GROUP CLASSIFICATION FOR A VARIABLE COEFFICIENT $K(m, n)$ EQUATION

K. Charalambous[1], C. Sophocleous[1],
O.O. Vaneeva[2]

[1]Department of Mathematics and Statistics, University of Cyprus
[2]Institute of Mathematics of National Academy of Sciences of Ukraine

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We consider the class of nonlinear equation

$$u_t + f(t)(u^m)_x + g(t)(u^n)_{xxx} = 0, \quad gn \neq 0, \tag{1}$$

which is of interest in Mathematical Physics. Special cases of this class have been used to model successfully physical situations in a wide range of fields. For example, we have the generalization of the KdV equation

$$u_t + (u^m)_x + \frac{1}{n}(u^n)_{xxx} = 0.$$

Equations of the above type with values of the parameters m and n are denoted by $K(m, n)$.

2. Equivalence Transformations

We call an equivalence transformation of a class of partial differential equations (PDEs), $E(t, x, u) = 0$, an invertible transformation of the variables t, x and u of the form

$$\tilde{t} = Q(t, x, u), \quad \tilde{x} = P(t, x, u), \quad \tilde{u} = R(t, x, u)$$

that maps every equation of the class into an equation of the same form, $E(\tilde{t}, \tilde{x}, \tilde{u}) = 0$. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group. Here we use the direct method to derive the desired equivalence transformations, which was used first by Lie.

We present the equivalence transformations of equation (1) in the next theorems.

Theorem

The usual equivalence group $G^{\sim}$ of class (1) *is formed by the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u,$$
$$\tilde{f} = \frac{\delta_1 \delta_3^{1-m}}{T_t} f, \quad \tilde{g} = \frac{\delta_1^3 \delta_3^{1-n}}{T_t} g, \quad \tilde{n} = n, \quad \tilde{m} = m,$$

where δ_j, $j = 1, 2, 3$, are arbitrary constants with $\delta_1 \delta_3 \neq 0$, $T(t)$ is an arbitrary smooth function with $T_t \neq 0$.

Theorem

If $m=0$ or $m=1$ then there exist additional equivalence transformations. If $m=1$, they have the form

$$\tilde t = T(t), \quad \tilde x = \delta_1\left(x - \int f(t)\,\mathrm{d}t + \delta_2\right), \quad \tilde u = \delta_3 u,$$

$$\tilde g = \frac{\delta_1^3\delta_3^{1-n}}{T_t}\, g, \quad \tilde n = n, \quad \tilde m = 0,$$

where δ_j, $j=1,2,3$, are arbitrary constants with $\delta_1\delta_3\neq 0$, $T(t)$ is an arbitrary smooth function with $T_t\neq 0$.
If $m=0$, they are written as

$$\tilde t = T(t), \quad \tilde x = \delta_1 x + \beta(t), \quad \tilde u = \delta_3 u,$$

$$\tilde g = \frac{\delta_1^3\delta_3^{1-n}}{T_t}\, g, \quad \tilde f = \frac{\beta_t}{T_t}, \quad \tilde n = n, \quad \tilde m = 1,$$

where δ_j, $j=1,2,3$, are arbitrary constants with $\delta_1\delta_3\neq 0$, $T(t)$ and $\beta(t)$ are arbitrary smooth functions with $\beta_t T_t\neq 0$.

A special case of the first Theorem that we have presented is

$$\tilde{t} = \int f(t)\mathrm{dt}, \quad \tilde{x} = x, \quad \tilde{u} = u,$$

that maps equation (1) into

$$\tilde{u}_{\tilde{t}} + (\tilde{u}^m)_{\tilde{x}} + \tilde{g}(\tilde{t})(\tilde{u}^n)_{\tilde{x}\tilde{x}\tilde{x}} = 0, \quad \text{where } \tilde{g}n \neq 0 \text{ and } \tilde{g} = \frac{g}{f}. \qquad (2)$$

Theorem

The usual equivalence group $G^{\sim}$ of class (2) is formed by the transformations

$$\tilde{t} = \delta_1 \delta_3^{1-m} t + \delta_0, \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u,$$
$$\tilde{g} = \delta_1^2 \delta_3^{m-n} g, \quad \tilde{n} = n, \quad \tilde{m} = m,$$

where δ_j, $j = 0, 1, 2, 3$, are arbitrary constants with $\delta_1 \delta_3 \neq 0$.

