The Magnetized Kepler Problems in Odd Dimensions

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June 7, 2013

§1. Kepler Problem (I. Newton, ~1660)

- **configuration space**: $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{\mathbf{0}\}$.
- **equation of motion:**

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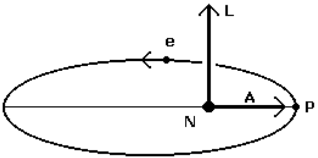
- **orbits.** Since $\mathbf{L} := \mathbf{r} \times \mathbf{r}'$ and $\mathbf{A} = \mathbf{L} \times \mathbf{r}' + \frac{\mathbf{r}}{r}$, $\mathbf{L} \cdot \mathbf{A} = 0$



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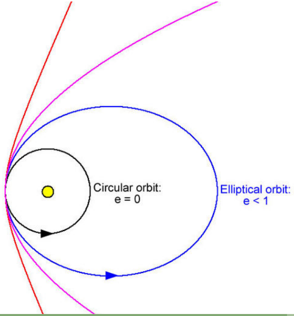
$$\mathbf{L} \cdot \mathbf{r} = 0, \quad r - \mathbf{A} \cdot \mathbf{r} = |\mathbf{L}|^2. \tag{2}$$

So a non-colliding orbit is a conic with eccentricity $e = |\mathbf{A}|$.



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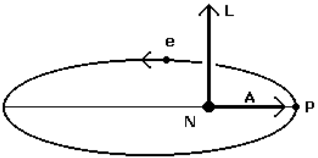

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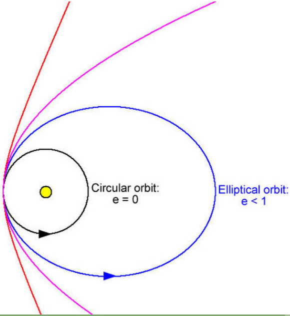

- **energy.** The energy $E := \frac{1}{2}|\mathbf{r}'|^2 - \frac{1}{r}$ can be expressed in terms of $\mathbf{L}$ and $\mathbf{A}$ provided that the orbit is non-colliding (i.e., $\mathbf{L} \neq \mathbf{0}$):

$$E = -\frac{1 - |\mathbf{A}|^2}{2|\mathbf{L}|^2}. \tag{3}$$

Proof.

$$\begin{aligned} |\mathbf{A}|^2 &= |\mathbf{L} \times \mathbf{r}'|^2 + 1 + 2\frac{\mathbf{r} \cdot (\mathbf{L} \times \mathbf{r}')}{r} \\ &= |\mathbf{L}|^2 |\mathbf{r}'|^2 + 1 - \frac{|\mathbf{L}|^2}{r} \\ &= 2|\mathbf{L}|^2 E + 1. \end{aligned}$$

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- angular momentum $L := \mathbf{r} \wedge \mathbf{r}'$.

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$$\begin{aligned} \mathbf{A}' &= \mathbf{r}'' \lrcorner L + \left(\frac{\mathbf{r}}{r}\right)' = -\frac{\mathbf{r} \lrcorner (\mathbf{r} \wedge \mathbf{r}')}{r^3} + \left(\frac{\mathbf{r}}{r}\right)' \\ &= -\frac{r^2 \mathbf{r}' - r r' \mathbf{r}}{r^3} + \left(\frac{\mathbf{r}}{r}\right)' = \mathbf{0}. \end{aligned}$$

And $L \wedge \mathbf{A} = 0$, $L \wedge \mathbf{r} = 0$, $r - \mathbf{A} \cdot \mathbf{r} = |L|^2$.

Note that this intrinsic formulation works in any dimension! So, in any dimension, the Kepler problem exists and its non-colliding orbits are conics.

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§3. Lightcone Reformulation (G.Meng, 2011)

The non-colliding orbit

$$\mathbf{L}\cdot\mathbf{r}=0, \quad r-\mathbf{A}\cdot\mathbf{r}=|\mathbf{L}|^2. \tag{4}$$

has an attractive lightcone reformulation. Let $x=(x_0,\mathbf{r})$ and

$$l=(0,\frac{\mathbf{L}}{|\mathbf{L}|}), \quad a=\frac{1}{|\mathbf{L}|^2}(1,\mathbf{A}). \tag{5}$$

Then $l^2=-1$, $l\cdot a=0$, $a_0>0$. The orbit (4) can be recast as the intersection of the affine plane

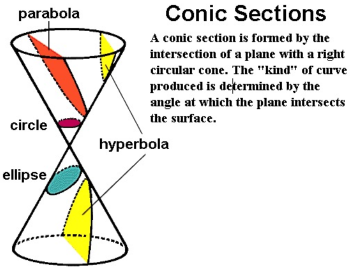
$$l\cdot x=0, \quad a\cdot x=1 \tag{6}$$

with the future lightcone

$$x^2=0, \quad x_0>0. \tag{7}$$

In this reformulation, the energy is $E=-\frac{a^2}{2a_0}$.

