Constraints and Symmetries in Mechanics of Affine Motion

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Plan of the presentation:

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• Affine motion and affine bodies

We describe the configuration of an affine body by

$$x^i\,(r,\varphi;t) = r^i(t) + \varphi^i{}_K(t)a^K,$$

where x^i are spatial variables, r^i are coordinates of the centre of mass, $\varphi^i{}_K$ are internal (relative) parameters, and a^K are material variables.

To describe equations of motion we use the following definitions:

- the total mass of the body and the co-moving (constant) tensor of inertia in the material space

$$M = \int d\mu, \qquad J^{KL} = \int a^K a^L d\mu(a)$$

- when the centre of mass is placed at $a^K = 0$, then

$$J^K = \int a^K d\mu(a) = 0$$

- the total force and the spatial components of the co-moving dipole of forces distribution

$$F^i = \int \mathcal{F}^i(a)d\mu(a), \qquad N^{ij} = \int \varphi^i{}_K\varphi^j{}_L a^K a^L d\mu(a) = \varphi^i{}_K\varphi^j{}_L \int a^K a^L d\mu(a).$$

• Affine motion and affine bodies (cont.)

The equations of motion can be written in the following form:

$$M\frac{d^2r^i}{dt^2} = F^i, \qquad \varphi^i{}_K \frac{d^2\varphi^j{}_L}{dt^2} J^{KL} = N^{ij}.$$

Alternative balance forms of the above equations of motion:

$$\frac{dp^i}{dt} = F^i, \qquad \frac{dK^{ij}}{dt} = \frac{d\varphi^i{}_K}{dt}\frac{d\varphi^j{}_L}{dt} J^{KL} + N^{ij},$$

where p^i is a linear momentum and K is an affine spin:

$$p^i = M\frac{dr^i}{dt}, \qquad K^{ij} = \varphi^i{}_K \frac{d\varphi^j{}_L}{dt} J^{KL}.$$

The angular momentum (spin) $S^{ij} = K^{ij} - K^{ji}$ is conserved, if N^{ij} is symmetric:

$$\frac{dS^{ij}}{dt} = N^{ij} - N^{ji}.$$

In other words:

$$\frac{dp^i}{dt} = F^i, \qquad \frac{dK^{ij}}{dt} = \Omega^i{}_m K^{mj} + N^{ij},$$

where the affine velocity, called also Eringen's "gyration", is

$$\Omega^i{}_j = \frac{d\varphi^i{}_A}{dt}\varphi^{-1A}{}_j, \qquad \widehat{\Omega}^A{}_B = \varphi^{-1A}{}_i \Omega^i{}_j \varphi^j{}_B.$$

• Affine motion and affine bodies (cont.)

If Lagrangian is given by

$$L = T - V\left(r^{i}, \varphi^{i}{}_{K}\right),$$

where the kinetic energy is

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{M}{2} g_{ij} \frac{dr^{i}}{dt} \frac{dr^{j}}{dt} + \frac{1}{2} g_{ij} \frac{d\varphi^{i}{}_{K}}{dt} \frac{d\varphi^{j}{}_{L}}{dt} J^{KL},$$

then the forces and the momentum of forces are

$$F^{i} = -g^{ij} \frac{\partial V}{\partial r^{j}}, \qquad N^{ij} = -\varphi^{i}{}_{A} \frac{\partial V}{\partial \varphi^{k}{}_{A}} g^{kj}.$$

There is also another formula:

$$\frac{dK^{ij}}{dt} = N^{ij} + 2\frac{\partial T_{int}}{\partial g_{ij}}.$$

When there exist dissipative forces non-derivable from Lagrangian, then there appear some additional terms. In the simplest case, we choose them just linear or quadratic in generalized velocities.

