Group Analysis as a Microscope of Mathematical Modelling

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1. INTRODUCTION

1.1. Mathematical symmetry versus intuitive notion of symmetry in everyday life

Intuitive, or naive, notion of **symmetry** implies a certain **regularity**. It can be understood as **invariance with respect to some group of transformations:**
translations, rotations, reflections, projections etc.
Investigation of mathematical models with respect to symmetry groups is called **GROUP ANALYSIS**.
An intuitive notion of symmetry and a mathematical approach are not identical. A good example is provided by celestial mechanics.

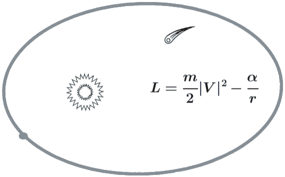


Figure: Kepler's first law: The orbit of a planet is an ellipse with the sun at one focus.

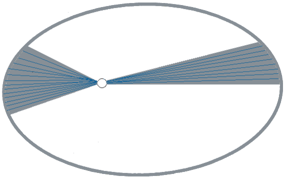


Figure: Kepler's second law: The areas swept out in equal times by the line joining the sun to a planet are equal.

Newton's equation:

$$\frac{d^2\mathbf{x}}{dt^2} + \frac{\alpha}{r^3}\mathbf{x} = 0$$

1.2. Group analysis of Kepler's laws

The first law "elliptic orbits" follows from the symmetry (NHI 1983)

$$\delta \mathbf{x} = \mathbf{x} \times (\mathbf{v} \times \mathbf{a}) + (\mathbf{x} \times \mathbf{v}) \times \mathbf{a}.$$

The second law "equality of areas" follows from the rotation symmetry

$$\delta \mathbf{x} = \mathbf{x} \times \mathbf{a}.$$

1.3. New approach to teaching differential equations and mathematical modelling

The new approach is essentially based on Lie group analysis. It

- **simplifies learning mathematics significantly. E.g. reduces about 1000 types of second-order ODEs usually solved by artificial substitutions to 4 types only!**
- **allows one to solve complicated problems in engineering, biomathematics, etc.**
- **attracts about 10 times more students than teaching with the traditional approach**

1.4. Teaching experience

- **Russia** (Novosibirsk State University, Ufa State Aviation Technical University, Moscow State University, etc.)
- **USA** (Georgia Tech, Standford University)
- **France** (Collége de France)
- **South Africa** (Wits University, University of North-West)
- **Turkey** (Istanbul Technical University)
- **Italy** (University of Catania)
- **Sweden** (Blekinge Institute of Technology)

This approach has been generously supported by educational authorities in several countries, e.g. NRF (National Research Foundation) in South Africa and IVA (Academy of Engineering Sciences) in Sweden. Last year we received a megagrant of the Russian Ministry of Education and organized a research laboratory "Group analysis of mathematical models in natural and engineering sciences."

In our laboratory we do not only use this approach in teaching, students also take an active part in research using group analysis. This year about 30 papers have been prepared for publication.

1.5. Textbook

Ibragimov, N.H. **A practical course in differential equations and mathematical modeling**. Higher Education Press (China), World Scientific (Singapore) and Imperial College Press (London), 2009. Translated into Chinese, Swedish, Russian.

2. GROUP ANALYSIS of ODEs

2.1. Definition of transformation group

Transformation group $G = \{T_a\}$ on (x, y) plane

$$T_a : \quad \bar{x} = \varphi(x, y, a), \quad \bar{y} = \psi(x, y, a). \tag{1}$$

Initial condition

$$T_0 : \quad \varphi(x, y, 0) = x, \quad \psi(x, y, 0) = y. \tag{2}$$

Group property $T_b T_a = T_{a+b}$, i.e.

$$\begin{aligned} \varphi\big(\varphi(x, y, a), \psi(x, y, a), b\big) &= \varphi(x, y, a + b), \\ \psi\big(\varphi(x, y, a), \psi(x, y, a), b\big) &= \psi(x, y, a + b). \end{aligned} \tag{3}$$

Infinitesimal transformation

$$\bar{x} \approx x + \xi(x, y)\, a, \quad \bar{y} \approx y + \eta(x, y)\, a$$

Group generator

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}, \tag{4}$$

Lie equations

$$\begin{aligned} \frac{d\bar{x}}{da} &= \xi(\bar{x}, \bar{y}), \quad \bar{x}\big|_{a=0} = x, \\ \frac{d\bar{y}}{da} &= \eta(\bar{x}, \bar{y}), \quad \bar{y}\big|_{a=0} = y. \end{aligned} \tag{5}$$

