A Recursion Operator for the Geodesic Flow on n-dimensional Sphere

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- Liouville proved that a system with n degrees of freedom is integrable by quadratures when there exist n independent first integrals in involution (cf. [1]).
- In classical mechanics, *a completely integrable system* in the sense of Liouville are called simply *an integrable system*.

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- The integrable system is characterized by the recursion operator $\boldsymbol{T}$ in the Hamiltonian dynamical system on the cotangent vector bundle $\boldsymbol{T^*\mathcal{M}}$ of a manifold $\boldsymbol{\mathcal{M}}$.
  $\boldsymbol{T}$ is a diagonalizable $\mathbf{(1,1)}$-tensor field which satisfies certain conditions. In particular, the recursion operator is written in the following form if we choose an action-angle variables $\boldsymbol{(J_k,\ \varphi^k)}$:

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$$T = \sum_k \lambda^k(J_k)\left(\frac{\partial}{\partial J_k} \otimes dJ_k + \frac{\partial}{\partial \varphi^k} \otimes d\varphi^k\right),^{(*)}$$

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- Functionally independent constants of the motion are obtained by taking each trace of T^k:

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Definition (Endomorphisms)

Let T be a $(1,1)$-tensor field on a manifold $\mathcal{M}$ and we write T : $(1,1)$-tensor $s.t.$

$$T = \sum_{i,j=1}^{n} T_i^{\ j} dx^i \otimes \frac{\partial}{\partial x^j}.$$

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Then we define endomorphisms $\hat{T}$ and $\check{T}$ by:

$$\hat{T} : T_p\mathcal{M} \ni X \mapsto \hat{T}X \in T_p\mathcal{M}, \quad \hat{T}X = \sum_{i,j=1}^{n} T_i^{\ j} X^i \frac{\partial}{\partial x^j},$$

$$\check{T} : T_p^*\mathcal{M} \ni \alpha \mapsto \check{T}\alpha \in T_p^*\mathcal{M}, \quad \check{T}\alpha = \sum_{i,j=1}^{n} \alpha_j T_i^{\ j} dx^i,$$

where a vector field X and a 1-form α $s.t.$

$$X = \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k}, \quad \alpha = \sum_{k=1}^{n} \alpha_k dx^k.$$

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A dynamical vector field Δ is said to be *separable on an open subset* $\mathcal{O} \subseteq \mathcal{M}$ when there exists a basis $\{e_i\}$ of local vector fields on $\mathcal{O}$ $s.t.$

$$\mathcal{L}_{e_i}\langle \Delta, \vartheta^j \rangle \neq 0 \Rightarrow i = j,$$

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where $\{\vartheta^j\}$ is the dual basis of $\{e_i\}$. If $\mathcal{O} = \mathcal{M}$, we call Δ is separable.

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Let a vector field $\boldsymbol{\Delta}$ is a dynamical vector field on a $\boldsymbol{2n}$-dimensional manifold $\boldsymbol{\mathcal{M}}$ and suppose $\boldsymbol{\Delta}$ admits a diagonalizable $\mathbf{(1,1)}$-tensor field $\boldsymbol{T}$ $\boldsymbol{s.t.}$

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Then, the vector field $\boldsymbol{\Delta}$ is separable, completely integrable and Hamiltonian with respect to a certain symplectic structure.

This $\boldsymbol{T}$ is called a recursion operator of the vector field $\boldsymbol{\Delta}$.

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4. Construction of a recursion operator for the geodesic flow on n-dimensional sphere

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And, the constants of motion is written by the trace of T^k:

$$\{Tr(T),\ Tr(T^2),\ \ldots,\ Tr(T^n)\}.$$

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1. **Considering the canonical Riemaniann metric on S^n**

Using the spherical polar coordinate for an n-dimensional sphere of radius a, we consider an embedding ϕ to the sphere:

$$\phi(q^1, \ldots, q^n) = \begin{pmatrix} a\cos q_1 \\ a\sin q_1 \cos q_2 \\ \cdots\cdots \\ a\sin q_1 \cdots \sin q_{n-2}\cos q_{n-1} \\ a\sin q_1 \cdots \sin q_{n-2}\sin q_{n-1} \end{pmatrix}.$$

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We see that

$$g_{ij} = \rho_i^2 \delta_{ij}, \quad \left(i,j = 1, \ldots, n,\ \rho_1 = a,\ \rho_\ell = a \prod_{k=1}^{\ell-1} \sin q_k,\ (\ell = 2, \ldots, n) \right).$$

