# Integrable Discretisations for a Class of NLS Equations on Grassmann Algebras

Georgi Grahovski

Department of Applied Mathematics, University of Leeds, UK
and
Institute for Nuclear Research and Nuclear Energy, BAS, Sofia, Bulgaria

**E-mail:** G.Grahovski@leeds.ac.uk grah@inrne.bas.bg

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- GGG, A. V. Mikhailov - E-print: `arXiv:1303.1853`

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# Motivation

- Noncommutative extensions of integrable equations: KdV, NLS, sin-Gordon, the KP equation, the Hirota-Miwa equation, two-dimensional Toda lattice equation and AKNS hierarchy.

B. A. Kupershmidt, *KP or mKP: Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems*, Math. Surv. and Monogr. **78**, AMS Providence, RI (2000).

- Supersymmetric systems – particular examples of noncommutative integrable systems.
- Perhaps the best known example of such equation is the Manin-Radul super-KdV equation.

Yu. I. Manin and A. O. Radul, Commun. Math. Phys. **98** (1985), 65–77.

- Soliton solutions for the Manin-Radul super-KdV equation $\rightarrow$ Darboux transformations.
- The theory of Darboux transformations was boosted by the dressing method.

# Darboux Transformations and Dscretisations

- Darboux transformations also play a role in constructing integrable discretizations of integrable equations.
- The Bianchi commutativity for Bäcklund-Darboux transformations is also known as a principle for nonlinear superposition.

  D. Levi, J. Phys. A **14** (1981) 1083–1098.

- The classifications of elementary Darboux transforms can be used as a tool to classify discrete systems related to a given Lax operator. These discrete systems will have Lax pairs provided by the set of two consistent Darboux transformations.
- The corresponding Bäcklund transformations will represent symmetries of the discrete (difference systems).

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# Noncommutative NLS equations

- *osp*(1|2)-invariant SUSY NLS models:

$$\begin{aligned} \mathrm{i}u_t + u_{xx} - 2u^\dagger u u - \Psi^\dagger \Psi u + \mathrm{i}\Psi\Psi_x = 0 \\ \mathrm{i}\Psi_t + \Psi_{xx} - u^\dagger u \Psi + \mathrm{i}(2u\Psi_x^\dagger + \Psi^\dagger u_x) = 0 \end{aligned},$$

Kulish P. P., *Quantum osp(1|2)-invariant nonlinear Schrödinger equation*, ICTP Preprint IC/85/39, Trieste (1985).

- $u$, $\Psi$ – smooth functions taking values in an infinite-dimensional Grassmann algebra

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1.$$

- The variables $u$ are called commuting (Bosonic) variables: $u_1 u_2 = u_2 u_1,\ u_1, u_2 \in \mathcal{G}_0$
- the variables $\Psi$ are called anti-commuting (Fermionic) ones: $\Psi_1\Psi_2 = -\Psi_2\Psi_1,\ \Psi_1, \Psi_2 \in \mathcal{G}_1$.

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# Grassmann Algebras

- Let $\mathcal{G}$ be a $\mathbb{Z}_2$-graded algebra over a field $K$ of characteristics zero.
- $\mathcal{G}$ as a linear space is a direct sum $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$, such that

$$\mathcal{G}_i \mathcal{G}_j \subseteq \mathcal{G}_{i+j} \pmod 2.$$

- Those elements of $\mathcal{G}$ that belong either to $\mathcal{G}_0$ or to $\mathcal{G}_1$ are called homogeneous, $\mathcal{G}_0$ – even, $\mathcal{G}_1$ – odd.
- The parity $|a|$ of an even homogeneous element $a$ is 0 and it is 1 for odd homogeneous elements: $|ab| = |a| + |b|$.
- Grassmann commutativity means that $ba = (-1)^{|a||b|} ab$ for any homogenous elements $a$ and $b$. In particular, $a_1^2 = 0$, for all $a_1 \in \mathcal{G}_1$.