Example. Let

$$X = y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}\,.$$

Lie equations (5)

$$\frac{d\bar{x}}{da} = \bar{y}, \qquad \bar{x}\big|_{a=0} = x,$$

$$\frac{d\bar{y}}{da} = -\bar{x}, \quad \bar{y}\big|_{a=0} = y,$$

give the rotation group

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a\,.$$

2.2. Prolongation of group generator

Second-order prolongation

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta_1 \frac{\partial}{\partial y'} + \zeta_2 \frac{\partial}{\partial y''} \tag{6}$$

prolongation formulae

$$\zeta_1 = D_x(\eta) - y' D_x(\xi), \quad \zeta_2 = D_x(\zeta_1) - y'' D_x(\xi), \tag{7}$$

D_x is total differentiation in x :

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + y''' \frac{\partial}{\partial y''} + \cdots .$$

2.3. Definition of symmetry group

G given by (1) is symmetry group of 2nd-order ODE

$$y'' = f(x, y, y') \tag{8}$$

if

$$\bar{y}'' = f(\bar{x}, \bar{y}, \bar{y}').$$

G is also known as a group admitted by Eq. (8). Generator X of G is called **infinitesimal symmetry** of Eq. (8).

Example. Equation

$$y'' + \mathrm{e}^{3y} y'^4 + y'^2 = 0 \tag{9}$$

admits translation group $\bar{x} = x + a, \quad \bar{y} = y$ as well as two other one-parameter groups:

$$\bar{x} = x\mathrm{e}^{3b}, \quad \bar{y} = y + 2b$$

and

$$\bar{x} = x, \quad \bar{y} = \ln\left(\mathrm{e}^{y} + c\right).$$

Their generators are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 3x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}, \quad X_3 = \mathrm{e}^{-y}\frac{\partial}{\partial y} \cdot \tag{10}$$

2.4. Determining equation. Lie algebra

All infinitesimal symmetries for Eq. (8), in particular (10) for Eq. (9), are found by solving the determining equation

$$X\Big(y'' - f(x, y, y')\Big)\Big|_{y''=f} = 0, \qquad (11)$$

where X is the prolonged generator (6). The solutions of Eq. (11) form a **Lie algebra**, i.e. a vector space $L_r =< X_1, \ldots, X_r >$ which is closed under the **commutator**:

$$[X_i, X_j] = c_{ij}^k X_k, \quad c_{ij}^k = \text{const.}$$

2.5. Lie's integration method

I	$[X_1,\ X_2] = 0,\ X_1 \vee X_2 \neq 0$	$X_1 = \dfrac{\partial}{\partial t},\ X_2 = \dfrac{\partial}{\partial u}$
II	$[X_1,\ X_2] = 0,\ X_1 \vee X_2 = 0$	$X_1 = \dfrac{\partial}{\partial u},\ X_2 = t\dfrac{\partial}{\partial u}$
III	$[X_1,\ X_2] = X_1,\ X_1 \vee X_2 \neq 0$	$X_1 = \dfrac{\partial}{\partial u},\ X_2 = t\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial u}$
IV	$[X_1,\ X_2] = X_1,\ X_1 \vee X_2 = 0$	$X_1 = \dfrac{\partial}{\partial u},\ X_2 = u\dfrac{\partial}{\partial u}$

Here $X_1 \vee X_2 = \xi_1\eta_2 - \eta_1\xi_2.$

Canonical variables are obtained solving equations

$$
\begin{aligned}
\textbf{Type I}: &\quad X_1(t) = 1,\ X_2(t) = 0; \quad X_1(u) = 0,\ X_2(u) = 1.\\
\textbf{Type II}: &\quad X_1(t) = 0,\ X_2(t) = 0; \quad X_1(u) = 1,\ X_2(u) = t. \quad (12)\\
\textbf{Type III}: &\quad X_1(t) = 0,\ X_2(t) = t; \quad X_1(u) = 1,\ X_2(u) = u.\\
\textbf{Type IV}: &\quad X_1(t) = 0,\ X_2(t) = 0; \quad X_1(u) = 1,\ X_2(u) = u.
\end{aligned}
$$