3. Lie Symmetries for Equation (2)

We have seen that if we use the equivalence transformations equation (1) can be mapped into (2). For this reason we are looking only for the Lie symmetries of equation (2).
From the definition, a PDE, $E(t, x, u, u_t, u_x, \dots) = 0$, possesses a Lie point symmetry,

$$\Gamma = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u,$$

if and only if

$$\Gamma^{(se)} E\Big|_{E=0} = 0$$

where $\Gamma^{(se)}$ means that Γ must be suitably extended. Equation (2) admits Lie point symmetries if and only if

$$\Gamma^{(3)}[u_t + (u^m)_x + g(t)(u^n)_{xxx}] = 0 \tag{3}$$

for $u_t = -(u^m)_x - g(t)(u^n)_{xxx}$.

After we have used the above expression we can eliminate u_t and equation (3) becomes an identity in the variables $u_x,\ u_{xx},\ u_{tx},\ u_{xxx}$ and u_{txx}. From coefficients of different powers of these variables, which must be equal to zero, we derive the determining equations on the coefficients τ, ξ and η. We use the general results on point transformations between evolution equations [*Kingston and Sophocleous 1998] and the forms of the coefficients can be simplified, that is,*

$$\tau = \tau(t) \textit{ and } \xi = \xi(t, x).$$

From the coefficient of u_{xxx} we have that

$$[g_t\tau + g(\tau_t - 3\xi_x)]\, u + (n - 1)g\eta = 0.$$

We deduce that the analysis needs to be split in two cases:

- $n \neq 1$ *and*
- $n = 1$.

$n \neq 1$

In this case the form of η is

$$\eta = -\frac{[g_t\tau + g(\tau_t - 3\xi_x)]u}{(n-1)g}$$

and the coefficients of u_{xx}, u_x^2, u_x and the term independent of derivatives in (3) produce the following determining equations, respectively,

$n(2n+1)g\xi_{xx} = 0$,
(The coefficient of u_{xx} is the same as the coefficient of u_x^2),

$$\begin{aligned}&[m(m-n)g\tau_t - m(3m-n-2)g\xi_x + m(m-1)g_t\tau]\, u^m \\ &-8n(n+1)g^2\xi_{xxx}u^n + (n-1)g\xi_t u = 0,\end{aligned}$$

$$\begin{aligned}&3mg^2\xi_{xx}u^m + 3ng^3\xi_{xxxx}u^n \\ &-\left[g^2\tau_{tt} + gg_t\tau_t + gg_{tt}\tau - g_t^2\tau - 3\xi_{tx}g^2\right] u = 0.\end{aligned}$$

After we have solved the above determining system , we take the forms of $\tau(t)$, $\xi(t,x)$ and the function $g(t)$. The Lie symmetries according the form of $g(t)$ are tabulated in the Table 1.

Table 1 : Classification of equation (2) ($n \neq 1$)

