha^{\pm}(\lambda)$.

2.2 The generating operators

Analogy between the standard Fourier transform and the expansions over the 'squared solutions'.

$$
\begin{aligned}
D_0 &= -i\frac{d}{dx} \qquad & D_0 e^{i\lambda x} &= \lambda e^{i\lambda x}, \\
\Lambda_\pm &= ? \qquad & \Lambda_\pm \boldsymbol{\Psi}^{+}_{-\alpha}(x,\lambda) &= \lambda \boldsymbol{\Psi}^{+}_{-\alpha}(x,\lambda).
\end{aligned}
$$

Therefore we introduce the generating operators $\Lambda_\pm$ through:

$$
\begin{aligned}
&(\Lambda_+ - \lambda)\boldsymbol{\Psi}^{+}_{-\alpha}(x,\lambda) = 0, \quad (\Lambda_+ - \lambda)\boldsymbol{\Psi}^{-}_{\alpha}(x,\lambda) = 0, \quad (\Lambda_+ - \lambda_j^{\pm})\boldsymbol{\Psi}^{+}_{\mp\alpha;j}(x) = 0, \\
&(\Lambda_- - \lambda)\boldsymbol{\Phi}^{+}_{\alpha}(x,\lambda) = 0, \quad (\Lambda_- - \lambda)\boldsymbol{\Phi}^{-}_{-\alpha}(x,\lambda) = 0, \quad (\Lambda_+ - \lambda_j^{\pm})\boldsymbol{\Phi}^{+}_{\pm\alpha;j}(x) = 0.
\end{aligned}
$$

The generating operators $\Lambda_\pm$ are given by:

$$
\Lambda_\pm X(x) \equiv \operatorname{ad}_J^{-1}\left(i\frac{\mathrm{d}X}{\mathrm{d}x} + i\left[Q(x), \int_{\pm\infty}^{x} \mathrm{d}y\, [Q(y), X(y)] \right] \right). \qquad (36)
$$

The completeness relation can be viewed as the spectral decompositions of the recursion operators $\Lambda_\pm$.

3 Resolvent and spectral decompositions in the spinor representation of $\mathfrak{g} \simeq B_r$

In the spinor representation the Lax operators take the form:

$$L_{\rm sp}\psi_{\rm sp} = i\frac{\partial\psi_{\rm sp}}{\partial x} + (Q_{\rm sp} - \lambda J_{\rm sp})\psi_{\rm sp}(x,\lambda) = 0, \qquad (37)$$

where $Q_{\rm sp}(x,t)$ and $J_{\rm sp}$ are $2^r \times 2^r$ matrices of the form:

$$Q_{\rm sp} = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{q}^\dagger & 0 \end{pmatrix}, \qquad J_{\rm sp} = \frac{1}{2}\begin{pmatrix} \mathbb{1}_{2^{r-1}} & 0 \\ 0 & -\mathbb{1}_{2^{r-1}} \end{pmatrix}, \qquad (38)$$

The spinor representations of $so(2r+1)$ are realized by symplectic (resp. orthogonal) matrices if $r(r+1)/2$ is odd (resp. even). Thus we can view the spinor representations of $so(2r+1)$ as imbedded in the typical representations of $sp(2^r)$ (resp. $so(2^r)$) algebra.

This spectral problem is technically more simple to treat.

$$\psi(x,\lambda) \underset{x\to\infty}{\simeq} e^{-i\lambda Jx}, \qquad \phi(x,\lambda) \underset{x\to-\infty}{\simeq} e^{-i\lambda Jx},$$
$$T(\lambda) = \begin{pmatrix} \boldsymbol{a}^{+} & -\boldsymbol{b}^{-} \\ \boldsymbol{b}^{+} & \boldsymbol{a}^{-} \end{pmatrix},$$
$$\psi(x,\lambda) = (\psi^{-}(x,\lambda), \psi^{+}(x,\lambda)), \qquad \phi(x,\lambda) = (\phi^{+}(x,\lambda), \phi^{-}(x,\lambda)),$$
$$\chi^{+}(x,\lambda) = (\phi^{+}(x,\lambda), \psi^{+}(x,\lambda)), \qquad \chi^{-}(x,\lambda) = (\psi^{-}(x,\lambda), \phi^{-}(x,\lambda)), \tag{39}$$

3.1 The Gauss factors in the spinor representation

The Gauss factors of $T_{\rm sp}(\lambda)$ and FAS:

$$\begin{aligned}\chi^{+}_{\rm sp}(x,\lambda) &\equiv \left(|\phi^{+}\rangle, |\psi^{+}\hat{c}^{+}\rangle\right)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^{+}_{\rm sp}(\lambda) = \psi_{\rm sp}(x,\lambda)\boldsymbol{T}^{-}_{\rm sp}(\lambda)D^{+}_{\rm sp}(\lambda), \\ \chi^{-}_{\rm sp}(x,\lambda) &\equiv \left(|\psi^{-}\hat{c}^{-}\rangle, |\phi^{-}\rangle\right)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^{-}_{\rm sp}(\lambda) = \psi_{\rm sp}(x,\lambda)\boldsymbol{T}^{+}_{\rm sp}(\lambda)D^{-}_{\rm sp}(\lambda),\end{aligned} \tag{40}$$

where the block-triangular functions $\boldsymbol{S}_{\rm sp}^{\pm}(\lambda)$ and $\boldsymbol{T}_{\rm sp}^{\pm}(\lambda)$ are given by:

$$\begin{aligned} \boldsymbol{S}_{\rm sp}^{+}(\lambda) &= \begin{pmatrix} \mathbb{1} & \boldsymbol{d}^{-}\hat{\boldsymbol{c}}^{+}(\lambda) \\ 0 & \mathbb{1} \end{pmatrix}, \qquad \boldsymbol{T}_{\rm sp}^{-}(\lambda) = \begin{pmatrix} \mathbb{1} & 0 \\ \boldsymbol{b}^{+}\hat{\boldsymbol{a}}^{+}(\lambda) & \mathbb{1} \end{pmatrix}, \\ \boldsymbol{S}_{\rm sp}^{-}(\lambda) &= \begin{pmatrix} \mathbb{1} & 0 \\ -\boldsymbol{d}^{+}\hat{\boldsymbol{c}}^{-}(\lambda) & \mathbb{1} \end{pmatrix}, \qquad \boldsymbol{T}_{\rm sp}^{+}(\lambda) = \begin{pmatrix} \mathbb{1} & -\boldsymbol{b}^{-}\hat{\boldsymbol{a}}^{-}(\lambda) \\ 0 & \mathbb{1} \end{pmatrix}, \end{aligned} \tag{41}$$

The matrices $D_{\rm sp}^{\pm}(\lambda)$ are block-diagonal and equal:

$$D_{\rm sp}^{+}(\lambda) = \begin{pmatrix} \boldsymbol{a}^{+}(\lambda) & 0 \\ 0 & \hat{\boldsymbol{c}}^{+}(\lambda) \end{pmatrix}, \qquad D_{\rm sp}^{-}(\lambda) = \begin{pmatrix} \hat{\boldsymbol{c}}^{-}(\lambda) & 0 \\ 0 & \boldsymbol{a}^{-}(\lambda) \end{pmatrix}. \tag{42}$$

The supper scripts $\pm$ here refer to their analyticity properties for $\lambda \in \mathbb{C}_{\pm}$.