Lie's four standard forms of Eqs. (8) admitting L_2

I	$X_1 = \frac{\partial}{\partial t}\,,\ X_2 = \frac{\partial}{\partial u}$	$u'' = f(u')$
II	$X_1 = \frac{\partial}{\partial u}\,,\ X_2 = t\frac{\partial}{\partial u}$	$u'' = f(t)$
III	$X_1 = \frac{\partial}{\partial u}\,,\ X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$	$u'' = \frac{1}{t}f(u')$
IV	$X_1 = \frac{\partial}{\partial u}\,,\ X_2 = u\frac{\partial}{\partial u}$	$u'' = f(t)u'$

Example. Equation

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}$$

admits $L_2 < X_1, X_2 >$

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$$

of Type III. Lie's method gives canonical variables

$$t = \left(\frac{y}{x}\right)^2, \quad u = -\frac{1}{x}$$

and the following standard integrable form of our equation:

$$u'' = -\frac{1}{t}\left(u' + \frac{1}{2}\right)u'$$

Solving the reduced equation and returning to the original variables one obtains the following general solution of the original equation:

$$y = Kx,$$

$$y = \pm\sqrt{2x + Cx^2},$$

$$C_1 y + C_2 x + x \ln\left|C_1 \frac{y}{x} - 1\right| + C_1^2 = 0.$$

The first two formulas define **internal singularities** and the third formula provides **implicit solutions** (next page)

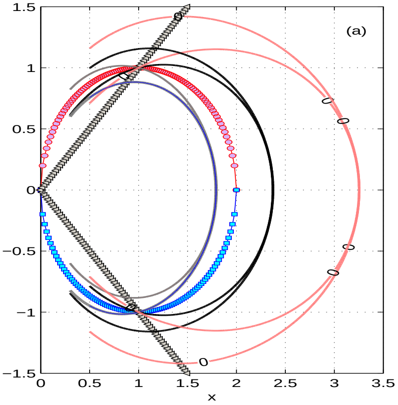


Рис.: Singular and implicit solutions

2.6. Linearization of ODEs

Preamble

$$y'' = 2\left(\frac{y'^2}{y} - \frac{xy'}{1+x^2}\right) \quad \Rrightarrow \quad u'' = 0$$

$$t = \frac{1}{y}\,, \quad u = \arctan x\,.$$

Lie's linearization theorem

$$y'' + F_3(x,y)y'^{\,3} + F_2(x,y)y'^{\,2} + F_1(x,y)y' + F(x,y) = 0 \quad (13)$$

$$
\begin{aligned}
&3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} \\
&= (3F_1F_3 - F_2^2)_x - 3(FF_3)_y - 3F_3F_y + F_2(F_1)_y, \\
&3F_{yy} - 2(F_1)_{xy} + (F_2)_{xx} \\
&= 3(FF_3)_x + (F_1^2 - 3FF_2)_y + 3F(F_3)_x - F_1(F_2)_x.
\end{aligned}
\tag{14}
$$

Example 1. The following equation is not linearizable:

$$
y'' + F(x, y) = 0, \quad F_{yy} \neq 0.
$$

Example 2. $y'' + y'^{\,2} = f(x) \quad \Rrightarrow \quad u'' = f(x)u, \quad u = \mathrm{e}^y.$

2.7. Linear ODEs reducible to algebraic equations

Constant coefficient and Euler's eqs., 1768

$$y'' + Ay' + By = 0, \quad y = \mathrm{e}^{\lambda x}, \tag{15}$$

$$x^2 y'' + (A+1)xy' + By = 0, \quad y = x^{\lambda}. \tag{16}$$

NHI, 2008:

$$\phi^2\, y'' + \big[A + \phi' - 2\sigma\big]\phi\, y' + \big[B - A\sigma + \sigma^2 - \phi\sigma'\big]y = 0, \tag{17}$$

$$\phi = \phi(x),\ \sigma = \sigma(x),\ A, B = \mathrm{const}.$$

$$y = \mathrm{e}^{\int \frac{\sigma(x)+\lambda}{\phi(x)}\, dx} \tag{18}$$

$$\lambda^2 + A\lambda + B = 0. \tag{19}$$

Real roots $\lambda_1 \neq \lambda_2$:

$$y = K_1 \, \mathrm{e}^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)} \, dx} + K_2 \, \mathrm{e}^{\int \frac{\sigma(x)+\lambda_2}{\phi(x)} \, dx} \tag{20}$$

