Hodge theory for bundles over C^* algebras

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Symplectic linear algebra

Symplectic vector space
(V, ω_0) - real/complex $2n$ dimensional vector space,
$\omega_0 : V \times V \to \mathbb{R}/\mathbb{C}$ non-degenerate antisymmetric
Symplectic group
$Sp(V, \omega_0) = \{A : V \to V \mid \omega_0(Av, Aw) = \omega_0(v, w) \text{ for each } v, w \in V\}$
Retractable onto $U(n)$ which is of the same homotopy type as S^1, $\pi_1(Sp(V, \omega_0)) = \mathbb{Z}$.
Possesses a non-universal connected 2-fold covering, the so called **Metaplectic group** $Mp(V, \omega_0)$, $\lambda : Mp(V, \omega_0) \to^{2:1} Sp(V, \omega_0)$
Universal covering would be infinitely many folded over $Sp(V, \omega_0)$.

Properties of the SSW representation

Segal-Shale-Weil representation of the metaplectic group.
Inventors:
David Shale (quantization of solutions to the Klein-Gordon equation)
André Weil in the mid of '60
Berezin used it at the infinitesimal level
- Underlying vector space $L^2(\mathbb{R}^n)$
- $\rho_0 : Mp(V, \omega_0) \to \mathcal{U}(L^2(\mathbb{R}^n))$ (continuous homomorphism)
- Non-trivial faithful unitary representation of $Mp(V, \omega_0)$
- Splits into 2 irreducible representations, odd and even L^2 functions on $\mathbb{R}^n$.
- There exists $g_0 \in Mp(V, \omega_0)$ such that
$\rho_0(g_0) = \mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ (continuous on $L^2(\mathbb{R}^n)$)

Properties of the SSW-representation

- Similar to the spinor representation of Spin groups - it is also not a representation of the underlying $Sp(V,\omega_0)$.

$$\begin{array}{ccc} Mp(V,\omega_0) & \xrightarrow{\rho_0} & \mathcal{U}(L^2(\mathbb{R}^n)) \\ \big\downarrow \lambda & \overset{\nexists}{\nearrow} & \\ Sp(V,\omega_0) & & \end{array}$$

- Smallest wr. to dimension and realized inside of the the holomorphic series
- Highest weights $(\frac{1}{2},\dots,\frac{1}{2},-\frac{1}{2})$, $(\frac{1}{2},\dots,\frac{1}{2},-\frac{3}{2})$

Symplectic manifolds

(M, ω) - M manifold, ω non-degenerate differential 2-form and $d\omega = 0$.

Examples:

1) T^*M, where M is any manifold, $\omega_U = \sum_{i=1}^{n} dp_i \wedge dq^i$, q^i local coordinates on the manifold, p_i coordinates at $T_{(q^1,\ldots,q^n)}M$
2) S^2 with $\omega = \text{vol} = r^2 \sin\vartheta d\phi \wedge d\vartheta$
3) even dimensional tori $\omega = d\phi_1 \wedge d\vartheta^1 + \ldots + d\phi_n \wedge d\vartheta^n$ (in mechanics: action-angle variables)
4) Kähler manifolds, $\omega(-,-) = h(-, J-)$

Symplectic connections

Darboux theorem: In a neighborhood of any point, one can choose coordinates in which $\omega = \sum_{i=1}^{n} dq^i \wedge dp_i$. In Riemannian, geometry the metric can be transformed into the "canonical" form only point-wise - curvature obstruction. Measured by the curvature tensor. In s.g., due to Darboux theorem, the connection cannot have such meaning.

Definition: A connection on a symplectic manifold (M, ω) equipped with a symplectic form ω is called symplectic if $\nabla\omega = 0$, and it is called Fedosov if in addition, it is torsion-free. (Boris Fedosov)

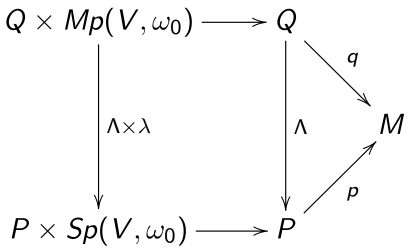
Metaplectic structure

Symplectic structure

(M, ω) symplectic manifold. At any point $m \in M$, consider the set $P_m = \{b = (e_1, \ldots, e_{2n}) | \text{bis a symplectic basis of } (T^*_m M, \omega_m)\}$.
$P = \bigcup_{m \in M} P_m$ the space of symplectic repères, $p : P \to M$ ("foot-point" projection).