F. A. Berezin, *Introduction to superanalysis*, D. Reidel Publishing, Dordrecht/Boston/Lancaster/Tokyo (1987).

D. A. Leites (ed), *Seminar on Supersymmetry*, Independent University Press, Moscow (2011) [in Russian].

# Grassmann NLS equation: Lax pair

- Consider a Lax operator of the form

$$L = \partial_x + U - \lambda h,$$

- the matrix $U$ has entries in a Grassmann algebra:

$$U = \begin{pmatrix} 0 & \psi & 2q \\ -\varkappa & 0 & \zeta \\ 2p & \phi & 0 \end{pmatrix}, \qquad h = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

- $p$ and $q$ are even elements of $\mathcal{G}$, $\zeta$, $\varkappa$, $\phi$ and $\psi$ –odd homogeneous elements; $\lambda \in \mathbb{C}$ – spectral parameter (even variable).
- We will be using the natural grading $U_{ij} \in \mathcal{G}_{i+j}$ (mod 2).

# Grassmann NLS equation: Zero Curvature Representation

- The zero curvature condition $[L, A] = 0$ gives:

$$\begin{aligned}
q_t &= -q_{xx} + \psi_x\zeta - \psi\zeta_x - 2q(\psi\varkappa - \phi\zeta) + 8q^2 p \\
p_t &= p_{xx} + \phi_x\varkappa - \phi\varkappa_x + 2p(\psi\varkappa - \phi\zeta) - 8p^2 q \\
\psi_t &= \psi_{xx} - q_x\phi - 2q\phi_x + 2pq\psi + \psi\phi\zeta \\
\zeta_t &= \zeta_{xx} - q_x\varkappa - 2q\varkappa_x + 2pq\zeta - \psi\varkappa\zeta \\
\varkappa_t &= -\varkappa_{xx} + p_x\zeta + 2p\zeta_x + 2pq\varkappa - \phi\varkappa\zeta \\
\phi_t &= -\phi_{xx} + p_x\psi + 2p\psi_x + 2pq\phi + \psi\phi\varkappa
\end{aligned}$$

- The second Lax operator $A$ is of the form:

$$A = \partial_t + V_0 + \lambda U - \lambda^2 h,$$

$$V_0 = \mathrm{ad}_h^{-1} U_x + \begin{pmatrix} 4pq - 2\psi\varkappa & 0 & 0 \\ 0 & -4pq - 2\phi\zeta & 0 \\ 0 & 0 & 2(\phi\zeta + \psi\varkappa) \end{pmatrix}.$$

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# Grassmann NLS equation: reductions and integrals of motion

- The reduction $\psi = \zeta = \phi^\dagger = \varkappa^\dagger$ and $p = q^\dagger$ leads to a system which after a re-scaling and a point transformation $t \to \mathrm{i}t$, $x \to \mathrm{i}x$ leads to the Kulish model.
- It can be shown that our Grassmann NLS model is a completely integrable Hamiltonian system.
- The first three constants of motion are of the form:

$$
\begin{aligned}
\mathcal{N} &= \int_{-\infty}^{\infty} \mathrm{d}x \ \{4pq + \phi\psi + \varkappa\zeta\}; \\
\mathcal{P} &= \int_{-\infty}^{\infty} \mathrm{d}x \ \{-2pq - \phi\psi_x - \varkappa\zeta_x - \phi_x\psi - \varkappa_x\zeta\}; \\
\mathcal{H} &= \int_{-\infty}^{\infty} \mathrm{d}x \ \{2p_x q_x + \phi_x\psi_x + \varkappa_x\zeta_x + 2p^2q^2 + pq(\phi\psi + \varkappa\zeta) \\
&\qquad\qquad -q(\phi\phi_x + \varkappa\varkappa_x) - p(\psi\psi_x + \zeta\zeta_x)\}.
\end{aligned}
$$

- $\mathcal{N}$ – "total number of particles"; $\mathcal{P}$ – "total momentum"; $\mathcal{H}$ – the Hamiltonian of the system.