The resolvent $R_{\rm sp}(\lambda)$ of $L_{\rm sp}$ is again expressed through the FAS

$$R_{\rm sp}(\lambda)f(x) = \int_{-\infty}^{\infty} R_{\rm sp}(x,y,\lambda)f(y). \tag{43}$$

where $R_{\rm sp}(x,y,\lambda)$ are given by:

$$R_{\rm sp}(x,y,\lambda) = \begin{cases} R_{\rm sp}^{+}(x,y,\lambda) & \text{for } \lambda \in \mathbb{C}^{+}, \\ R_{\rm sp}^{-}(x,y,\lambda) & \text{for } \lambda \in \mathbb{C}^{-}, \end{cases} \tag{44}$$

and

$$R^{\pm}_{\rm sp}(x,y,\lambda) = \pm i\chi^{\pm}_{\rm sp}(x,\lambda)\Theta^{\pm}(x-y)\hat{\chi}^{\pm}_{\rm sp}(y,\lambda), \qquad \Theta^{\pm}(z) = \begin{pmatrix} \theta(\mp z)\mathbb{1} & 0 \\ 0 & -\theta(\pm z)\mathbb{1} \end{pmatrix} \tag{45}$$

4 MNLS with Constant Boundary Conditions

Require: i) regular behaviour of the solutions for $t \to \pm\infty$;
ii) require that the spectrum of the two asymptotic operators $L_{\pm} = id/dx + U_{\pm}(\lambda)$ have the same spectrum. Here

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \qquad U_{\pm}(\lambda) \equiv \lim_{x\to\pm\infty} U(x,t,\lambda) = Q_{\pm} - \lambda J. \tag{46}$$

The first requirement can be satisfied by regularizing the MNLS, i.e. by conveniently adding linear in $\boldsymbol{q}$ terms. The corresponding regularized MNLS have the form:

$$i\boldsymbol{q}_t + \boldsymbol{q}_{xx} - 2\boldsymbol{q}\boldsymbol{q}^{\dagger}\boldsymbol{q} + \boldsymbol{q}\mu + \bar{\mu}\boldsymbol{q} = 0, \tag{47}$$

$$\lim_{x\to\pm\infty} \boldsymbol{q}(x,t) = \boldsymbol{q}_\pm, \qquad \mu = \boldsymbol{q}_+^\dagger \boldsymbol{q}_+ = \boldsymbol{q}_-^\dagger \boldsymbol{q}_-, \qquad \overline{\mu} = \boldsymbol{q}_+ \boldsymbol{q}_+^\dagger = \boldsymbol{q}_- \boldsymbol{q}_-^\dagger.$$

$$\text{ii)} \qquad Q_+ = u_\theta^{-1} Q_- u_\theta.$$

then $U_+(\lambda)$ and $U_-(\lambda)$ have the same sets of eigenvalues.

The M-operators of the MNLS with CBC contains additional terms

$$V_0(x,t) = -[Q, \mathrm{ad}_J^{-1} Q] + 2i\mathrm{ad}_J^{-1} Q_x(x,t) + [Q_\pm, \mathrm{ad}_J^{-1} Q_\pm]. \qquad (48)$$

with $Q_\pm$ which ensure the regular behavior of the solutions for large t.

The Lax operator can be associated with a symmetric spaces if

A.II $\mathfrak{g} \simeq A_{N-1} \equiv sl(N)$, $J = H_{\vec{a}}$, where the vector $\vec{a}$ in the root space $\mathbb{E}^r$ dual to J is given by $\vec{a} = \sum_{k=1}^s e_k - \sum_{k=s+1}^N e_k$;

In the next two cases $s = r$ and $N = 2r$ is even.

C.II $\mathfrak{g} \simeq C_r \equiv sp(2r)$, $J = H_{\vec{a}}$, where the vector $\vec{a}$ in the root space $\mathbb{E}^r$ dual to J is given by $\vec{a} = \sum_{k=1}^r e_k$;

D.III $\mathfrak{g} \simeq D_r \equiv so(2r)$, $J = H_{\vec{a}}$, where the vector $\vec{a}$ in the root space $\mathbb{E}^r$ dual to J is given by $\vec{a} = \sum_{k=1}^r e_k$.

BD.I $\mathfrak{g} \simeq D_r \equiv so(2r)$ for $N = 2r$ and $\mathfrak{g} \simeq B_r \equiv so(2r+1)$ for $N = 2r+1$, $J = H_{e_1}$.