Metaplectic structure Q

- Formally: (Q, Λ), $q : Q \to M$ bundle over M, $\Lambda : Q \to P$
- At any point, $Q_m \simeq Mp(V, \omega)$
- Compatibility with the symplectic structure:



Vector bundles

Associated bundles:
At any point - replace Q_m by $L^2(\mathbb{R}^n)$ and do it equivariantly with respect to the Segal-Shale-Weil representation (in algebraic geometry called base change). Physics - the (principle) bundle of repères is replaced by the bundle of observed quantities (elements of vector spaces). Formally, a **notion from differential geometry** - associated bundle:
$\mathcal{S} = (Q \times L^2(\mathbb{R}^n))/(rg, f) \sim (r, \rho_0(g)f)$, $g \in Mp(V, \omega_0)$
In this case, introduced by Bertram Kostant: symplectic spinor bundle
Associated connections:
From a symplectic connection ∇, it is possible to construct the associated connection $\nabla^{\mathcal{S}}$ on the sections of the associated **symplectic spinor bundle** $\mathcal{S} \to M$
Physics: The Fermi-Walker connection is constructed from the Riemannian connection of the Einstein metric. It transports the fermions in the classical spinor bundle.

Work of K. Haberemann

Operators generated by symplectic connections

Symplectic Dirac operators:

$$\mathfrak{D} : \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S}), \;\; \mathfrak{D}s = \sum_{i=1}^{2n} e_i . \nabla^S_{e_i} s$$

$$(e_i . s)(x) = \imath x^i s(x), \;\; (e_{i+n} . s)(x) = \frac{\partial s}{\partial x^i}(x) \; (quantization).$$

Symplectic Dirac is self-adjoint, its symbol is fiber-wise linear injective map.

Its associated second order operator $\mathcal{P} = \mathfrak{D}\mathfrak{D}^*$ is self-adjoint and elliptic (elliptic = its symbol is an isomorphism).

But: Kernel of $\mathcal{P}$ on $S^2 = CP^1$ is infinite dimensional (see Habermann, Habermann).

Aim: Normalize/repair the theory and restore the classical theory: Ellipticity $\Longrightarrow$ "finiteness" of kernels.

Hodge theory for elliptic complexes

Objects:

1) $\mathcal{E}^i \to M$ vector bundles over M, each of finite rank ($=$ dimension of the fiber over $\mathbb{R}$ or $\mathbb{C}$ **is finite**).
2) M compact
3) D_i differential operators of finite order ($=$ do "not allow" infinitely many differentiating)
4) $\sigma(D_i;\xi) =: \sigma_i^\xi : \mathcal{E}^i \to \mathcal{E}^i$, symbol of D_i (vector bundle homomorphism), $\xi \in T^*M$.

Examples:

symbol of exterior differentiation $\sigma^\xi(d_i)\alpha = \xi \wedge \alpha$
symbol of Laplace-Beltrami operator $\sigma^\xi(\triangle)f = (\sum_{i=1}^n (\xi_i)^2)f$
symbol of the Dolbeault operator $\sigma^\xi(\overline{\partial})\alpha = \imath\xi^{(0,1)} \wedge \alpha$

Hodge theory for elliptic complexes

Definition: The sequence

$$0 \to \Gamma(\mathcal{E}^0, M) \xrightarrow{D_0} \Gamma(\mathcal{E}^1, M) \xrightarrow{D_1} \dots \xrightarrow{D_{n-1}} \Gamma(\mathcal{E}^n, M)$$

is called complex if $D_{i+1}D_i = 0$ for all i.

Definition: For any $m \in M$ and any nonzero covector $\xi \in T_m^*M \setminus \{0\}$, the complex

$$0 \to \Gamma(\mathcal{E}^0, M) \xrightarrow{D_0} \Gamma(\mathcal{E}^1, M) \xrightarrow{D_1} \dots \xrightarrow{D_{n-1}} \Gamma(\mathcal{E}^n, M)$$

is called elliptic, iff the symbol sequence

$$0 \to \mathcal{E}^0 \xrightarrow{\sigma_0^\xi} \mathcal{E}^1 \xrightarrow{\sigma_1^\xi} \dots \xrightarrow{\sigma_{n-1}^\xi} \mathcal{E}^n$$

is **exact** (kernel of any map is the image of the preceding map).

Hodge theory for elliptic complexes

Definition: Cohomology of the complex is the group (vector space)

$$H^i(D,\mathbb{C}) = \frac{\mathrm{Ker}(D_i : \Gamma(\mathcal{E}^i, M) \to \Gamma(\mathcal{E}^{i+1}, M))}{\mathrm{Im}(D_{i-1} : \Gamma(\mathcal{E}^{i-1}, M) \to \Gamma(\mathcal{E}^i, M))}.$$

Hodge's trick: Construct the associated Laplacians
$\triangle_i = d_i^* d_i + d_{i-1} d_{i-1}^* : \Gamma(M, \mathcal{E}^i) \to \Gamma(M, \mathcal{E}^i)$
Theorem (Hodge theory): If the fibers are finite dimensional (over $\mathbb{C}/\mathbb{R}$), then

1) $\dim(\mathrm{Ker}\triangle_i) < +\infty$
2) $\mathrm{H}^i(D,\mathbb{C}) \simeq \mathrm{Ker}\,\triangle_i$
3) $H^i(D,\mathbb{C})$ is a norm complete vector topological space (trivial information if known 1) and 2))

Proof - classically due to W. Hodge (see, e.g., R . O. Wells, Analysis on complex manifolds, Springer). It is possible to avoid Hodge star operators; the proof needs to choose an (auxiliary) Riemannian metric.