Cases	n	m	$g(t)$	Conditions	Basis of $A^{\max}$
			$n \neq 1$		
1	$\forall$	$\forall$	$\forall$		∂_x
2	$3m-n-2=0$		$\forall$		$\partial_x,\ x\partial_x+\frac{3u}{n-1}\partial_u$
3	$\forall$	0	$\forall$		$\frac{1}{g}\partial_t,\ \partial_x,\ x\partial_x+\frac{3u}{n-1}\partial_u,\ 3\frac{\int g\mathrm{d}t}{g}\partial_t+x\partial_x$
4	$-\frac{1}{2}$	0	$\forall$		$\frac{1}{g}\partial_t,\ \partial_x,\ x\partial_x-2u\partial_u,\ 3\frac{\int g\mathrm{d}t}{g}\partial_t+x\partial_x,$ $x^2\partial_x-4xu\partial_u$
5	$\forall$	$\forall$	constant		$\partial_t,\ \partial_x,\ (3m-n-2)t\partial_t+(m-n)x\partial_x-2u\partial_u$
6	$-\frac{1}{2}$	$-\frac{1}{2}$	constant	 $g>0$ $g<0$	$\partial_t,\ \partial_x,\ 3t\partial_t+2u\partial_u,$ $\sqrt{g}\sin\left(\frac{1}{\sqrt{g}}x\right)\partial_x-2u\cos\left(\frac{1}{\sqrt{g}}x\right)\partial_u,$ $\sqrt{g}\cos\left(\frac{1}{\sqrt{g}}x\right)\partial_x+2u\sin\left(\frac{1}{\sqrt{g}}x\right)\partial_u$ $\sqrt{\lvert g\rvert}e^{\frac{1}{\sqrt{\lvert g\rvert}}x}\partial_x-2ue^{\frac{1}{\sqrt{\lvert g\rvert}}x}\partial_u,$ $\sqrt{\lvert g\rvert}e^{-\frac{1}{\sqrt{\lvert g\rvert}}x}\partial_x+2ue^{-\frac{1}{\sqrt{\lvert g\rvert}}x}\partial_u$
7	$\forall$	$\forall$	t^k	$k \neq 2$	$\partial_x,\ (3m-n-2)t\partial_t+(km-k+m-n)x\partial_x$ $+(k-2)u\partial_u$
8	$3m-n-2=0$		t^k	$k=2$	$\partial_x,\ x\partial_x+\frac{3u}{n-1}\partial_u,\ t\partial_t+x\partial_x$
9	$\forall$	$\forall$	e^{kt}		$\partial_x,\ (3m-n-2)\partial_t+k(m-1)x\partial_x+ku\partial_u$

For the Cases 3 and 4, for which $m=0$, we can introduce a new time $T=\int g\mathrm{d}t$.

$n = 1$

In this case, from the coefficient of $u_x u_{xx}$ we have that $\eta_{uu} = 0$, so

$$\eta = a_1(t,x)u + a_2(t,x).$$

We use the fact that $\tau = \tau(t)$, $\xi = \xi(t,x)$ and the form for η and from (3) we obtain the following determining equations

$$g_t\tau + g(\tau_t - 3\xi_x) = 0,$$

$$a_{1\,x} - \xi_{xx} = 0,$$

$$m\left[\tau_t - \xi_x + (m-1)a_1\right]u^{m+1}$$
$$+m(m-1)a_2u^m + (3ga_{1\,xx} - \xi_t - g\xi_{xxx})u^2 = 0,$$

$$ma_{1\,x}u^{m+1} + ma_{2\,x}u^m + (a_{1\,t} + ga_{1\,xxx})u^2 + (a_{2\,t} + ga_{2\,xxx})u = 0.$$

We solve the system and in Table 2 we present the different forms for the Lie algebra according to the possible forms of the function $g(t)$.

Table 2 : Classification of equation (2) ($n = 1$)