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# Darboux transformations

- By a Darboux transformation we understand a map

$$L \to L_1 = MLM^{-1}$$

where the Lax operator $L_1$ has an updated potential $U_1$:

$$L_1 = \partial_x + U_1 - \lambda h, \qquad U_1 = \begin{pmatrix} 0 & \psi_1 & 2q_1 \\ -\varkappa_1 & 0 & \zeta_1 \\ 2p_1 & \phi_1 & 0 \end{pmatrix}.$$

- Here $M$ and $M_{ij} \in \mathcal{G}_{i+j}$ (mod 2) are rational functions of $\lambda$ and differentiable functions of $x$.
- Dressing equation: $M_x + U_1 M - MU = 0$.
- A composition of Darboux transformations is again a Darboux transformation with more complicated rational dependence in $\lambda$.

# Elementary Darboux transformations

- We are interested in elementary Darboux transformations which cannot be decomposed further. Thus, we restrict ourselves by linear in $\lambda$ Darboux matrices:

$$M = M_0 + \lambda M_1.$$

- The substitution of $M$ in Dressing equation results in:

$$\begin{aligned} [h, M_1] &= 0, \\ M_{1,x} + U_1 M_1 - M_1 U + [h, M_0] &= 0, \\ M_{0,x} + U_1 M_0 - M_0 U &= 0. \end{aligned}$$

- Let us consider the simplest case of $\lambda$- independent Darboux transformations ($M_1 = 0$).
  From the second eqn. above, it follows that $M_0$ is a diagonal matrix.
  The third eqn, implies that $M_0$ is a constant diagonal matrix, and that $U_1 M_0 = M_0 U$. The later is nothing but a Lie point symmetry transformation, which does not lead to non-trivial results.

# Elementary Darboux transformations ... (cont'd)

- If $M_1 \neq 0$, then it follows that $M_1$ is a diagonal matrix: $M_1 = \mathrm{diag}\,(\alpha, \beta, \gamma)$.
- Furthermore, the second equation implies that $\alpha$, $\beta$ and $\gamma$ are constants.
- We will describe the elementary Darboux transformations for the special case when the matrix $M_1$ has rank one and $\alpha = 1$, $\beta = 0$, $\gamma = 0$. In this case it followsthat $M_{0,22}$ and $M_{0,33}$ are constants.
- Further analysis shows that there are two essentially different cases: (1) $M_{0,22} = 0$ and $M_{0,33} = 1$; (2) $M_{0,22} = M_{0,33} = 1$.
  For a sake of convenience, from now on, we will denote the matrix element $M_{11}$ by $F$.

# Case 1: $M_{22} = 1$ and $M_{33} = 0$

- The $\lambda$-term of the compatibility condition gives:

$$M_{12} = \psi, \quad M_{21} = -\varkappa_1, \quad M_{13} = p, \quad M_{31} = p_1, \quad M_{23} = M_{32} = 0.$$

- Therefore, the Darboux matrix $M$ takes the form

$$M = \begin{pmatrix} F + \lambda & \psi & q \\ -\varkappa_1 & 1 & 0 \\ p_1 & 0 & 0 \end{pmatrix}.$$

- The $\lambda$-independent term leads to the set of algebraic constrains:

$$\phi_1 = -p_1\psi, \qquad \zeta = -q\varkappa_1, \qquad p_1 q = 1.$$

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# Case 1: $M_{22} = 1$ and $M_{33} = 0$ ... (cont'd)

- Dressing chain of equations:

$$q_x = -(\psi \varkappa_1 + 2F)q, \quad F_x = 2\left(\frac{q_1}{q} - \frac{q}{q_{-1}}\right) + \psi\varkappa - \psi_1\varkappa_1,$$
$$\psi_x = \psi_1 - \psi F + \frac{q}{q_{-1}}\psi_{-1}, \quad \varkappa_{1,x} = -\varkappa + \varkappa_1 F + \frac{q_1}{q}\varkappa_2.$$