A associative algebra over $\mathbb{C}$ with a norm $|\,| : A \to \mathbb{R}_0^+$, i.e.,

$$|ab| \leq |a||b|$$
$$|\lambda a| = |\lambda||b|$$
$$|a| = 0 \implies a = 0$$

is called **normed algebra**.

Definition: Normed algebra A equipped by involution $* : A \to A$ such that

$$|aa^*| = |a||a^*|$$

is called a C^*-**algebra**.
We suppose A contains a unit, $1a = a1 = a$ (**unital** C^*-algebra).
Examples:

1) $\mathcal{C}_c^0(X)$, where X is a topological space
2) H a Hilbert space, $A := End(H)$, $*A := A^*$, $|\,|$ - sup norm
3) $\mathrm{Mat}(\mathbb{C}^n)$, $*A = A^\dagger$, $|A| = \max\{|\lambda|, \lambda \in \mathrm{spec}(A)\}$

Modules over A

A a unital C^*-algebra, 1 unit
spec$(a) = \{\lambda \in \mathbb{C} | a - \lambda 1$ does not possesses inverse (in A)$\}$
$a = a^* \implies \text{spec}(a) \subseteq \mathbb{R}$
$A_0^+ = \{a \in A | a = a^* \text{ and } \text{spec}(a) \subseteq \mathbb{R}_0^+\}$ - positive elements.
U a vector space with a left action on A (no continuity supposed) equipped by $(,) : U \times U \to A$ (mimics the Hilbert product) such that for each $u, v, w \in U$ and $r \in A$

1) $(u + rv, w) = (u, w) + r(u, w)$
2) $(a.u, v) = a(u, v)$
3) $(u, v) = (v, u)^*$
4) $(u, u) \in A_0^+$ and $(u, u) = 0 \implies u = 0$

and such that $A \ni a \mapsto |(a, a)|^{1/2}$ makes U a **complete** normed space is called an A-Hilbert module.

A-Hilbert bundles

A-Hilbert bundle $=$ Banach bundle (bundle the fibers of which are Banach spaces) such that the fibers have the structure a fixed A-Hilbert module, and the structure group (image of the transition maps) is a subgroup of the group of $\mathrm{Aut}_A(H)$.

U is finitely generated projective $:=$ means $U \oplus U^{\perp} = A^n$, A^n is the free-object, n is the upper bound for the number of the generators. The classical proof of Hodge cannot be used for finitely generated projective bundles cannot be used. "Sobolev completions are more complicated in the infinite rank case."

Cohomology characterization

(V,ω_0) vector space of dimension $2n$
(M^{2n},ω) admits a metaplectic structure P and a flat Fedosov connection ∇
$E^i = H \times \bigwedge^i V^*$ is a representation of $Mp(V,\omega_0)$

$$\rho(g)(\alpha \otimes s) = \lambda(g)^*\alpha \otimes \rho_0(g)s,\ g \in Mp(V,\omega_0).$$

$\mathcal{E}^i = P \times_\rho E^i$ - exterior forms with values in the symplectic spinor bundle
∇ induces ∇^S which can be extended to

$$d_i^\nabla : \Gamma(M,\mathcal{E}^i) \to \Gamma(M,\mathcal{E}^{i+1}) \quad \text{exterior covariant derivative}$$

Cohomology of the deRham complex twisted by the SSW-representation

Consider the sequence

$$0 \to \Gamma(\mathcal{E}^0, M) \stackrel{d_0^\nabla}{\to} \Gamma(\mathcal{E}^1, M) \stackrel{d_1^\nabla}{\to} \dots \stackrel{d_{n-1}^\nabla}{\to} \Gamma(\mathcal{E}^n, M)$$

(deRham complex twisted by the SSW-module.)

Similar to the complex of R. Penrose (deRham tensored by orthogonal spinors), where higher spin particles occur.

Theorem A: (M, ω) symplectic and ∇ flat connection. Suppose M is compact, admits a metaplectic structure and the extensions of the associated Laplacians $\triangle_i$ to the Sobolev completions of the respective smooth section spaces have closed images. Then each cohomology group $H^i(d, A)$ is a

1) finitely generated A-module and
2) a normed space

Questions: Can one drop the closeness assumption in certain cases? Is the cohomology a finitely generated **Hilbert** A-module?

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