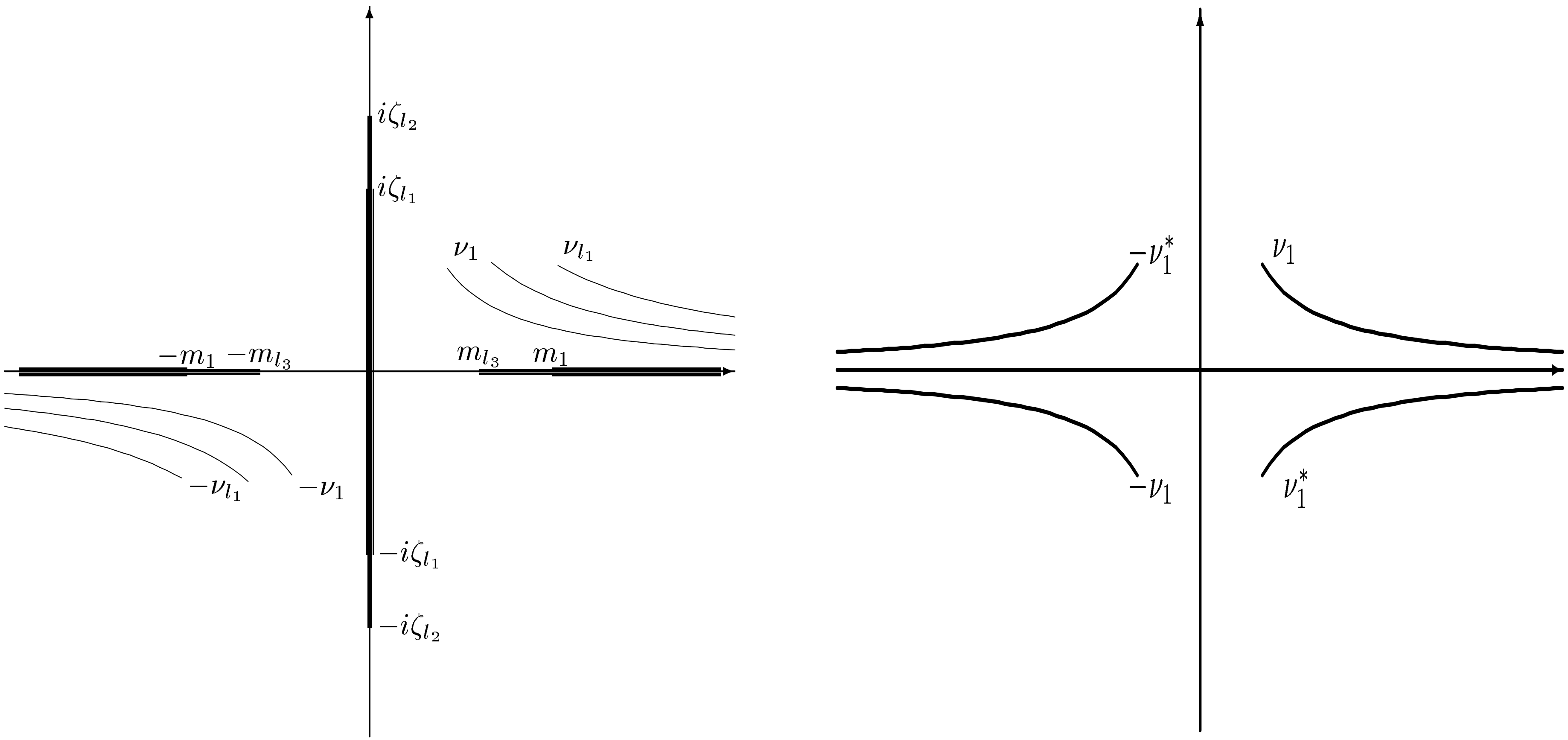
The spectrum of the asymptotic operators $L_\pm$ is purely continuous and is determined by the the eigenvalues of $Q_\pm$ which generically may be arbitrary complex numbers. The spectra of A-type symmetric spaces were described by VSG, Kulish (1983).

a) $\nu_k \neq \pm\nu_k^*$, $k = 1, \dots, l_1$ – two branches of two-fold spectrum filling up the hyperbola's arcs $\operatorname{Re}\lambda \operatorname{Im}\lambda = \operatorname{Re}\nu_k \operatorname{Im}\nu_k$ on which $|\operatorname{Re}\lambda| \geq |\operatorname{Re}\nu_k|$;

b) $\nu_{l_1+k} = -\nu_{l_1+k}^* = i\zeta_k$, $k = 1, \dots, l_2$ – two branches of two-fold spectrum filling up the real axis and the segment $|\operatorname{Im}\lambda| \leq |\zeta_k|$ of the imaginary axis;

c) $\nu_{l_1+l_2+k} = \nu_{l_1+l_2+k}^* = m_k$, $k = 1, \dots, l_3 = r - l_1 - l_2 + 1$ – two branches of two-fold spectrum filling up the segments $|\operatorname{Re}\lambda| \geq |m_k|$ of the real axis;

For $C.II$- and $D.III$-type symmetric spaces the spectra consist of four branches filling up the hyperbola's arcs $\mathrm{Re}\,\lambda\,\mathrm{Im}\,\lambda = \pm\mathrm{Re}\,\nu_1\mathrm{Im}\,\nu_1$ on which $|\mathrm{Re}\,\lambda| \geq |\mathrm{Re}\,\nu_1|$, see the right panel of the figure - VSG 2004.



Фигура 2: Left panel: the continuous spectrum of L, generic case; Right panel: the continuous spectrum of the $sp(4)$ and $so(8)$ MNLS with CBC for $D < 0$; the only difference is that while the multiplicity of the spectra of $sp(4)$ is 2 the one for $so(8)$ is 4.

4.1 Spectral properties of $sp(4)$–MNLS with CBC

As mentioned in Section 3, the continuous spectrum of the GZS system is determined by the set of eigenvalues $\{\nu_j, j = 1, 2\}$ of the matrices $q_+r_+ = q_-r_-$. These eigenvalues for $Q_\pm$ with $r = 2$ satisfy the characteristic equation:

$$\nu^2 - K_0\nu + K_1 = 0, \qquad K_0 = \frac{1}{2}\operatorname{tr} Q_\pm^2, \qquad K_1 = \det Q_\pm. \tag{49}$$

and determine the end points of the spectrum. If we impose on $Q(x, t)$, and consequently on $Q_\pm$ the involution ($\mathbb{Z}_2$-reduction):

$$B_1^{-1}Q^\dagger B_1 = Q, \qquad B_1 = \operatorname{diag}(1, \epsilon, \epsilon, 1), \qquad \epsilon = \pm 1. \tag{50}$$

which in components takes the form:

$$r_1 = \epsilon q_1^*, \qquad r_2 = q_2^*, \qquad r_3 = q_3^*. \tag{51}$$

Then the coefficients K_0 and K_1 equal:

$$K_0 = 2\epsilon|q_1^\pm|^2 + |q_2^\pm|^2 + |q_3^\pm|^2, \qquad K_1 = |(q_1^\pm)^2 + q_2^\pm q_3^\pm|^2 \tag{52}$$

We have three possibilities for the roots ν_1, ν_2 of eq. (49) depending on the sign of the discriminant:

$$D = \frac{1}{4} K_0^2 - 4K_1. \qquad (53)$$

a) $D > 0$**,** i.e. the roots $\nu_1 > \nu_2$ are different and real. The continuous spectrum of L fills up two pairs of rays on the real axis $|\mathrm{Re}\,\lambda| > \nu_1$ and $|\mathrm{Re}\,\lambda| > \mathrm{Re}\,\nu_2$;

b) $D = 0$**,** i.e. the roots $\nu_1 = \nu_2$; the two pairs of rays in a) now coincide; the total multiplicity of the spectrum is 4;

c) $D < 0$**,** i.e. the roots ν_j are complex-valued and $\nu_1 = \nu_2^*$; The continuous spectrum of L fils up two branches of two-fold spectrum along the hyperbola's arcs $\mathrm{Re}\,\lambda\,\mathrm{Im}\,\lambda = \mathrm{Re}\,\nu_k \mathrm{Im}\,\nu_k$, see the right panel of fig. 2;

In the generic case there are no apriory limitations as to the positions of the discrete eigenvalues. Such may come up if we consider potentials

$Q = -Q^\dagger$; then the GZS system become equivalent to a formally self-adjoint linear problem whose spectrum should be confined to the real λ-axis only. The formal self-adjointness takes place for $\epsilon = 1$.