4. Construction of a recursion op. for the geodesic flow on S^n - 3

2. **Calculating the Hamiltonian H from the metric**

② **Calculating the Hamiltonian H from the metric**

The corresponding Hamiltonian function H is

$$H(q,p) = \frac{1}{2a^2}\sum_{k=1}^{n} P_k \cdot p_k^2, \quad P_k = \begin{cases} 1, & (k=1), \\ \displaystyle\prod_{i=1}^{k-1} \frac{1}{\sin^2 q_i}, & (otherwise). \end{cases} \tag{1}$$

4. Construction of a recursion op. for the geodesic flow on S^n - 4

3. **Discribing the Hamiltonian system $(H,\ \Delta,\ \omega)$ - 1**

❸ **Discribing the Hamiltonian system (H, Δ, ω) - 1**

The Hamilton-Jacobi equation for (1) is

$$E = \frac{1}{2a^2} \sum_{k=1}^{n} P_k \left(\frac{dS_k}{dq_k} \right)^2, \quad S = \sum_{i=1}^{n} S_i(q_i),$$

where S is generating function.

4. Construction of a recursion op. for the geodesic flow on S^n - 5

3 **Discribing the Hamiltonian system $(H,\ \Delta,\ \omega)$ - 2**

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By the variable transformation, we assume the following:

$$Q_\ell := \left\{ R_\ell - \left(\frac{dS_\ell}{dq_\ell} \right)^2 \right\} \sin^2 q_\ell = \sum_{k=1}^{n-\ell} P_k \left(\frac{dS_{\ell+k}}{dq_{\ell+k}} \right)^2 ,$$

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Q_ℓ ,E and a are constants, so we can set $\alpha_1 := \sqrt{2a^2 E}$ and $\alpha_\ell := \sqrt{Q_{\ell-1}}$, therefore,

$$p_\ell = \frac{dS_\ell}{dq_\ell} = \begin{cases} \sqrt{\alpha_\ell^2 - \dfrac{\alpha_{\ell+1}^2}{\sin^2 q_\ell}}, & (\ell = 1, \ldots, n-1), \\ \alpha_\ell, & (\ell = n). \end{cases} \tag{2}$$

4. Construction of a recursion op. for the geodesic flow on S^n - 6

3. **Discribing the Hamiltonian system $(H,\ \Delta,\ \omega)$ - 3**

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Then, let $J_\ell := \dfrac{1}{2\pi}\oint p_\ell dq_\ell$, we obtain the action variables $J_\ell(q, p)$:

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hence

$$\alpha_\ell = \sum_{k=\ell}^{n} J_k. \tag{3}$$

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and the symplectic form ω is

$$\omega = \sum_{i=1}^{n} dJ_i \wedge d\varphi^i. \tag{4}$$

From the above,

4. Construction of a recursion op. for the geodesic flow on S^n - 8

From the above, the tensor field $\boldsymbol{T}$ defined by

$$\boldsymbol{T} = \frac{1}{2} \sum_{i,\ell} \left\{ \left({}^t\mathcal{S} \right)^{\ell}{}_{i} \frac{\partial}{\partial J_i} \otimes dJ_\ell + \mathcal{S}^{i}{}_{\ell} \frac{\partial}{\partial \varphi^i} \otimes d\varphi^\ell \right\} \tag{5}$$

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where

$$\begin{cases} \mathcal{S}^{1}{}_{1} = J_1, & \\ \mathcal{S}^{i}{}_{1} = -\sum_k J_k + J_1 + 2J_i, & (i > 2), \\ \mathcal{S}^{\ell}{}_{\ell} = \sum_k J_k - J_\ell, & (\ell > 2), \\ \mathcal{S}^{i}{}_{\ell} = J_\ell & (otherwise) \end{cases} \tag{6}$$

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$$\begin{cases} \mathcal{S}^{1}{}_{1} = J_1, \\ \mathcal{S}^{i}{}_{1} = -\sum_k J_k + J_1 + 2J_i, & (i > 2), \\ \mathcal{S}^{\ell}{}_{\ell} = \sum_k J_k - J_\ell, & (\ell > 2), \\ \mathcal{S}^{i}{}_{\ell} = J_\ell & (otherwise) \end{cases} \tag{6}$$

fullfills the conditions for the recursion operator.