- Introduce new variables $v$, $\phi$ and $\psi$ (forward $v_1$ and backward $v_{-1}$ shifts) $q = \mathrm{e}^{v}, p = \mathrm{e}^{v_{-1}}, \psi = \eta \mathrm{e}^{v/2}, \varkappa_1 = \varphi \mathrm{e}^{-v/2}$, one can eliminate the function $F$ and cast the dressing chain into:

$$\begin{aligned} v_{xx} &= 4\left(\mathrm{e}^{v_1 - v} - \mathrm{e}^{v - v_{-1}}\right) + (\varphi\eta_{-1} + \varphi_{-1}\eta)\mathrm{e}^{(v - v_{-1})/2} \\ &\quad - (\varphi\eta_1 + \varphi_1\eta)\mathrm{e}^{(v_1 - v)/2}, \\ \varphi_x &= \varphi_1 \mathrm{e}^{(v_1 - v)/2} + \varphi_{-1}\mathrm{e}^{(v - v_{-1})/2}, \\ \eta_x &= -\eta_1 \mathrm{e}^{(v_1 - v)/2} - \eta_{-1}\mathrm{e}^{(v - v_{-1})/2}. \end{aligned}$$

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## Case 1: $M_{22} = 1$ and $M_{33} = 0$ ... (cont'd)

- The above system is an integrable noncommutative extension of the Toda chain: the reduction $\xi = \eta = 0$ leads to the standard Toda chain:

$$v_{xx} = 4\mathrm{e}^{v_1 - v} - 4\mathrm{e}^{v - v_{-1}}.$$

- The system also has a Lagrangian formulation with a Lagrangian:

$$\mathcal{L}(v, \xi, \eta) = \int \mathrm{d}x$$
$$\times \left( \frac{v_x^2}{2} - 4\mathrm{e}^{v - v_{-1}} + 2(\varphi\eta_{-1} + \varphi_{-1}\eta)\mathrm{e}^{(v - v_{-1})/2} + \varphi\eta_x - \varphi_x\eta \right).$$

# Case 2: $M_{22} = 1$, $M_{33} = 1$

- Here we have$M_{12} = \psi$, $M_{21} = -\varkappa_1$, $M_{13} = q$, $M_{31} = p_1$ and $M_{23} = M_{32} = 0$.
- Due to Abel's theorem, the Wronskian does not depend on $x$ (since the potential $U$ is a traceless matrix) and thus

$$(F - p_1 q + \psi\varkappa_1)_x = 0 \quad \rightarrow \quad F = p_1 q - \psi\varkappa_1 + \mu.$$

- As a result, the Darboux matrix $M$ takes the form

$$M = \begin{pmatrix} \mu + p_1 q - \psi\varkappa_1 + \lambda & \psi & q \\ -\varkappa_1 & 1 & 0 \\ p_1 & 0 & 1 \end{pmatrix}.$$

- Algebraic constraints:

$$(1 - T)\zeta = (q\varkappa_1), \qquad (1 - T)\phi = (p_1\psi).$$

## Case 2: $M_{22} = 1$, $M_{33} = 1$ ... (cont'd)

- Dressing chain equations:

$$\begin{aligned} q_x &= 2q(\psi\varkappa_1 - p_1 q - \mu) + 2q_1 - (1 - T)^{-1}(q\varkappa_1)\psi \\ p_x &= -2p(\psi_{-1}\varkappa - pq_{-1} - \mu) - 2p_{-1} - (1 - T)^{-1}(p_1\psi)\varkappa \\ \psi_x &= \psi_1 - q(1 - T)^{-1}(p_1\psi) - (\mu + p_1 q)\psi \\ \varkappa_x &= -\varkappa_{-1} - p(1 - T)^{-1}(q\varkappa_1) + (\mu + pq_{-1})\varkappa \end{aligned}$$