4.2 Spectral properties of $so(8)$-MNLS with CBC

The characteristic equation for $q_\pm r_\pm$ takes more simple form:

$$\det(q_\pm r_\pm - \nu) = (\nu^2 - K_0\nu + K_1)^2, \qquad (54)$$

where the coefficients K_j now are given by:

$$K_0 = \frac{1}{2}\mathrm{tr}\,(q_\pm r_\pm) = \sum_{1\leq i<j\leq 4} q_{ij}^\pm r_{ij}^\pm, \qquad (55)$$

$$K_1 = (\det(q_\pm r_\pm))^{1/2} = (q_{13}^\pm q_{24}^\pm - q_{34}^\pm q_{12}^\pm - q_{23}^\pm q_{14}^\pm)(r_{13}^\pm r_{24}^\pm - r_{34}^\pm r_{12}^\pm - r_{23}^\pm r_{14}^\pm).$$

An involution of the type (50) gives $r_{ij} = \epsilon_i\epsilon_j q_{ij}^*$ with $\epsilon_j = \pm 1$ and makes the coefficients K_0, K_1 real. Besides now each of the eigenvalues ν_j, $j = 1, 2$ is two-fold. Again we have the three possibilities depending on the value of D; the only difference is that the multiplicity of each of the branches is 4. This imposes certain symmetry on the locations of the eigenvalues of ν_j which in fact determine the end-points of the continuous spectra of L.

5 BD.I-type MNLS with CBC

The BD.I-type MNLS with CBC take the form

$$\begin{aligned} &i\vec{q}_t + \vec{q}_{xx} + 2\epsilon \left((\vec{q}^\dagger, \vec{p}) - \eta_0 \right) \vec{q} - \left((\vec{q}, s_0\vec{q}) - \tilde{\eta}_0 \right) s_0 \vec{q}^* = 0, \\ &\eta_0 = \lim_{x\to\pm\infty} (\vec{q}^\dagger, \vec{p}), \qquad \tilde{\eta}_0 = \lim_{x\to\pm\infty} (\vec{q}^T s_0 \vec{q}), \end{aligned} \tag{56}$$

The Lax pair has different spectral properties.

$$L_\pm = i\frac{d}{dx} + U_\pm(\lambda), \qquad U(x,\lambda) = q(x,t) - \lambda J,$$

$$q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad U_\pm = \lim_{x\to\pm\infty} U(x,\lambda) = q_\pm - \lambda J.$$

Additional reduction:

$$\vec{p}(x,t) = K_1 \vec{q}^*, \qquad K_1^2 = \mathbb{1}.$$

Request that

$$(\vec{q}_+^\dagger, \vec{p}_+) = (\vec{q}_-^\dagger, \vec{p}_-), \qquad (\vec{q}_+^T, s_0 \vec{q}_+) = (\vec{q}_-^T, s_0 \vec{q}_-), \qquad (\vec{p}_+^T, s_0 \vec{p}_+) = (\vec{p}_-^T, s_0 \vec{p}_-).$$

This condition means that the asymptotic Lax operators:

$$L_\pm = i \frac{d}{dx} + U_\pm(\lambda)$$

have the same spectrum determined by the roots of the characteristic polynomial:

$$\begin{aligned} &\mu^{n-2}(\mu^4 - \mu^2(2f_0 + \lambda^2) + f_0^2 - f_1) = 0, \\ &f_0 = (\vec{q}_\pm^T, \vec{p}_\pm), \qquad f_1 = (\vec{q}_\pm^T s_0 \vec{q}_\pm)(\vec{p}_\pm^T s_0 \vec{p}_\pm), \end{aligned} \qquad (57)$$

The nontrivial roots of this polynomial are given by:

$$\mu_{1,2}^2 = \frac{\lambda^2}{2} + f_0 \pm \sqrt{\lambda^4 + 4 f_0 \lambda^2 + 4 f_1}$$

and the continuous spectrum of $L_{\rm as}$ lies on those lines in the complex λ-plane on which

$$\operatorname{Im} \mu_j(\lambda) = 0.$$

The Jost solutions are determined by their asymptotics for $x \to \pm\infty$ as follows:

$$\psi(x,\lambda) \longrightarrow u_{0,+} e^{i\mu(\lambda)x} \hat{u}_{0,+}, \quad \text{for } x \to \infty;$$

$$\phi(x,\lambda) \longrightarrow u_{0,-} e^{i\mu(\lambda)x} \hat{u}_{0,-}, \quad \text{for } x \to -\infty;$$

$$Q_{\pm} - \lambda J = u_{0,\pm} \mu(\lambda) \hat{u}_{0,\pm}, \qquad \mu(\lambda) = \operatorname{diag}(\mu_1(\lambda), \dots, \mu_n(\lambda)),$$