• Gyroscopic constraints

There are some additional geometric, namely group-implied, forces imposed on the system. For example, gyroscopic constraints (pseudo-holonomic constraints of rigid motion) imply that $\Omega^i{}_j$, $\widehat{\Omega}^A{}_B$ are respectively g- and η-skew-symmetric angular velocities in spatial and co-moving representations,

$$\Omega^i{}_j = -\Omega_j{}^i = -g_{jk}\Omega^k{}_l g^{li}, \qquad \widehat{\Omega}^A{}_B = -\widehat{\Omega}_B{}^A = -\eta_{BC}\widehat{\Omega}^C{}_D\eta^{DA},$$

where g is the metric tensor of the physical space and η is the material metric.

It is easy to see that the above conditions are holonomic and may be written down as the conditions of isometry,

$$g_{ij}\varphi^i{}_A\varphi^j{}_B = \eta_{AB}.$$

Then the reaction moments N_R are symmetric,

$$N_{Rij} = N_{Rji}$$

and our equations are independent of explicitly non-specified reactions. Of course, gyroscopic reactions do not vanish, but their full tensor contractions with skew-symmetric affine virtual velocities (angular velocities) are vanishing in virtue of constraints.

So, if we are taking the skew-symmetric part of original equations, we can eliminate reaction moments and then obtain the effective equations of motion.

• Isochoric constraints (incompressible body)

In the case of incompressible body (isochoric constraints) the traces of affine velocities vanish:

$$\mathrm{Tr}\ \Omega = \Omega^{i}{}_{i} = 0.$$

The total contractions of such virtual Ω-s with the reaction affine moment N_R must vanish:

$$N_R{}^{ij}\Omega_{ji} = N_R{}^{ij}\Omega^{k}{}_{i}g_{jk} = 0.$$

It is easy to see that then reactions are pure traces,

$$N_R{}^{i}{}_{j} = \lambda\delta^{i}{}_{j}, \qquad N_R{}^{ij} = \lambda g^{ij},$$

where

$$\lambda = \frac{1}{n}\ \mathrm{Tr}\ N_R = \frac{1}{n}\ g_{ij}N_R{}^{ij}.$$

So, to eliminate the Lagrange multiplier λ, we must take the constraints condition (i.e., $\det\varphi = \mathrm{const}$) jointly with the g-traceless part of the initial equation itself:

$$\varphi^{i}{}_{A}\frac{d^2\varphi^{j}{}_{B}}{dt^2}J^{AB} - \frac{1}{n}g_{ab}\varphi^{a}{}_{A}\frac{d^2\varphi^{b}{}_{B}}{dt^2}J^{AB}g^{ij} = N^{ij} - \frac{1}{n}g_{ab}N^{ab}g^{ij}.$$

- **Constraints implied by linear conformal group (rotations and dilatations)**

In such a case an affine velocity (gyration) has the form:

$$\Omega^i{}_j = \omega^i{}_j + \alpha\delta^i{}_j$$

where $\omega^i{}_j$ is the g-skew-symmetric angular velocity, and α is an arbitrary real, dilatational parameter, so that

$$g_{ij}\varphi^i{}_A\varphi^j{}_B = \lambda\eta_{AB}, \quad \lambda > 0.$$

The reaction-free equations of motion consist of the skew-symmetric part of the original equation and of the g-trace of that equation, and reaction moments $N_R{}^{ij}$ are symmetric and g-traceless:

$$\begin{aligned} \varphi^i{}_A\frac{d^2\varphi^j{}_B}{dt^2}J^{AB} - \varphi^j{}_A\frac{d^2\varphi^i{}_B}{dt^2}J^{AB} &= N^{ij} - N^{ji} \\ g_{ij}\varphi^i{}_A\frac{d^2\varphi^j{}_B}{dt^2}J^{AB} &= g_{ij}N^{ij}. \end{aligned}$$