Complex roots $\lambda_1 = \gamma + i\theta, \; \lambda_2 = \gamma - i\theta$:

$$y = \left[K_1 \cos\left(\theta \int \frac{dx}{\phi(x)}\right) + K_2 \sin\left(\theta \int \frac{dx}{\phi(x)}\right)\right] \mathrm{e}^{\int \frac{\sigma(x)+\gamma}{\phi(x)} \, dx} \tag{21}$$

Equal roots $\lambda_1 = \lambda_2$:

$$y = \left[K_1 + K_2 \int \frac{dx}{\phi(x)} \right] \mathrm{e}^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)}\, dx} \qquad (22)$$

Example 1. $\phi(x) = 1,\ \sigma(x) = 0 \Rightarrow$ Eq. (15).
Example 2. $\phi(x) = x,\ \sigma(x) = 0 \Rightarrow$ Eq. (16).
Example 3. $\phi = 1 + x^2,\ \sigma = x,\ A = 0,\ B = \omega^2$

$$\mathrm{Eq.}(17): \quad \left(1 + x^2\right)^2 y'' + (\omega^2 - 1)y = 0$$

Solution (22):

$$y(x) = \sqrt{1 + x^2}\, \left[K_1 \cos\left(\omega \arctan x\right) + K_2 \sin\left(\omega \arctan x\right) \right].$$

2.8. Nonlinear superposition for systems of ODEs

Riccati equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(t) + Q(t)x + R(t)x^2. \qquad (23)$$

Nonlinear superposition

$$\frac{(x - x_2)(x_3 - x_1)}{(x_1 - x)(x_2 - x_3)} = C$$

x_1, x_2, x_3 are particular solutions, x is the particular solution.

Vessiot-Guldberg-Lie theory

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = T_1(t)\xi_1^i(x) + \cdots + T_r(t)\xi_r^i(x), \quad i = 1, \ldots, n, \tag{24}$$

$$X_\alpha = \xi_\alpha^i(x)\frac{\partial}{\partial x^i}, \quad \alpha = 1, \ldots, r,$$

span a Lie algebra L_r, $\quad r < \infty$.
Nonlinear superposition $\quad (nm \geq r)$

$$J_i(x^1, \ldots, x^n;\ x_1^1, \ldots, x_1^n; \ldots,\ x_m^1, \ldots, x_m^n) = C_i, \quad i = 1, \ldots, n,$$

If $r \leqq 2$ Eq. (24) can be integrated by Lie's method.

Example (Arises from laser physics)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy^2 - \frac{x}{2t}, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x^2y - \frac{y}{2t} \tag{25}$$

$$X_1 = xy^2\frac{\partial}{\partial x} + x^2y\frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$r = 2$. Nonlinear superposition $(m = 1)$

$$x^2 - y^2 = C_1(x_1^2 - y_1^2), \quad \ln(x/y) = C_2 + C_1\ln(x_1/y_1).$$

General solution

$$x = \sqrt{\frac{k}{t(1-\zeta^2)}}, \quad y = \zeta\sqrt{\frac{k}{t(1-\zeta^2)}}, \quad \zeta = Ct^k.$$

3. GROUP ANALYSIS of PDEs

3.1. Invariant solutions: an irrigation system

Model

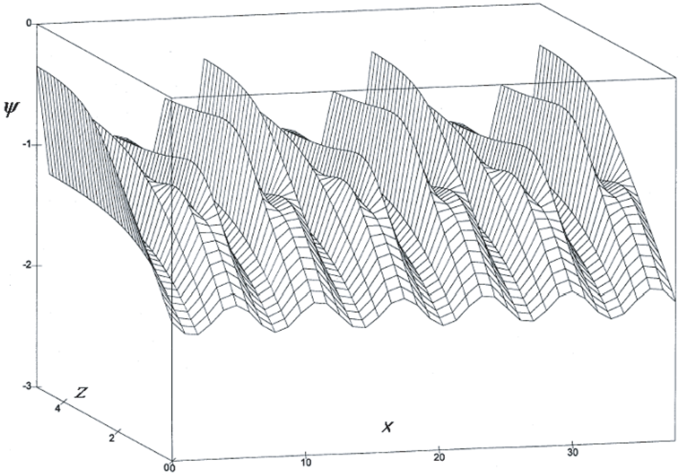
$$C(\psi)\,\psi_t = [K(\psi)\psi_x]_x + [K(\psi)\,(\psi_z - 1)]_z - S(\psi).$$