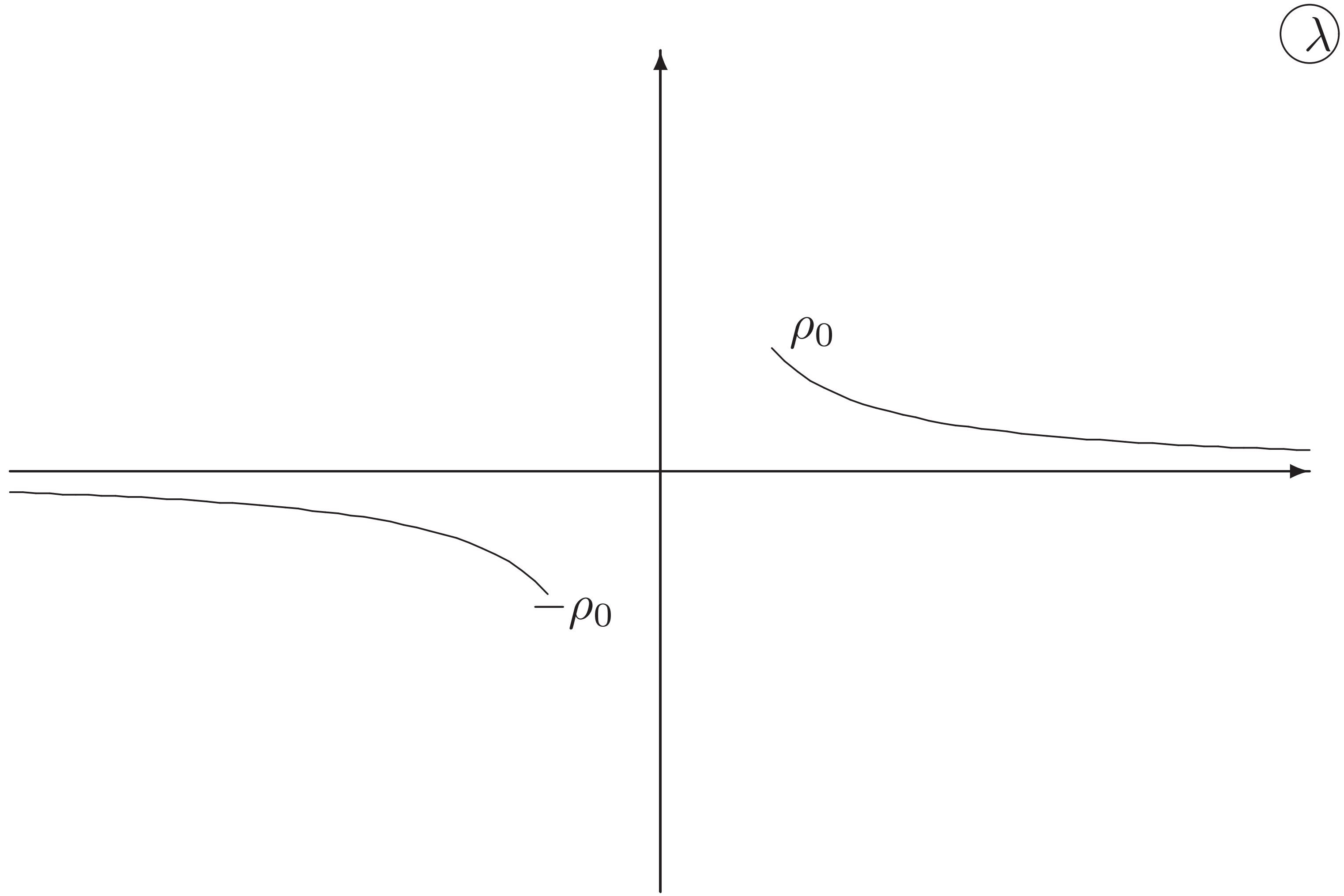
$$\mu_{1,n}^2(\lambda) = \frac{\lambda^2 + 2a + \sqrt{\lambda^4 + 4a\lambda^2 + b}}{2}, \quad \mu_{2,n-1}^2(\lambda) = \frac{\lambda^2 + 2a - \sqrt{\lambda^4 + 4a\lambda^2 + b}}{2},$$

$$\mu_{3,4,\dots,n-2} = 0, \quad a = (\vec{r}_{\pm}, \vec{q}_{\pm}), \qquad b = 4(\vec{r}_{\pm}, s_0 \vec{r}_{\pm})(\vec{q}_{\pm}, s_0 \vec{q}_{\pm}).$$

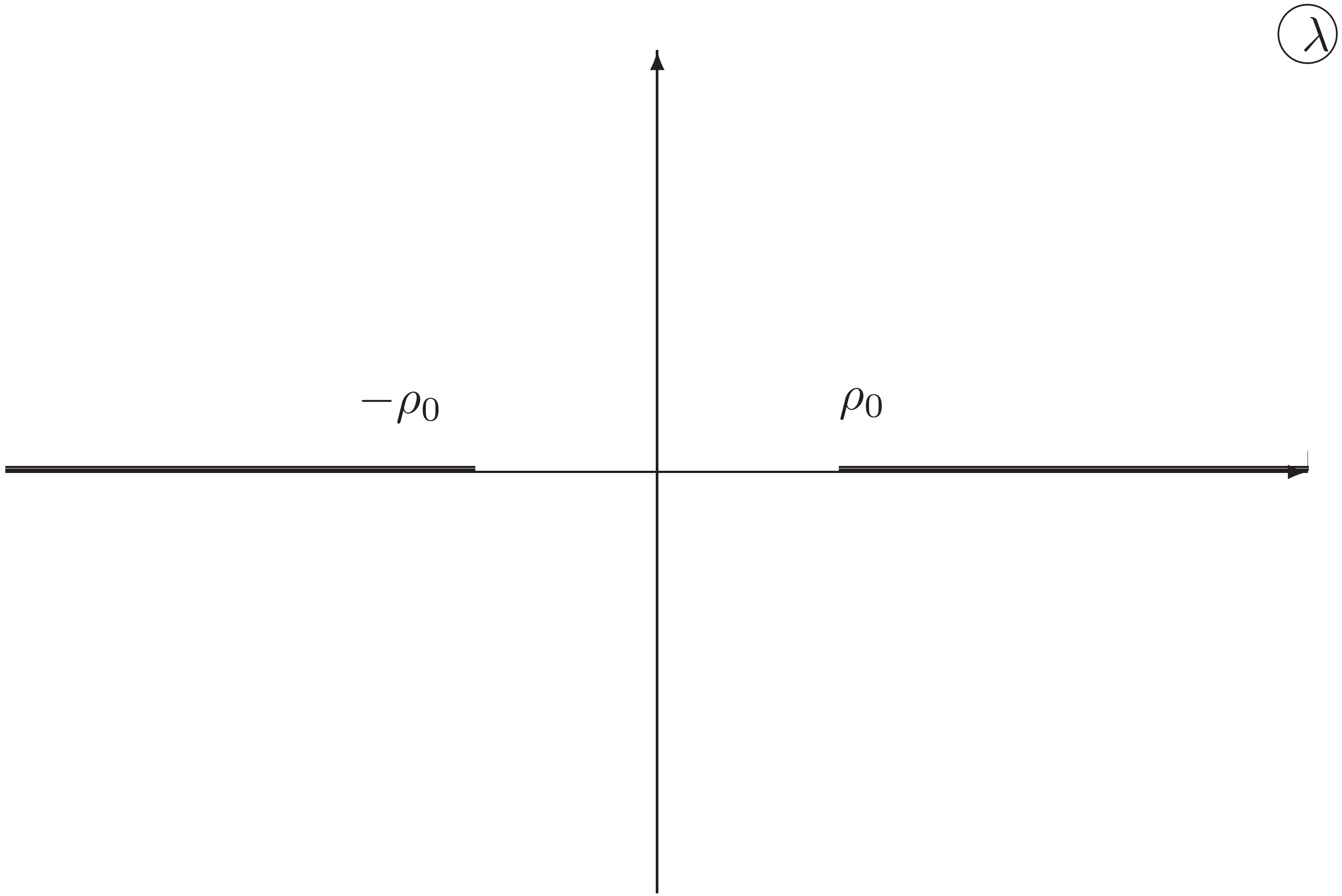
The continuous spectrum of L is determined by $\operatorname{Re} \mu_k(\lambda) = 0$. If $b = 4a^2$ this simplifies

$$\mu_{1,n}^2 = \lambda^2 + 2a, \qquad \mu_{2,3,\dots,n-1} = 0.$$

With the reduction $\vec{r} = -\vec{q}^{*}$ we get that $a = -m_0^2/2 < 0$ and the spectrum fills in the two semiaxis $|\lambda| > m_0$.