- **Constraints of purely rotation-free affine motion**

It is a very interesting example of nonholonomic constraints, when Ω is g-symmetric (the only geometrically correct definition):

$$\Omega^i{}_j - \Omega_j{}^i = \Omega^i{}_j - g_{jk}g^{il}\Omega^k{}_l = 0.$$

Then the reactions forces are anti-symmetric. So, the above equation must be joined with the symmetric part of equations of motion as balance laws:

$$\varphi^i{}_A\frac{d^2\varphi^j{}_B}{dt^2}J^{AB} + \varphi^j{}_A\frac{d^2\varphi^i{}_B}{dt^2}J^{AB} = N^{ij} + N^{ji}.$$

• Elimination of reaction forces: d'Alembert prescription

Let Lagrangian of the dynamical system be $L(q, \dot{q})$, i.e., it is a function of generalized coordinates $q^1, \ldots, q^n$ and their velocities, but we can also take the time into a consideration explicitly, i.e., $L(t, q, \dot{q})$.

Then the constraints are given by the following expressions:

$$F_a(q, \dot{q}) = 0 \qquad (F_a(t, q, \dot{q}) = 0), \qquad a = 1, \ldots, m.$$

In applications mostly often we have the constraints linear in velocities:

$$F_a(q, \dot{q}) = \omega_{ai}(q)\, \dot{q}^i.$$

Then the d'Alembert principle give us the following equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = R_i,$$

where R_i are reaction forces, which vanish on velocities compatible with constraints:

$$\omega_{ai}(q)\, \dot{q}^i = 0, \qquad \text{i.e.,} \qquad R_i \dot{q}^i = 0.$$

This implies that

$$R_i = \lambda^a \omega_{ai}.$$

• d'Alembert prescription (cont.)

By analogy the similar expressions can be written also for systems with dissipative forces. The non-constrained dynamics is given by the following equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = D_i,$$

where D_i are covariant vectors of non-variational, e.g., friction forces.

The corresponding constrained systems is given by the expressions:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} &= D_i + R_i \\ F_a\left(q(t), \dot{q}(t)\right) &= \omega_{ai}(q)\dot{q}^i = 0 \end{aligned}$$

where R_i are the reaction forces.

- **Elimination of reaction forces: Vakonomic prescription**

The variational principle constrained by $F_a = 0$ is given by the following expressions:

$$\delta \int L\left(q(t), \dot{q}(t)\right) dt = 0, \qquad F_a\left(q(t), \dot{q}(t)\right) = 0,$$

where the variations $\delta q^i\left(t\right)$ are subject to constraints.

The Lusternik theorem give us that the above variational principle is equivalent to the corresponding non-restricted principle:

$$\delta \int L\left[\mu\right]\left(q(t), \dot{q}(t)\right)dt = 0$$

where μ is the Lagrange multiplier and $L\left[\mu\right]$ is given by

$$L\left[\mu\right]\left(q(t), \dot{q}(t)\right) = L(q(t), \dot{q}(t)) - \mu^a F_a(q(t), \dot{q}(t)).$$

Mathematically here μ^a are some a priori unknown functions of time.

• Linear vakonomic constraints

The variational principle for $L\,[\mu]$ implies that for constraints that are linear in velocities,

$$F_a(q(t),\dot{q}(t)) = \omega_{ai}(q(t))\dot{q}^i(t)$$

we can write the following equations of motion:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} &= \frac{d\mu^a}{dt}\omega_{ai} - \mu^a\left(\frac{\partial \omega_{aj}}{\partial q^i} - \frac{\partial \omega_{ai}}{\partial q^j}\right)\dot{q}^j \\ F_a(q(t),\dot{q}(t)) &= \omega_{ai}(q(t))\dot{q}^i(t) = 0. \end{aligned}$$

This is the system of $(n+m)$ differential equations for the $(n+m)$ variables $q^i\,(t)$ and $\mu^a\,(t)$ as functions of time.