- Here $T$ is the shift operator: $U_1 = TU = \mathrm{Ad}_{M_1} U$.
- The presence of the operator $(1 - T)^{-1}$ in the dressing chain leads to a non-local dressing chain.
- It can be rewritten into a local form but will lead to non-evolutionary dressing chain equations for the odd variables.
- In the Bosonic limit, it reduces to the standard NLS dressing chain:

$$q_x = -2q(p_1 q + \mu) + 2q_1, \quad p_{1,x} = 2p_1(p_1 q + \mu) - 2p$$

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# Bianchi Commutativity

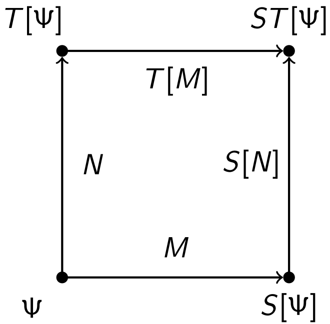
- Discrete systems appear as consistency conditions of two Darboux matrices $M$ and $N$ around a square (the Bianchi commutativity).
- Introduce lattice variables $(k, m)$ $(k, m \in \mathbb{Z})$: generic even $v_{k,m}$ and odd variables $\tau_{k,m}$ are defined on an integer lattice $\mathbb{Z} \times \mathbb{Z}$.
- Introduce the shift operators $S$ and $T$. For example

$$Sq_{k,m} = q_{k+1,m}, \qquad T\zeta_{k,m} = \tau_{k,m+1}, \qquad TS = ST.$$

- Consider two Darboux transformations $M(\lambda)$ and $N(\lambda)$. On the space of fundamental solutions $\{\Psi\}$ of $L(\lambda)$ they act as follows:

$$S[\Psi(\lambda)] = M(\lambda)\Psi(\lambda), \qquad T[\Psi(\lambda)] = N(\lambda)\Psi(\lambda).$$

# Bianchi Commutativity ... (cont'd)




- The compatibility of the transformations $[S, T] = 0$ implies

$$S[N(\lambda)]M(\lambda) = T[M(\lambda)]N(\lambda),$$

  and leads to a set of algebraic relations between $U$, $S[U]$, $T[U]$ and $TS[U]$.

- In this setting, Darboux transformations can be considered as a discrete Lax pair associated with $L(\lambda)$.
- The system (coming from Bianchi commutativity) is an integrable discretisation of the hierarchy of $L(\lambda)$.

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# Bianchi Commutativity ... (cont'd)

- The differential equations for the Bäcklund transformation, coming from the derivation of the Darboux matrices can be considered as symmetries of the difference system.
- We will describe the set of integrable discretisations obtained from imposing a consistency of two elementary Darboux transformations.

# Case A

- Consider two Darboux matrices of the type described in case 2:

$$M = \begin{pmatrix} \mu + p_{10} q - \psi \varkappa_{10} + \lambda & \psi & q \\ -\varkappa_{10} & 1 & 0 \\ p_{10} & 0 & 1 \end{pmatrix};$$

$$N = \begin{pmatrix} \nu + p_{01} q - \psi \varkappa_{01} + \lambda & \psi & q \\ -\varkappa_{01} & 1 & 0 \\ p_{01} & 0 & 1 \end{pmatrix},$$

The consistency condition leads to:

$$p_{01} - p_{10} = \frac{\mu - \nu}{(1 + p_{11} q)^2} (1 + p_{11} q + \psi \varkappa_{11}) p_{11},$$

$$q_{01} - q_{10} = -\frac{\mu - \nu}{(1 + p_{11} q)^2} (1 + p_{11} q + \psi \varkappa_{11}) q,$$

$$\varkappa_{01} - \varkappa_{10} = \frac{\mu - \nu}{1 + p_{11} q} \varkappa_{11}, \quad \psi_{01} - \psi_{10} = -\frac{\mu - \nu}{1 + p_{11} q} \psi.$$

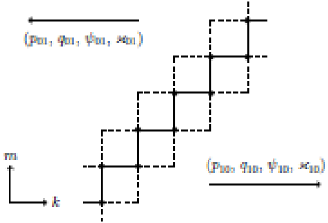
# Case A ... (cont'd)

- If all odd variables vanish, this system of difference equations reduced to a familiar two-component system:

$$p_{01} - p_{10} = \frac{(\mu - \nu)p_{11}}{1 + p_{11}q}, \quad q_{01} - q_{10} = -\frac{(\mu - \nu)q}{1 + p_{11}q}.$$


V. E. Adler, Physica D **73** (1994) 335–351.

V. E. Adler. Phys. Lett. A **190** (1994) 53–58.


- One can pose an initial value problem with initial conditions on a staircase. For a given set of initial data on the staircase, a solution of the difference system can be found recursively.