The continuous spectrum of $L_\pm$ for **BD.I**-type MNLS with non-typical reductions and $f_1 = f_0^2$ and $\rho_0 = \sqrt{-2f_0}$.



The continuous spectrum of $L_\pm$ for **BD.I**-type MNLS with typical reduction and $f_1 = f_0^2$. Here $\rho_0 = \sqrt{-2f_0}$ and $f_0 < 0$.

6 Generalized Zakharov-Shabat systems with deep reductions

6.1 Mikhailov's reduction group

Lax representation:

$$[L(\lambda), M(\lambda)] = 0,$$

$$L(\lambda) = i\frac{d}{dx} + U(x,\lambda), \qquad M(\lambda) = i\frac{d}{dt} + V(x,\lambda), \qquad U(x,\lambda),\ V(x,\lambda) \in \mathfrak{g}$$

G_R – finite group of Aut $\mathfrak{g}$ × Conf_λ

$$G_R \begin{array}{l} \nearrow \ \mathrm{Aut}\,\mathfrak{g} \\ \searrow \ \mathrm{Conf}\,\mathbb{C} \end{array},$$

$$C_k(U(\Gamma_k(\lambda))) = \eta_k U(\lambda), \quad C_k(V(\Gamma_k(\lambda))) = \eta_k V(\lambda), \qquad (58)$$

For each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$.

Finite subgroups of Conf_λ: $\quad \mathbb{Z}_h, \mathbb{D}_h, \mathbb{T}, \mathbb{O}, \mathbb{I}$

Examples for all these groups constructed by Mikhailov in (1978) - (1980). 2d Toda field theory.

$$\begin{array}{llll} 1) & C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda), & & C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda), \\ 2) & C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), & & C_2(V^T(\kappa_2(\lambda))) = -V(\lambda), \\ 3) & C_3(U^*(\kappa_1(\lambda))) = -U(\lambda), & & C_3(V^*(\kappa_1(\lambda))) = -V(\lambda), \\ 4) & C_4(U(\kappa_2(\lambda))) = U(\lambda), & & C_4(V(\kappa_2(\lambda))) = V(\lambda), \end{array}$$

We will illustrate these reductions on two basic examples:
A) generalized Zakharov-Shabat systems related to homogeneous spaces:

$$U(x,t,\lambda) = [J, Q(x,t)] - \lambda J, \qquad V(x,t,\lambda) = [I, Q(x,t)] - \lambda I,$$

where $J = \mathrm{diag}\,(a_1, \dots, a_n)$, $a_1 > a_2 > \cdots > a_n$;
used first by Zakharov and Manakov (1974) to solve the N-wave equations;
B) generalized Zakharov-Shabat systems related to symmetric spaces:

$$L\psi(x,t,\lambda) \equiv i\partial_x\psi + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0.$$

$$M\psi(x,t,\lambda) \equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0,$$
$$V_1(x,t) = Q(x,t), \qquad V_0(x,t) = i\mathrm{ad}_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}_J^{-1}Q, Q(x,t)\right].$$

used first by Manakov (1974) to solve the first multicomponent NLS system; general theory for MNLS developed later by Fordy and Kulish (1983).

7 $\mathbb{Z}_h$-reductions

The $\mathbb{Z}_h$-reduction condition is introduced by:

$$C(\tilde{U}(x,t,\lambda\omega)) = \tilde{U}(x,t,\lambda), \qquad C(\tilde{V}(x,t,\lambda\omega)) = \tilde{V}(x,t,\lambda),$$

$$C^h = \mathbb{1}, \qquad \kappa(\lambda) = \lambda\omega, \qquad \omega = \exp(2\pi i/h).$$

and h - Coxeter number of $\mathfrak{g}$; C- Coxeter automorphism.

Important NLEE obtained with this reduction:

2-dim Toda field theories (Mikhailov 1980)

$$\frac{\partial^2 q_k}{\partial x \partial t} = e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}}, \qquad k = 1, \dots, n, \qquad e^{q_{n+1}} \equiv e^{q_1}.$$

$\mathbb{Z}_h$-NLS

$$i\frac{\partial q_k}{\partial t} + \gamma \coth \frac{\pi k}{n} \frac{\partial^2 q_k}{\partial x^2} + i\gamma \sum_{p=1}^{n-1} \frac{d}{dx} (q_p q_{k-p}) = 0, \qquad k = 1, \dots, n, \tag{59}$$

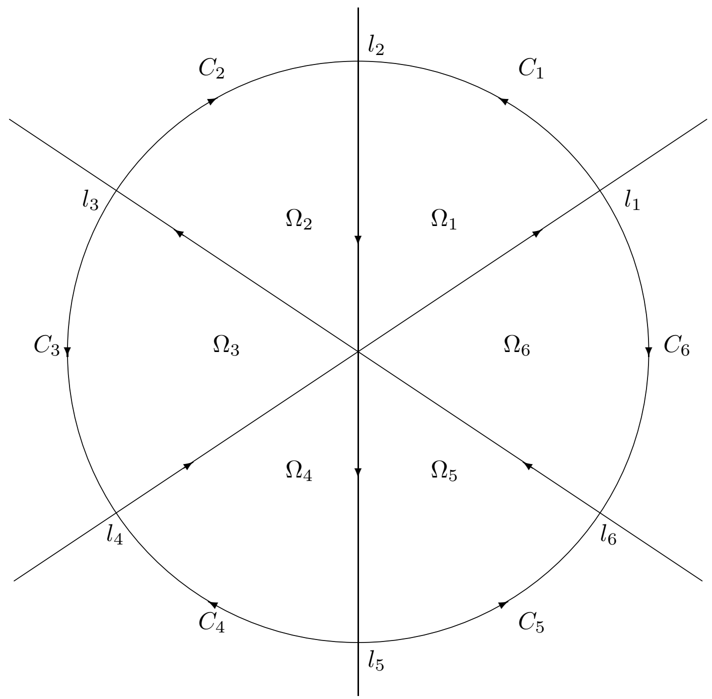
and $k - p$ is understood modulo n and $q_0 = q_n = 0$.