Correspondingly the constraints reactions are given as follows:

$$R_i = \frac{d\mu^a}{dt}\omega_{ai} + \mu^a\left(\frac{\partial \omega_{ai}}{\partial q^j} - \frac{\partial \omega_{aj}}{\partial q^i}\right)\frac{dq^j}{dt}.$$

• Elimination of reaction forces: Linear constraints (summary)

So, there are two prescriptions for calculating R_i, namely:

1. **d'Alembert prescription:**

$$R_i = \lambda^a \omega_{ai}, \qquad \text{i.e.,} \qquad R_i \dot{q}^i = 0$$

   for every virtual velocity satisfying the constraints,

2. **Vaconomic prescription:**

$$R_i = \frac{d\mu^a}{dt}\omega_{ai} + \mu^a \left(\frac{\partial \omega_{ai}}{\partial q^j} - \frac{\partial \omega_{aj}}{\partial q^i} \right) \frac{dq^j}{dt}.$$

• Holonomic constraints

For the holonomic constraints

$$F_a(q) = 0, \qquad a = 1, \ldots, m$$

in the reaction forces survives only the first term and then they are given by the usual d'Alembert expression

$$R_i = \lambda^a \omega_{ai} \qquad \text{with the multiplier} \qquad \lambda^a = \frac{d\mu^a}{dt}.$$

• Nonholonomic constraints of rotation-free affine motion

Let us remind that the affine velocity and its co-moving counterpart are given by the expressions:

$$\Omega^i{}_j = \frac{d\varphi^i{}_A}{dt}\varphi^{-1A}{}_j, \qquad \widehat{\Omega}^A{}_B = \varphi^{-1A}{}_i \frac{d\varphi^i{}_B}{dt} = \varphi^{-1A}{}_i \Omega^i{}_j \varphi^j{}_B.$$

For the gyroscopic (metrically rigid) motion we have that

$$\Omega^i{}_j + \Omega_j{}^i = \Omega^i{}_j + g_{ja}\Omega^a{}_b g^{bi} = 0$$

i.e., they are g-antisymmetric. This is nonholonomic description of holonomic constraints. Skew-symmetric matrices form a Lie algebra and those equations are integrated to the orthogonal group.

By analogy, the rotation-free motion is primarily described by

$$\Omega^i{}_j - \Omega_j{}^i = \Omega^i{}_j - g_{jk}g^{il}\Omega^k{}_l = 0$$

i.e., by the g-symmetry. But symmetric matrices do not form a Lie algebra. Moreover, those are truly nonholonomic constraints and they are not integrated to any submanifold.

• Polar decomposition

The polar decomposition of φ can be written as follows:

$$\varphi = UA,$$

where U is an orthogonal (isometric) matrix and A is an η-symmetric one:

$$U \in \mathrm{O}\left(U, \eta; V, g\right), \qquad A \in \mathrm{Symm}\left(U, \eta\right), \qquad \text{i.e.,} \qquad \eta_{AB} = g_{ij}\varphi^{i}{}_{A}\varphi^{j}{}_{B}, \qquad \eta_{AC}A^{C}{}_{B} = \eta_{BC}A^{C}{}_{A}.$$

The co-moving angular velocity $\widehat{\omega}$ of the U-rotator is given by

$$\widehat{\omega} = U^{-1}\frac{dU}{dt}.$$

The kinetic energy can be written as the sum of the translational and internal (relative) terms:

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{M}{2}g_{ij}\frac{dr^{i}}{dt}\frac{dr^{j}}{dt} + \frac{1}{2}g_{ij}\frac{d\varphi^{i}{}_{A}}{dt}\frac{d\varphi^{j}{}_{B}}{dt}J^{AB}.$$

In the polar decomposition the internal kinetic energy T_{int} becomes as follows:

$$T_{\mathrm{int}} = \frac{1}{2}\eta_{KL}\frac{dA^{K}{}_{A}}{dt}\frac{dA^{L}{}_{B}}{dt}J^{AB} + \eta_{KL}\widehat{\omega}^{K}{}_{C}A^{C}{}_{A}\frac{dA^{L}{}_{B}}{dt}J^{AB} + \frac{1}{2}\eta_{KL}\widehat{\omega}^{K}{}_{C}\widehat{\omega}^{L}{}_{D}A^{C}{}_{A}A^{D}{}_{B}J^{AB}.$$

• Polar decomposition (cont.)