Here is the ancient Greek's original definition of conics:



Question: Is there any relation between the restricted Lorentz group $\mathrm{SO}^+(1,3)$ and the orbits?

Answer: Yes, provided that we also include the orbits of the magnetized companions of the Kepler problem (MICZ-Kepler problems).

§4. MICZ Kepler problems

The MICZ-Kepler problem (H. McIntosh and A. Cisneros, D. Zwanziger, 1960s) with magnetic charge $\mu \in \mathbb{R}$ is a magnetized version of the Kepler problem with the equation of motion

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3} - \mathbf{r}' \times \mu \frac{\mathbf{r}}{r^3} + \frac{\mu^2}{r^4}\mathbf{r}.$$

Just as in the Kepler problem, the conserved quantities are

$$\mathbf{L} = \mathbf{r} \times \mathbf{r}' + \mu \frac{\mathbf{r}}{r}, \quad \mathbf{A} = \mathbf{L} \times \mathbf{r}' + \frac{\mathbf{r}}{r}$$

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Description of MICZ-Kepler orbits

The Kepler orbits are described by equations

$$\mathbf{L}\cdot\mathbf{r} = \mu r, \quad r - \mathbf{A}\cdot\mathbf{r} = |\mathbf{L}|^2 - \mu^2, \tag{8}$$

In the light cone formulation, they are described as the intersection of the future light cone with the affine plane

$$l\cdot x = 0, \quad a\cdot x = 1. \tag{9}$$

Here,

$$l = \frac{1}{\sqrt{|\mathbf{L}|^2 - \mu^2}}(\mu, \mathbf{L}), \quad a = \frac{1}{|\mathbf{L}|^2 - \mu^2}(1, \mathbf{A}). \tag{10}$$

Note that $l^2 = -1$, $l\cdot a = 0$, $a_0 > 0$, and

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The Lie group $\mathrm{SO}^+(1,3)\times\mathbb{R}_+$ *acts transitively on both the set of oriented elliptic MICZ-Kepler orbits and the set of oriented parabolic MICZ-Kepler orbits.*

Remark. The Lightcone formulation suggests that 1) *a 2nd temporal dimension appears naturally*, 2) the *magnetic charge is relative*.

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The intrinsic lightcone formulation of the affine plane suggests an extension of the magnetized Kepler problems to higher dimensions. However, this extension is not obvious at all, as one can see from the following facts:

- The extension to dimension 5 was done 20 years later, by T. Iwai in 1990.
- For this extension to dimension 5, it was not even clear what the non-colliding orbits are.
- The extension to any other dimension was not believed to be possible.

Besides the lightcone formulation, we have another clue: an extension to higher dimensions was carried out at the quantum level, cf. G. W. Meng, J. Math. Phys. 48, 032105 (2007).

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Magnetized Kepler Problems in Odd Dimensions (G. W. Meng, 2012)

a) The configuration space P_μ: it fibers over $X := \mathbb{R}_*^{2k+1}$, with the fiber being a certain coadjoint orbit $\mathcal{O}_\mu$ of $G := \mathrm{SO}(2k)$. This principal G-bundle $P \to X$ is the pullback bundle of the principal G-bundle $\mathrm{SO}(2k+1) \to \mathrm{S}^{2k}$ under the map

$$\begin{array}{ccc} X & \to & \mathrm{S}^{2k} \\ \mathbf{r} & \mapsto & \frac{\mathbf{r}}{r} \end{array} \tag{11}$$

The bundle $P \to X$ is equipped with a canonical connection, i.e. the pull back of the connection

$$\mathrm{Proj}_{\mathfrak{so}(2k)}(g^{-1}\,dg)$$

on $\mathrm{SO}(2k+1) \to \mathrm{S}^{2k}$.

b) The Equation of Motion. Let $E \to X$ be the adjoint bundle of $P \to X$, $\Omega \in \Gamma(\wedge^2 T^*X \otimes E)$ be the curvature of the canonical connection. The equation of motion is

$$\begin{cases} \mathbf{r}'' = -\frac{\mathbf{r}}{r^3} + \frac{\mu^2}{k}\frac{\mathbf{r}}{r^4} + \mathbf{r}' \lrcorner \langle \xi, \Omega \rangle, \\ \frac{D\xi}{dt} = 0. \end{cases} \tag{12}$$

Here ξ is a lifting of the map $\mathbf{r}\colon \mathbb{R} \to X$:

$$\begin{array}{ccc} & & P_\mu \\ & \xi \nearrow & \downarrow \\ \mathbb{R} & \xrightarrow{\mathbf{r}} & X \end{array}$$

and $\frac{D\xi}{dt}$ is the covariant derivative of ξ.