Group analysis reveals 29 types of irrigation systems, i.e. coefficients $C(\psi), K(\psi), S(\psi)$ with extended symmetries. One of them is

$$\frac{4}{Me^{4\psi} - 1}\,\psi_t = \left(e^{-4\psi}\psi_x\right)_x + \left(e^{-4\psi}\psi_z\right)_z + 4e^{-4\psi}\psi_z + M - e^{-4\psi}.$$

Invariant solution

$\psi = -\frac{1}{4}\,\ln\left|M + \frac{e^{-2z}}{t}\left(2e^{2z}\sin^2 x + l_1 e^z \sin x + l_2\right)\right|.$


ψ
0
-1
-2
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Z
4
2
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0
10
20
30
X

3.2. Invariant solutions: tumour growth model

In healthy tissue, balance is preserved between cellular reproduction and cell death. A change of DNA caused by genetic, chemical or other environmental reasons, can give rise to a malignant tumour cell which disrupts this balance and causes an uncontrolled reproduction of cells followed by infiltration into neighboring or remote tissues (metastasis). Several authors, motivated by observations in tumour biology, suggested a mathematical model

$$u_t = f(u) - (uc_x)_x\,, \quad c_t = -g(c, u), \tag{26}$$

where $f(u) > 0, \quad g_c(c, u) > 0, \quad g_u(c, u) > 0$. Here u and c represent concentrations of invasive cells and extracellular matrix, respectively.

Eqs. (26) with $f = \alpha u, \quad g = u\mathrm{e}^{-c}$ have nontrivial symmetry

$$X = \frac{\partial}{\partial t} + \frac{\partial}{\partial c} + u\frac{\partial}{\partial u} \cdot$$

Using it we (NHI and N. Säfström, *Comm. Nonlin. Sci. Numerical Simulation*, vol. 9(1), 2004) have obtained the invariant solution

$$\begin{aligned} c(t,x) &= t + \ln\left|A_2 \cos\left(\sqrt{1-\alpha}(A_1 - x)\right)\right|, \\ u(t,x) &= -\mathrm{e}^t\left|A_2 \cos\left(\sqrt{1-\alpha}(A_1 - x)\right)\right|, \quad \alpha < 0. \end{aligned} \tag{27}$$

It satisfies the conditions of the problem:

$$f(u) = \alpha u > 0, \quad g_c(c,u) = -u\mathrm{e}^{-c} > 0, \quad g_u(c,u) = \mathrm{e}^{-c} > 0.$$

3.3. Propagation of light in curved space-times

$$V_4 : \quad ds^2 = g_{ij}(x)dx^i dx^j$$

Wave equation in V_4 :

$$g^{ij}(x)u_{,ij} + \frac{1}{6}\,Ru = 0,$$

where $u_{,ij}$ second covariant derivative, R scalar curvature.
Sharp rear front of light if V_4 has **nontrivial** conformal group.
Then wave equation is written

$$u_{tt} - u_{xx} - f(x-t)u_{yy} - 2\varphi(x-t)u_{yz} - u_{zz} = 0, \quad f - \varphi^2 > 0.$$

Cauchy's problem:

$$t = 0: \;\; u = g(x, y, z), \;\; u_t = h(x, y, z).$$

Solution:

$$u = T[h - g_x] + \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) T[g],$$

where the operator T is defined below, e.g.

$$T[g](t, x, y, z) = \frac{1}{4\pi} \int\limits_{x-t}^{x+t} d\xi \int\limits_{0}^{2\pi} g(\xi, y + A\cos\theta, \; z + B\cos\theta + C\sin\theta) d\theta$$

with

$$A = \sqrt{(x + t - \xi)[F(\xi) - F(x - t)]}$$

$$B = \frac{x + t - \xi}{A}\Big[\Phi(\xi) - \Phi(x - t)\Big]$$

$$C = \sqrt{t^2 - (x - \xi)^2 - B^2}$$

and

$$F'(\tau) = f(\tau), \quad \Phi'(\tau) = \varphi(\tau).$$

4. SYMMETRIES AND CONSERVATION LAWS

4.1. Noether's theorem

4.2. Method of nonlinear self-adjointness

4.3. Application in gasdynamics, oceanology, nonlinear optics

4.4. Method of conservation laws for constructing solutions of systems of nonlinear PDEs

4.5. Application of the method of conservation laws to gas flows and diffusion in anisotropic media

You are welcome to our Laboratory in Ufa (Russia) and Research center ALGA in Karlskrona (Sweden) for cooperation.

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Thank you!