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# Case A ... (cont'd)

- One can define the Elimination map and express any variable on the lattice in terms of a finite subset of the initial set of variables on the staircase.

  A. V. Mikhailov, J. P. Wang and P. Xenitidis, Nonlinearity **24** (2011) 2079–2097.

- It is clear, that these expressions are rational functions of the even initial variables and multi-linear function of the odd ones.

# Case B

- Combine two Darboux transformations of types 1 and 2:

$$M = \begin{pmatrix} \mu + p_{10} q - \psi \varkappa_{10} + \lambda & \psi & q \\ -\varkappa_{10} & 1 & 0 \\ p_{10} & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} F + \lambda & \psi & q \\ -\varkappa_{01} & 1 & 0 \\ p_{01} & 0 & 0 \end{pmatrix};$$

- The Bianchi commutativity gives the quadrilateral system:

$$\begin{aligned} p_{01} &= (\mu - F + p_{10} q - \psi \varkappa_{10}) p_{11} \\ q_{10} &= (\mu - F_{10} + p_{11} q_{01} - \psi_{01} \varkappa_{11}) q \\ \psi_{01} - \psi_{10} &= -(\mu - F_{10} + p_{11} q_{01} - \psi_{01} \varkappa_{11}) \psi \\ \varkappa_{01} - \varkappa_{10} &= (\mu - F + p_{10} q - \psi \varkappa_{10}) \varkappa_{11} \\ F(\mu + p_{11} q_{01} - \psi_{01} \varkappa_{11}) &= F_{10}(\mu + p_{10} q - \psi \varkappa_{10}) \\ &\quad + p_{10} q_{10} - p_{01} q_{01} - \psi_{10} \varkappa_{10} + \psi_{01} \varkappa_{01}, \end{aligned}$$

and the condition: $p_{01} q = 1$ which enables us to eliminate $p$.

# Case B ... (cont'd)

- One can solve the last system with respect to $F$ and its shift $F_{10}$:

$$F = \mu - \frac{q_{10}}{q} + \frac{q}{q_{1,-1}} - \psi \varkappa_{10}, \qquad F_{10} = \mu - \frac{q_{01}}{q} + \frac{q_{01}}{q_{10}} - \psi_{01} \varkappa_{11}.$$

- Then, one can eliminate $F$. The compatibility condition $S(F) = F_{10}$ read:

$$\frac{q}{q_{-1,0}} + \frac{q}{q_{1,-1}} - \frac{q_{-1,1}}{q} - \frac{q_{1,0}}{q} + \psi_{-1,1}\xi - \psi\xi_{1,-1} + \mu - \mu_1 = 0,$$

$$\psi_{1,0} - \psi_{01} = \frac{q_{10}}{q}\psi,$$

$$\xi - \xi_{1,-1} = \frac{q_{10}}{q}\xi_{10}.$$

Here $\xi = \varkappa_{0,1}$.

# Case B ... (cont'd)

- After setting $q = \mathrm{e}^{v}$, where $v$ is an even variable, one can easily recognize a non-commutative extension of the fully discrete Toda chain:

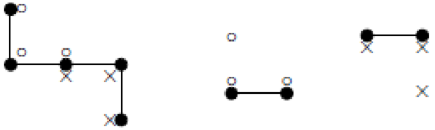
$$\begin{aligned} \mathrm{e}^{v_{1,-1}-v} + \mathrm{e}^{v_{-1,0}-v} - \mathrm{e}^{v-v_{1,0}} - \mathrm{e}^{v-v_{-1,1}} \\ +\psi_{-1,1}\xi - \psi\xi_{1,-1} + \mu_1 - \mu = 0, \\ \psi_{1,0} - \psi_{01} = \mathrm{e}^{v_{1,0}-v}\psi, \\ \xi - \xi_{1,-1} = \mathrm{e}^{v_{1,0}-v}\xi_{10}. \end{aligned}$$