Lax representations.

$$[L(\lambda), M(\lambda)] = 0, \tag{60}$$

$$L(\lambda)\psi(x,t,\lambda) = \left(i\frac{d}{dx} + Q(x,t) - \lambda J \right) \psi(x,t,\lambda) = 0; \tag{61}$$

$$M_1(\lambda)\psi = \left(i\frac{d}{dt} + V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2 \right) \psi(x,t,\lambda) = \lambda^2 \psi(x,t,\lambda) V_2^{\rm as};$$



Фигура 3: Spectral properties of $\mathbb{Z}_h$ reduced Lax operators $(h = 3)$.

$$M_2(\lambda)\psi = \left(i\frac{d}{dt} + V_0(x,t) + \frac{1}{\lambda}V_{-1}(x,t) \right) \psi(x,t,\lambda) = \frac{1}{\lambda}\psi(x,t,\lambda)V_{-1}^{\rm as};$$

where $V_2^{\rm as} = \lim_{x\to\pm\infty} V_2(x,t)$ and $V_{-1}^{\rm as} = \lim_{x\to\pm\infty} V_{-1}(x,t)$.

Two FAS $\chi^{\pm}(x,\lambda)$, $\lambda \in \mathbb{C}_{\pm}$
Eigenvalues of J are all real
$J = \operatorname{diag}(J_1, J_2, \dots, J_n)$
Continuous spectrum:
$\operatorname{Im} \lambda(J_i - J_k) = 0 \Rightarrow \mathfrak{S} \equiv \mathbb{R}$
RHP on $\mathbb{R}$:

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda) G(\lambda), \quad \lambda \in \mathbb{R}$$

Eigenvalues come in pairs:
$\lambda_k^{+}, \lambda_k^{-} = (\lambda_k^{+})^*$
$G \in \mathcal{G}$

$2h$ FAS $\chi_\nu(x,\lambda)$, $\lambda \in \Omega_\nu$
Eigenvalues of J are not real:
$J = \operatorname{diag}(1, \omega, \omega^2, \dots, \omega^{h-1})$
Continuous spectrum:
$\operatorname{Im} \lambda(\omega^i - \omega^k) = 0 \Rightarrow \mathfrak{S} \equiv \cup_{\nu=0}^{h-1} l_\nu$
RHP on $\cup_{\nu=0}^{h-1} l_\nu$:

$$\chi_{\nu+1}(x,\lambda) = \chi_\nu(x,\lambda) G_\nu(\lambda), \quad \lambda \in l_\nu$$

Eigenvalues come in $2h$-tuples:
$\lambda_k^{+}\omega^s, \lambda_k^{-}\omega^s,, \; s = 0, 1, \dots, h-1$
$G \in \mathcal{G}_\nu = \otimes SL(2)$

Algebraic structures: graded Lie and Kac-Moody algebras

$$\mathfrak{g} = \mathop{\oplus}_{k=0}^{h-1} \mathfrak{g}^{(k)}, \qquad (62)$$

which are eigensubspaces of C, i.e. if

$$X^{(k)} \in \mathfrak{g}^{(k)} \qquad \Leftrightarrow \qquad C(X^{(k)}) = \omega^{-k} X^{(k)}, \qquad (63)$$

Grading condition:

$$\left[X^{(k)}, X^{(m)} \right] = X^{(k+m)} \in \mathfrak{g}^{(k+m)}. \qquad (64)$$

8 Fundamental analytic solutions and spectral properties of L

The $\mathbb{Z}_h$-symmetry imposes the following constraints on the FAS and on the scattering matrix and its factors:

$$\xi^{\nu}(x, \lambda\omega) = \psi^{\nu}(x, \lambda\omega) T_{\nu}(\lambda) = \phi^{\nu}(x, \lambda\omega) S_{\nu}(\lambda), \qquad (65a)$$

$$C_0\xi^{\nu}(x,\lambda\omega)C_0^{-1} = \xi^{\nu-2}(x,\lambda), \qquad C_0T_{\nu}(\lambda\omega)C_0^{-1} = T_{\nu-2}(\lambda), \tag{65б}$$

$$C_0S_{\nu}^{\pm}(\lambda\omega)C_0^{-1} = S_{\nu-2}^{\pm}(\lambda), \qquad C_0D_{\nu}^{\pm}(\lambda\omega)C_0^{-1} = D_{\nu-2}^{\pm}(\lambda), \tag{65в}$$

where the index $\nu - 2$ should be taken modulo $2n$. Independent data – only on two rays, e.g. on l_1 and $l_{2n} \equiv l_0$.

9 Expansions over the squared solutions

The ‘squared solutions’

$$e_{\nu,\beta}^{\pm}(x,\lambda) = \chi_{\nu}E_{\beta}\hat{\chi}_{\nu}(x,\lambda), \qquad \boldsymbol{e}_{\nu,\beta}^{\pm}(x,\lambda) = P_{0J}(\chi_{\nu}E_{\beta}\hat{\chi}_{\nu}(x,\lambda)),$$

$P_{0J} = \mathrm{ad}\,_J^{-1}\mathrm{ad}\,_J$ – the projector onto the off-diagonal part of the corresponding matrix-valued function.

The squared solution are complete set of functions.

$$Q(x) = -\frac{i}{\pi}\sum_{\nu=0}^{h-1}(-1)^{\nu}\int_{l_{\nu}} d\lambda \sum_{\alpha\in\delta_n u^+} (\tau_{\nu,\alpha}(\lambda)\boldsymbol{e}_{\nu,\alpha}(x,\lambda) - \tau_{\nu,\alpha}(\lambda)\boldsymbol{e}_{\nu,-\alpha}(x,\lambda))$$

$$Q(x) \leftrightarrows \{\tau_{\alpha}^{\nu,\pm}(\lambda),\ \alpha \in \delta_{\nu}^{+} \cup \delta_{\nu}^{-}\},$$

$$\mathrm{ad}\,_J^{-1}\delta Q(x) = \frac{i}{\pi}\sum_{\nu=0}^{h-1}(-1)^{\nu}\int_{l_\nu} d\lambda \sum_{\alpha\in\delta_n u^+} (\delta\tau_{\nu,\alpha}(\lambda)\boldsymbol{e}_{\nu,\alpha}(x,\lambda) + \delta\tau_{\nu,\alpha}(\lambda)\boldsymbol{e}_{\nu,-\alpha}(x,\lambda))$$

and similarly:

$$\delta Q(x) \leftrightarrows \{\delta\tau_{\alpha}^{\nu,\pm}(\lambda),\ \alpha \in \delta_{\nu}^{+} \cup \delta_{\nu}^{-}\},$$

$$\mathrm{ad}\,_J^{-1}\frac{dQ}{dt} = \frac{i}{\pi}\sum_{\nu=0}^{h-1}(-1)^{\nu}\int_{l_\nu} d\lambda \sum_{\alpha\in\delta_n u^+} \left(\frac{\tau_{\nu,\alpha}}{dt}(\lambda)\boldsymbol{e}_{\nu,\alpha}(x,\lambda) + \frac{\tau_{\nu,\alpha}}{dt}(\lambda)\boldsymbol{e}_{\nu,-\alpha}(x,\lambda)\right) \tag{66}$$