Obviously, $\widehat{\omega}$ is η-skew-symmetric:

$$\eta_{AC}\widehat{\omega}^{C}{}_{B} = -\eta_{BC}\widehat{\omega}^{C}{}_{A}.$$

The g-symmetry constraints on Ω imply that

$$\widehat{\omega} = \frac{1}{2}\left[A^{-1}, \frac{dA}{dt}\right] = \frac{1}{2}\left(A^{-1}\frac{dA}{dt} - \frac{dA}{dt}A^{-1}\right).$$

Substituting this to the expression for the internal kinetic energy T_{int}, we obtain that

$$\begin{aligned} T^{\mathrm{Vak}}_{\mathrm{int}} &= \frac{1}{2}\eta_{KL}\frac{dA^{K}{}_{A}}{dt}\frac{dA^{L}{}_{B}}{dt}J^{AB} + \frac{1}{4}\eta_{KL}A^{-1K}{}_{D}\frac{dA^{D}{}_{C}}{dt}A^{C}{}_{A}\frac{dA^{L}{}_{B}}{dt}J^{AB} \\ &+ \frac{1}{8}\eta_{KL}A^{-1K}{}_{E}\frac{dA^{E}{}_{C}}{dt}A^{C}{}_{A}A^{-1L}{}_{F}\frac{dA^{F}{}_{D}}{dt}A^{D}{}_{B}J^{AB}. \end{aligned}$$

The simplest vakonomic Lagrangian is obtained by putting:

$$L^{\mathrm{Vak}}_{\mathrm{int}} = T^{\mathrm{Vak}}_{\mathrm{int}} + V\left(G\right),$$

where the potential V depends on the Green deformation tensor G:

$$G_{AB} = g_{ij}\varphi^{i}{}_{A}\varphi^{j}{}_{B} = \eta_{CD}A^{C}{}_{A}A^{D}{}_{B}.$$

• Vakonomic lagrangian and resulting equations of motion

The variational derivative of $T_{\text{int}}^{\text{Vak}}$ with respect to the symmetric tensor $A_{AB} = \eta_{AC} A^C{}_B = A_{BA}$ is given by

$$\left.\frac{\delta T_{\text{int}}^{\text{Vak}}}{\delta A_{AB}}\right|_{\text{symm}} = -\frac{1}{4}\frac{d^2}{dt^2} A^{(A}{}_L J^{B)L} - \frac{1}{4}\frac{d}{dt}\left(\left(A^{-1}\right)^{(A}{}_E J^{B)L} \frac{dA^E{}_C}{dt} A^C{}_L\right)$$
$$-\frac{1}{4}\eta_{KL}\frac{d}{dt}\left(\frac{dA^K{}_E}{dt}\left(A^{-1}\right)^{L(A} A^{B)}{}_D\right) J^{ED}$$
$$-\frac{1}{4}\eta_{KL}\frac{d}{dt}\left(\left(A^{-1}\right)^K{}_E \frac{dA^E{}_C}{dt} A^C{}_F \left(A^{-1}\right)^{L(A} A^{B)}{}_D\right) J^{FD}$$
$$-\frac{1}{4}\eta_{KL}\frac{dA^K{}_E}{dt}\frac{dA^F{}_D}{dt} A^D{}_G \left(A^{-1}\right)^{L(A}\left(A^{-1}\right)^{B)}{}_F J^{EG}$$
$$-\frac{1}{4}\eta_{KL}\left(A^{-1}\right)^K{}_E \frac{dA^E{}_C}{dt} A^C{}_M \frac{dA^F{}_D}{dt} A^D{}_N \left(A^{-1}\right)^{L(A}\left(A^{-1}\right)^{B)}{}_F J^{MN}$$
$$+\frac{1}{4}\eta_{KL}\frac{dA^K{}_D}{dt}\left(A^{-1}\right)^L{}_E \frac{dA^{E(A}}{dt} J^{B)D}$$
$$+\frac{1}{4}\eta_{KL}\left(A^{-1}\right)^K{}_E \frac{dA^E{}_C}{dt} A^C{}_D \left(A^{-1}\right)^L{}_F \frac{dA^{F(A}}{dt} J^{B)D}.$$