Further Elaborations

$$\begin{cases} \mathbf{r}'' = -\frac{\mathbf{r}}{r^3} + \frac{\mu^2}{k}\frac{\mathbf{r}}{r^4} + \mathbf{r}' \lrcorner \langle \xi, \Omega \rangle, \\ \frac{D\xi}{dt} = 0. \end{cases} \tag{13}$$

- $\langle , \rangle$ is the pairing of $\mathfrak{g} := \mathfrak{so}(2k)$ with its dual $\mathfrak{g}^*$.
- two-vectors and two-forms are identified via the standard euclidean structure of $\mathbb{R}^{2k+1}$.
- The adjoint orbit $\mathcal{O}_\mu$ is

$$G \cdot \frac{|\mu|}{\sqrt{k}}(M_{1,2} + \cdots + M_{2k-3,2k-2} + \mathrm{sign}\,(\mu) M_{2k-1,2k}) \tag{14}$$

where $M_{i,j}$ is the element of $\mathfrak{so}(2k)$ corresponding to the antisymmetric matrix whose (i,j)-entry is 1, (j,i)-entry is -1, and all other entries are zero.

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- two-vectors and two-forms are identified via the standard euclidean structure of $\mathbb{R}^{2k+1}$.
- The adjoint orbit $\mathcal{O}_\mu$ is

$$G \cdot \frac{|\mu|}{\sqrt{k}} (M_{1,2} + \cdots + M_{2k-3,2k-2} + \operatorname{sign}(\mu) M_{2k-1,2k}) \tag{14}$$

where $M_{i,j}$ is the element of $\mathfrak{so}(2k)$ corresponding to the antisymmetric matrix whose (i, j)-entry is 1, (j, i)-entry is -1, and all other entries are zero.

Further Elaborations

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To understand an orbit inside P_μ, we need to understand its projection onto $X := \mathbb{R}^{2k+1}_*$ and then take a lift via the connection. The projection of an orbit, if it is non-colliding, shall be referred to as a **MICZ-Kepler orbit**. Here are a few known facts prior to our investigation:

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Theorem (Z.Q. Bai, G.W. Meng and E.X. Wang, 2013)

For the magnetized Kepler problems in dimension $2k+1$, a MICZ-Kepler orbit is an ellipse, a parabola, and a branch of a hyperbola according as the energy is negative, zero, and positive. Moreover, $\mathrm{SO}^+(1, 2k+1) \times \mathbb{R}_+$ acts transitively on both the set of oriented elliptic orbits and the set of oriented parabolic orbits.

Remark.

- The proof is not obvious.
- The key is to find the effective angular momentum $\bar{L}$ out of A and L, such that $\bar{L}$ is a decomposable 2-vector which becomes L if the dimension is 3 or the magnetic charge is 0.
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Sketch proof

Proof.

With the help of $\bar{L}$ and $\mathbf{A}$, again one has a lightcone formulation for the MICZ-Kepler orbits: each is the intersection of the future lightcone with an affine plane

$$m \wedge x = 0, \quad a \cdot x = 1 \tag{15}$$

in $\mathbb{R}^{1,2k+1}$. Here, *m is a decomposable 3-vector in the Lorentz space $\mathbb{R}^{1,2k+1}$ such that $(m, m) = 1$, and $a = (a_0, \mathbf{a})$ is a vector in the Lorentz space $\mathbb{R}^{1,2k+1}$ with $a_0 > 0$ and $a \wedge m = 0$.* In fact, in terms of $\bar{L}$ and $A := (1, \mathbf{A}) = e_0 + \mathbf{A}$, we have

$$m = \frac{\bar{L} \wedge A}{|\bar{L} \wedge A|}, \quad a = \frac{A}{|\bar{L} \wedge A|^2}.$$

□

Let us summarize this talk with a diagram.

Kepler Problem	Intrinsic Formulation $\Longrightarrow$	Higher Dim. KP
lightcone Formulation $\downarrow$		$\downarrow$ lightcone Formulation
MICZ KP	Intrinsic Formulation $\longrightarrow$	Higher Dim. Magnetized KP

Thanks!