10 Recursion operators

$$\boldsymbol{e}_{\nu,\alpha}(x,\lambda) = \sum_{k=0}^{h-1}\boldsymbol{e}_{\nu,\alpha}^{(k)}(x,\lambda), \qquad \boldsymbol{e}_{\nu,\alpha}^{(k)}(x,\lambda) \in \mathfrak{g}^{(k)},$$

In addition we have to split each of the projections $\boldsymbol{e}^{(k)}_{\nu,\alpha}(x,\lambda)$ into diagonal and off-diagonal parts:

$$\boldsymbol{e}^{(k)}_{\nu,\alpha}(x,\lambda) = \boldsymbol{e}^{(k),\mathrm{d}}_{\nu,\alpha}(x,\lambda) + \boldsymbol{e}^{(k),\mathrm{f}}_{\nu,\alpha}(x,\lambda),$$

This requires that we have to establish which of the linear subspaces $\mathfrak{g}^{(k)}$ have nontrivial section with $\mathfrak{h}$. To this end we make use of the explicit form of the Coxeter element C of the Weyl group and its eigenvectors. It is most effective to use the dihedral realization of C in the form:

$$C = w_0 w_1, \qquad w_0^2 = \mathbb{1}, \qquad w_1^2 = \mathbb{1}, \qquad C^h = \mathbb{1}.$$

Evaluate the action of C in the root space $\mathbb{E}^r$ and determine its eigenvectors:

$$C\vec{x}^{(k)} = \omega^{m_k}\vec{x}^{(k)}, \qquad \omega = \exp(2\pi i/h).$$

The integers m_k, $k = 1,\dots,r$ are called the exponents of $\mathfrak{g}$. Next we consider the elements $H^{(k)}$ of the Cartan subalgebra $\mathfrak{h}$ that are dual to $\vec{x}^{(k)}$. They obviously satisfy:

$$C(H^{(k)}) = \omega^{m_k} H^{(k)}, \qquad \text{i.e.} \qquad H^{(k)} \in \mathfrak{g}^{(m_k)}.$$

Let $\mathfrak{g} \simeq B_r, C_r$. Then $m_k = 2k-1$, $k = 1, \dots, r$; also $h = 2r$.

$$\dim(\mathfrak{g}^{(2k-1)} \cap \mathfrak{h}) = 1, \qquad \dim(\mathfrak{g}^{(2k)} \cap \mathfrak{h}) = 0.$$

Choose $J = H^{(m_1)}$, then $H^{(m_k)} = J^{m_k}$ and:

$$e^{(2k)}_{\nu,\alpha}(x,\lambda) \equiv \boldsymbol{e}^{(2k)}_{\nu,\alpha}(x,\lambda), \qquad e^{\nu}_{\alpha,m_k}(x,\lambda) = e^{\nu,\mathrm{d}}_{\alpha,m_k}(x,\lambda) + \boldsymbol{e}^{\nu}_{\alpha,m_k}(x,\lambda),$$

Thus we get:

$$\Lambda^{\pm}_{m_k} \boldsymbol{e}^{\nu}_{\alpha,m_k}(x,\lambda) = \lambda \boldsymbol{e}^{\nu}_{\alpha,m_k-1}(x,\lambda), \qquad \Lambda_0 \boldsymbol{e}^{\nu}_{\alpha,m_k+1}(x,\lambda) = \lambda \boldsymbol{e}^{\nu}_{\alpha,m_k}(x,\lambda), \tag{67}$$

where

$$\begin{aligned} \Lambda^{\pm}_{m_k} X(x) \equiv \mathrm{ad}_J^{-1} \Bigg(i \frac{dX}{dx} + P_{0J}[Q(x), X(x)] \\ + i\,[Q(x), J^{m_k}] \int_{\pm\infty}^{x} dy \, \left\langle J^{h-m_k}, [Q(y), X(y)] \right\rangle \Bigg), \end{aligned} \tag{68}$$

$$\Lambda_0 X(x) \equiv \mathrm{ad}_J^{-1} \left(i \frac{dX}{dx} + [Q(x), X(y)] \right).$$

Thus we get that for $\mathbb{Z}_h$-reduced systems the recursion operators factorize as follows:

$$\begin{aligned} \Lambda^{\pm}_{m_1}\Lambda_0\Lambda^{\pm}_{m_2}\Lambda_0\cdots\Lambda^{\pm}_{m_{r-1}}\Lambda_0\Lambda^{\pm}_{m_r}\Lambda_0\boldsymbol{e}^{\nu}_{\mp\alpha,0}(x,\lambda) = \lambda^h\boldsymbol{e}^{\nu}_{\mp\alpha,0}(x,\lambda), \\ \Lambda_0\Lambda^{\pm}_{m_2}\Lambda_0\Lambda^{\pm}_{m_3}\cdots\Lambda_0\Lambda^{\pm}_{m_r}\Lambda_0\Lambda^{\pm}_{m_1}\boldsymbol{e}^{\nu}_{\mp\alpha,1}(x,\lambda) = \lambda^h\boldsymbol{e}^{\nu}_{\mp\alpha,1}(x,\lambda), \end{aligned} \tag{69}$$

i.e.

$$\boldsymbol{\Lambda}^{\pm}_0 = \Lambda^{\pm}_{m_1}\Lambda_0\Lambda^{\pm}_{m_2}\Lambda_0\cdots\Lambda^{\pm}_{m_r}\Lambda_0, \qquad \boldsymbol{\Lambda}^{\pm}_1 = \Lambda_0\Lambda^{\pm}_{m_2}\Lambda_0\Lambda^{\pm}_{m_3}\cdots\Lambda^{\pm}_{m_r}\Lambda_0\Lambda^{\pm}_{m_1} \tag{70}$$

and similar expressions for the operators $\boldsymbol{\Lambda}^{\pm}_k$ with $k > 1$. Similar, but more complicated factorizations exist also for D_r and for the exceptional Lie algebras.

11 Conclusions and perspectives

We described the spectral properties of wide class of Lax operators and showed that they crucially depend on

- the choice of the representation of $\mathfrak{g}$

- on the choice of the boundary conditions for the potential;
- on the choice of the group of reductions $\mathbb{Z}_h$, $h > 2$;
- demonstrated the factorization properties of Λ-operators for $\mathbb{Z}_h$-reduced systems

Perspectives:

Analyze new classes of NLEE whose Lax operators have reduction groups $\mathbb{D}_h$ and

- describe the spectral properties of new classes of Lax operators with reduction groups $\mathbb{D}_h$
- derive their soliton solutions
- derive completeness relations for the 'squared solutions'
- derive their recursion operators

This will allow us to formulate all fundamental properties of the NLEE.

Thank you for your attention!