When there are hyperelastic forces derivable from the potential V depending only on the Green deformation tensor G, then equations of motion have the following form:

$$\left.\frac{\delta T_{\text{int}}^{\text{Vak}}}{\delta A_{AB}}\right|_{\text{symm}} = -A_{KC}\eta^{K(A}\widehat{N}^{B)C}.$$

• Usual (non-vakonomic) constraints and equations of motion

One can show that for the usual (non-vakonomic) constraints of the rotation-free motion the evolution of the system is given by the symmetric part of the following tensor equation:

$$AJ_\eta \frac{d^2A}{dt^2} - \frac{1}{2}AJ_\eta A\frac{d}{dt}\left[A^{-1},\frac{dA}{dt}\right] - AJ_\eta\frac{d}{dt}\left[A^{-1},\frac{dA}{dt}\right] + \frac{1}{4}AJ_\eta A\frac{d}{dt}\left[A^{-1},\frac{dA}{dt}\right]^2 = \overline{N},$$

where

$$J_\eta{}^K{}_L = J^{KM}\eta_{ML}, \qquad \overline{N}^{KL} = A^K{}_M A^L{}_N \widehat{N}^{MN},$$
$$\widehat{N}^{AB} = \varphi^{-1A}{}_i\varphi^{-1B}{}_j N^{ij}, \qquad N^{ij} = -g^{jk}\varphi^i{}_M\frac{\partial V}{\partial\varphi^k{}_M}.$$

In the explicit form the equations of motion are written as follows:

$$\begin{aligned}
J^{AB}\frac{d^2A^{B(C}}{dt^2}A^{D)}{}_A \quad &- \quad J^A{}_B A^B{}_E \frac{d}{dt}\frac{1}{2}\left(\left(A^{-1}\right)^E{}_F\frac{d}{dt}\left(A^{F(C}\right)A^{D)}{}_A - \frac{d}{dt}\left(A^E{}_F\right)\left(A^{-1}\right)^{F(C}A^{D)}{}_A\right)\\
&- \quad J^A{}_B\frac{dA^B{}_E}{dt}\left(\left(A^{-1}\right)^E{}_F\frac{d}{dt}\left(A^{F(C}\right)A^{D)}{}_A - \frac{d}{dt}\left(A^E{}_F\right)\left(A^{-1}\right)^{F(C}A^{D)}{}_A\right)\\
&+ \quad \frac{1}{4}J^A{}_B A^B{}_E\left(\left(A^{-1}\right)^E{}_G\frac{d}{dt}\left(A^G{}_F\right) - \frac{d}{dt}\left(A^E{}_G\right)\left(A^{-1}\right)^G{}_F\right)\\
&\cdot \quad \left(\left(A^{-1}\right)^F{}_H\frac{d}{dt}\left(A^{H(C}\right)A^{D)A} - \frac{d}{dt}\left(A^F{}_H\right)\left(A^{-1}\right)^{H(C}A^{D)}{}_A\right) = \overline{N}^{(CD)}.
\end{aligned}$$

The structures of vakonomic and d'Alembert equations are evidently different.

- **Solving the above equations of motion:**
  - solving the symmetric or vakonomic part of our equations of motion, we find $A(t)$.
  - then we substitute it to $\widehat{\omega}$ and solving equation
    $$\frac{dU}{dt} = U\widehat{\omega}$$
    we find $U(t)$.
  - finally, substituting it to
    $$\varphi(t) = U(t)A(t)$$
    we solve the problem, at least in principle.