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Proposition (HT)

When we introduced a canonical Riemannian metric $\boldsymbol{g}$ on $\boldsymbol{S^n}$, the geodesic flow of $\boldsymbol{T^*S^n}$ has a recursion operator $\boldsymbol{T}$. $\boldsymbol{T}$ is written by means of action-angle variables $(\boldsymbol{J}(\boldsymbol{q},\boldsymbol{p}),\boldsymbol{\varphi}(\boldsymbol{q},\boldsymbol{p}))$:

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where $(\boldsymbol{q},\boldsymbol{p})$ is a local coordinate system on $\boldsymbol{T^*S^n}$.

4. Construction of a recursion op. for the geodesic flow on S^n - 10

Thus, we obtained the following proposition from the above:

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When we introduced a canonical Riemannian metric g *on* S^n*, the geodesic flow of* T^*S^n *has a recursion operator* T. T *is written by means of action-angle variables* $(J(q,p), \varphi(q,p))$*:*

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Example (3-dimensional case)

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Example (3-dimensional case)

$$T = \frac{1}{2}\begin{pmatrix} J_1 & J_2 - J_3 & J_3 - J_2 & & & \\ J_2 & J_1 + J_3 & J_2 & & O & \\ J_3 & J_3 & J_1 + J_2 & & & \\ & & & J_1 & J_2 & J_3 \\ & & & J_2 - J_3 & J_1 + J_3 & J_3 \\ O & & & J_3 - J_2 & J_2 & J_1 + J_2 \end{pmatrix}.$$

5. Application of the recursion operator for the geodesic flow on n-dimensional sphere

5. Application of a recursion operator for the geodesic flow on S^n - 1

Constants of the motion

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The constants of motion F_k of the geodesic flow is:

$$F_k = Tr(T^k) = 2\sum_{i=1}^{n} \lambda_i^k, \quad (k = 1, \ldots, n),$$

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$$\begin{cases} F_1 & = 3J_1 + J_2 + J_3 & = \lambda_1 + \lambda_2 + \lambda_3, \\ F_2 & = 3(J_1^2 + J_2^2 + J_3^2) + 2(J_1J_2 - J_2J_3 + J_3J_1) & = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ F_3 & = \cdots\cdots & = \lambda_1^3 + \lambda_2^3 + \lambda_3^3. \end{cases}$$

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Thus, the constants of motion are obtained by traces of the kth powers of T.

5. Application of a recursion operator for the geodesic flow on S^n - 2

The symplectic form ω_1, which is induced by (5) and (6), is written as follows:

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therefore,

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A sequence of Abelian symmetries (**Γ**-scheme[5])

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A sequence of Abelian symmetries ($\boldsymbol{\Gamma}$-scheme[5])

We can set the new vector fields $\boldsymbol{\Delta}_{i+1}$ as follows:

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so that

$$\Delta_{i+1} = (-1)^{i+1}\frac{(i+1)!\,(J_1 + \cdots + J_n)^{i+2}}{2^{i+1}a^2}\left(\frac{\partial}{\partial\varphi_1} + \cdots + \frac{\partial}{\partial\varphi_n}\right), \ (i = 0, \dots, n-1).$$

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The generated Δ_i by the commutator are the Hamiltonian vector field which commute between them:

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5. Application of a recursion operator for the geodesic flow on S^n - 4

In general, from a recursion operator $\boldsymbol{T}$, given by (5) and (6), the commutator generates a sequence of Abelian symmetries between each $\boldsymbol{\Delta_{i+1}}$ $\boldsymbol{s.t.}$

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5. Application of a recursion operator for the geodesic flow on S^n - 4

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Thus, we obtained the following proposition:

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Thus, we obtained the following proposition:

Proposition (1)

*In the **n**-dimensional sphere case, there exists the sequence of Abelian symmetries, generated by* (7), ***s.t.***

$$\begin{cases} \Delta_0 = \Delta = \dfrac{J_1 + \cdots + J_n}{2a^2}\left(\dfrac{\partial}{\partial\varphi_1} + \cdots + \dfrac{\partial}{\partial\varphi_n}\right), \\ \Delta_{i+1} = (-1)^{i+1}\dfrac{(i+1)!\,(J_1 + \cdots + J_n)^{i+2}}{2^{i+1}a^2}\left(\dfrac{\partial}{\partial\varphi_1} + \cdots + \dfrac{\partial}{\partial\varphi_n}\right), (i = 0, \ldots, n-1). \end{cases}$$

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Thank you for your attention!