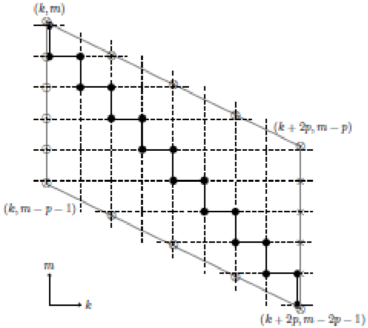
- In the special case when all anti-commuting variables vanish, it reduces to the discrete Toda lattice:

$$\mathrm{e}^{v_{1,-1}-v} + \mathrm{e}^{v_{-1,0}-v} - \mathrm{e}^{v-v_{1,0}} - \mathrm{e}^{v-v_{-1,1}} + \mu_1 - \mu = 0. \qquad (1)$$

# Case B ... (cont'd)

- On the lattice each equation can be represented by a graph:






For the commutative Toda lattice one can solve an initial value problem with initial data given on the staircase $W_0$:

$$W_0 = \{(k+n, m-n),\ (k+n, m-n-1) \ :\ n \in \{0, \dots, 2p\}\}.$$

# Case B ... (cont'd)

- In the case of noncommutative equations one also needs to define boundary odd variables. Taking some $p \in \mathbb{N}$ we define a parallelogram $W$ with boundaries

$$
\begin{aligned}
W_1 &= \{(k+2n, m-n), (k+2n, m-n-p-1) \,|\, n \in \{0, \dots, p\}\} \\
W_2 &= \{(k, m-n) \,|\, n \in \{0, \dots, p+1\}\}, \\
W_3 &= \{(k+2p, m-n-p) \,|\, n \in \{0, \dots, p+1\}\}.
\end{aligned}
$$

- The set of boundary variables $\psi_{km}^{(0)}$ are defined on $W_1 \cup W_2$ and the boundary variables $\xi_{km}^{(0)}$ are defined on $W_1 \cup W_3$.
- The variables on the boundary $W_1 \cup W_2 \cup W_3$ and inside $W$ can be expressed as rational functions of the even variables given on the staircase $W_0$ inscribed into the parallelogram $W$ and multi-linear functions of the odd boundary variables.

# Case B ... (cont'd)

- Indeed, the system with such initial boundary conditions can be solved by a finite sequence of iterations.
- For the first iteration we set all odd variables inside the parallelogram to zero and find the first approximation of even variables for all points of $W$.
- Then, using the boundary conditions for odd variables, one can solve the full system to update the values of odd variables inside $W$.
- Starting from these data, we repeat the sequence of iterations. This sequence will stabilise after a finite number of steps since the solution $(q_{k_1,m_1}, \psi_{k_1,m_1}, \xi_{k_1,m_1})$, $(k_1, m_1) \in W$ is a multi-linear function of the odd boundary data.

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# Summary

- We studied integrable difference equations associated with Grassmann extensions of the nonlinear Schrödinger equation.
- We constructed two elementary Darboux transformations. As a result, new Grassmann generalisations of the Toda lattice and the NLS dressing chain are obtained.
- We obtained difference integrable systems as a compatibility (Bianchi commutativity) of these Darboux transformations. Such systems can be viewed as Grassmamm generalisations of the difference Toda and NLS equations.
- The $osp(1|2)$-invariant supersymmetric NLS model of Kulish can be obtained by imposing the reduction $p = q^\dagger$ and $\psi = \zeta = \phi^\dagger = \varkappa^\dagger$.
- Our Darboux transformations is not applicable to the Kulish system, because they do not respect the reduction to $osp(1|2)$ superalgebra.

# Open problems

The results obtained here can be developed in several directions:

- to study the corresponding Yang-Baxter maps;
- To derive the recursion operators and study the associates multi-Hamiltonian structures;
- This can be generalised to other integrable hierarchies.

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# Thank you!

G.Grahovski@leeds.ac.uk grah@inrne.bas.bg

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