Cases	$g(t)$	Conditions	Basis of $A^{\max}$
		$n = 1$	
		$m \neq 2$	
1	$\forall$		∂_x
2	constant		$\partial_t,\ \partial_x,\ 3(m-1)t\partial_t + (m-1)x\partial_x - 2u\partial_u$
3	t^k		$\partial_x,\ 3(m-1)t\partial_t + (m-1)(k+1)x\partial_x + (k-2)u\partial_u$
4	e^{kt}		$\partial_x,\ 3(m-1)\partial_t + k(m-1)x\partial_x + ku\partial_u$
		$m = 2$	
5	$\forall$		$\partial_x,\ 2t\partial_x + \partial_u$
6	constant		$\partial_t,\ \partial_x,\ 2t\partial_x + \partial_u,\ 3t\partial_t + x\partial_x - 2u\partial_u$
7	t^k	$k \neq 1$	$\partial_x,\ 2t\partial_x + \partial_u,\ 3t\partial_t + (k+1)x\partial_x + (k-2)u\partial_u$
8	t^k	$k = 1$	$\partial_x,\ 2t\partial_x + \partial_u,\ 3t\partial_t + 2x\partial_x - u\partial_u,$ $2t^2\partial_t + 2tx\partial_x + (x-2tu)\partial_u$
9	e^{kt}		$\partial_x,\ 2t\partial_x + \partial_u,\ 3\partial_t + kx\partial_x + ku\partial_u$
10	$g_1(t)$	$p^2 - 4q - 4r^2 \neq 0$	$\partial_x,\ 2t\partial_x + \partial_u,\ 6(t^2 + pt + q)\partial_t + (6t + 2r + 3p)x\partial_x$ $-(6tu - 2ru + 3pu - 3x)\partial_u$

In Case 10, $g_1(t) = \sqrt{t^2 + pt + q}\exp\left(\int \frac{r\mathrm{d}t}{t^2+pt+q}\right)$ and p, q and r are arbitrary constants such that $p^2 - 4q - 4r^2 \neq 0$ because then we revert to Case 8.

4. Boundary Value Problem for a Generalized $K(m,n)$ Equation

We consider the following initial and boundary value problem

$$u_t + (u^m)_x + t^k (u^n)_{xxx} = 0, \qquad t > 0, x > 0, \tag{4}$$

$$\begin{aligned} u(x,0) &= 0, & x &> 0,\\ u(0,t) &= q(t), & t &> 0,\\ u_x(0,t) &= 0, & t &> 0,\\ u_{xx}(0,t) &= 0, & t &> 0. \end{aligned} \tag{5}$$

Equation (4) admits for arbitrary n, m and k two-dimensional Lie symmetry algebra with basis operators

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = (3m-n-2)t\partial_t + (km-k+m-n)x\partial_x + (k-2)u\partial_u.$$

To determine the symmetry we take the linear combination

$$\Gamma = \alpha_1\Gamma_1 + \alpha_2\Gamma_2.$$

In this case we have

$$\Gamma = \alpha_1\partial_x + \alpha_2\left[(3m-n-2)t\partial_t + (km-k+m-n)x\partial_x + (k-2)u\partial_u\right].$$

Application of Γ to the first boundary condition

$$x = 0, \; u(0,t) = q(t)$$

gives

$$\alpha_1 = 0 \quad \text{and} \quad q(t) = \gamma t^{\frac{k-2}{3m-n-2}}.$$

Using the second extension of Γ,

$$\begin{aligned}\Gamma^{(2)} &= (3m-n-2)t\partial_t + (km-k+m-n)x\partial_x + (k-2)u\partial_u \\ &+(2k-mk-m+n-2)u_x\partial_{u_x} \\ &+(3k-2mk-2m+2n-2)u_{xx}\partial_{u_{xx}},\end{aligned}$$

where the unused terms have been ignored, it can be shown that it leaves invariant the initial condition and the remaining two boundary conditions of (5).

Finally, symmetry Γ produces the transformation

$$u = t^{\frac{k-2}{3m-n-2}}\phi(\omega), \quad \omega = xt^{-\frac{km-k+m-n}{3m-n-2}},$$

which reduces the problem (4)–(5) into

$$(\phi^n)''' + (\phi^m)' - \frac{km-k+m-n}{3m-n-2}\omega\phi' + \frac{k-2}{3m-n-2}\phi = 0,$$
$$\phi(0) = \gamma, \qquad \phi'(0) = 0, \qquad \phi''(0) = 0.$$

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