• Nonlinear vakonomic constraints

In the case when there is no dissipation, calculating Euler-Lagrange equations for the modified Lagrangian $L[\mu] = L + \mu^a F_a$ we obtain the system for the $(n+m)$ variables $q^i(t),\ \mu^a(t)$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \mu^a \frac{\partial F_a}{\partial q^i} - \frac{d}{dt}\left(\mu^a \frac{\partial F_a}{\partial \dot{q}^i}\right), \qquad i = 1, \ldots, n,$$

where $\mu^a(t),\ a = 1, \ldots, m$, are Lagrange multipliers and $F_a(q, \dot{q}) = 0$.

The reactions forces:

$$R_i = \mu^a \frac{\partial F_a}{\partial q^i} - \frac{d}{dt}\left(\mu^a \frac{\partial F_a}{\partial \dot{q}^i}\right) = \mu^a \frac{\partial F_a}{\partial q^i} - \frac{d\mu^a}{dt}\frac{\partial F_a}{\partial \dot{q}^i} - \mu^a \frac{\partial^2 F_a}{\partial \dot{q}^i \partial q^j}\frac{dq^j}{dt} - \mu^a \frac{\partial^2 F_a}{\partial \dot{q}^i \partial \dot{q}^j}\frac{d^2 q^j}{dt^2}.$$

In general, such reactions need not be adiabatic. The equations of constrained motion have the form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = D_i + R_i, \quad F_a\left(q(t), \dot{q}(t)\right) = 0.$$

Nonlinearity of nonholonomic constraints with respect to velocities has a qualitative effect on the dynamical structure of reactions R_i (contains the term with second derivatives). Such acceleration-dependent forces modify the inertial properties of the object. Besides, nonlinearity of M may influence the energy balance because, in general, the above reactions R_i need not annihilate the velocity vectors. After calculating the power of the reactions along curves in Q compatible with constraints M we obtain

$$R_i \frac{dq^i}{dt} = \mu^a \frac{dF_a}{dt} - \frac{d}{dt}\left(\mu^a \dot{q}^i \frac{\partial F_a}{\partial \dot{q}^i}\right).$$

• Nonlinear vakonomic constraints (cont.)

The first term vanishes in virtue of constraints equations, so finally

$$R_i \dot{q}^i = -\frac{d}{dt}\left(\mu^a \frac{\partial F_a}{\partial \dot{q}^i}\dot{q}^i\right).$$

Then the energy balance has the form:

$$\frac{d}{dt}\left(E + \mu^a \dot{q}^i \frac{\partial F_a}{\partial \dot{q}^i}\right) = D_i \dot{q}^i$$

The balanced quantity

$$E[L,M] := E + \mu^a \dot{q}^i \frac{\partial F_a}{\partial \dot{q}^i}$$

can be interpreted as the effective energy of the system constrained by the manifold M.

When M is fixed, $E[L,M]$ does not depend on the particular choice of functions F_a, used as left-hand sides of equations of M.

- **Nonlinear vakonomic constraints (cont.)**

The quantity $E[L, M]$ contains two parts:

- the natural energy

$$E[L] := \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L$$

  of the unconstrained system and

- the energy of constraints

$$E[M] := \mu^a \frac{\partial F_a}{\partial \dot{q}^i} \dot{q}^i.$$

In the case with no dissipative forces, the total energy $E[L, M]$ is a constant of motion. The existence of this constant of motion is just the peculiarity and distinguishing feature of the Hamilton-Lusternik algorithm.

$E[L, M]$ can be directly obtained from the modified Lusternik Lagrangian $L[\mu]$:

$$E[L[\mu]] := \dot{q}^i \frac{\partial L[\mu]}{\partial \dot{q}^i} - L[\mu] = E[L] + \mu^a \frac{\partial F_a}{\partial \dot{q}^i} \dot{q}^i - \mu^a F_a,$$

where the last term vanishes on constraints M.

The mechanical work done by Hamilton-Lusternik reactions has a variational structure; it can be interpreted as the exchange of energy between the system in question and the constraining object.

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The end.

Thank you for your attention!

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