POSITIVE DEFINITE KERNELS AND QUANTIZATION

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Hamiltonian Dynamical Systems	Quantum Dynamical Systems
• Symplectic manifold (M, ω) • Hamiltonian flow $\sigma_t : M \longrightarrow M$ • defined by Hamilton equation $X \llcorner \omega = dF$, where $F \in C^\infty(M, \mathbb{R})$ and $X \in \Gamma^\infty(M, TM)$ is tangent to $\{\sigma\}_{t\in\mathbb{R}}$.	• $\mathcal{H}$ — Hilbert space • Unitary flow $U_t = e^{it\hat{F}}$ where $\hat{F}$ — a selfadjoint operator $\hat{F} : \mathcal{D}(\hat{F}) \longrightarrow \mathcal{H}$ unbounded in general

quantization

Hamiltonian Dynamical System	$\Longrightarrow$ $\Longleftarrow$	Quantum Dynamical System

dequantization

Example

$$(\mathcal{H}, e^{it\hat{F}}, \hat{F} : \mathcal{D}(\hat{F}) \longrightarrow \mathcal{H}) \Longrightarrow (M, \omega, F)$$

$\hat{F} = \int \lambda dE(\lambda)$ — selfadjoint operator with semisimple spectrum

Thus
• $\mathcal{H} \cong L^2(\mathbb{R}, d\sigma)$ where $d\sigma(\lambda) = \langle 0|dE(\lambda)|0\rangle$ and $|0\rangle$ — cyclic for $\hat{F}$
• $|n\rangle := P_n(\hat{F})|0\rangle$, $n = 0, 1, \ldots$ — orthonormal basis in $\mathcal{H}$, where P_n — orthogonal polynomials with respect to $d\sigma$

We assume the condition

$$\limsup_{n \longrightarrow \infty} \frac{\sqrt[n]{|\mu|_n}}{n} < +\infty$$

on the absolute moments

$$|\mu|_n := \int_{\mathbb{R}} |\omega|^n d\sigma(\omega) = \frac{1}{P_0^2} \langle 0| \, |\hat{F}| \, |0\rangle$$

of the operator $\hat{F}$.

Then, there exists the open strip $\Sigma \subset \mathbb{C}$ in complex plane $\mathbb{C}$, which is invariant under the translations

$$\tau_t z := z + t$$

$t \in \mathbb{R}$ and such that the characteristic functions

$$\chi(s) = \int_{\mathbb{R}} e^{-i\omega s} d\sigma(\omega),$$

$s \in \mathbb{R}$, of the measure $d\sigma$ posses holomorphic prolongation χ_Σ on Σ.

Hence, one has the positive definite kernel on Σ

$$K_\Sigma(\bar{z}, v) := \chi_\Sigma(\bar{z} - v).$$

The map $\mathfrak{K}_\Sigma : \Sigma \longrightarrow \mathcal{H} \cong \mathcal{B}(\mathbb{C}, \mathcal{H})$ defined by

$$\mathfrak{K}_\Sigma(z) := \sum_{n=0}^{\infty} \chi_n(z)|n\rangle$$

where

$$\chi_n(z) := \int e^{-iz\omega} P_n(\omega) d\sigma(\omega),$$

for $z \in \Sigma$, gives factorization

$$K_\Sigma(\bar{z}, v) = \mathfrak{K}_\Sigma(z)^* \mathfrak{K}_\Sigma(v)$$

of the kernel K_Σ.

One has

$$e^{-it\hat{F}}\mathfrak{K}_\Sigma(z) = \mathfrak{K}_\Sigma(z+t).$$

The states $\mathfrak{K}_\Sigma(z)$, $z \in \mathbb{Z}$, span an essential domain $\mathcal{D}(\hat{F})$ of $\hat{F}$ and

$$\hat{F}\mathfrak{K}_\Sigma(z) = i\frac{d}{dz}\mathfrak{K}_\Sigma(z).$$

The function

$$F = (\log\circ\chi_\Sigma)'(\bar{z} - z).$$

and the vector field tangent to the translation flow $\tau(t)$

$$X = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

satisfy

$$X \llcorner \Omega_\Sigma = dF$$

for symplectic form

$$\Omega_\Sigma = i\partial\bar{\partial}(\log\circ K_\Sigma)(\bar{z}, z) = i(\log\circ\chi_\Sigma)''(\bar{z} - z)d\bar{z}\wedge dz.$$

Applying the geometric quantization to Hamiltonian system $(M = \Sigma, \omega = \Omega_\Sigma, F)$ we back to the initial quantum system $(\mathcal{H}, e^{it\hat{F}}, \hat{F} : \mathcal{D}(\hat{F}) \longrightarrow \mathcal{H})$.

$$(M = \Sigma, \omega = \Omega_\Sigma, F) \quad \overset{\substack{\textit{geometric} \\ \textit{quantization}}}{\Longrightarrow} \quad (\mathcal{H}, e^{it\hat{F}}, \hat{F} : \mathcal{D}(\hat{F}) \longrightarrow \mathcal{H})$$

Positive definite kernels on the principal bundles

- P — a set
- V and $\mathcal{H}$ — Hilbert spaces
- $\mathcal{B}(V, \mathcal{H})$ — Banach space of bounded linear operators from V into $\mathcal{H}$

(i) The $\mathcal{B}(V)$-**valued positive definite kernels**, i.e. maps $K : P \times P \to \mathcal{B}(V)$ such that for any finite sequences $p_1, \ldots, p_J \in P$ and $v_1, \ldots, v_J \in V$ one has

$$\sum_{i,j=1}^{J} \langle v_i, K(p_i, p_j) v_j \rangle \geqq 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in V.
One has

$$K(q, p) = K(p, q)^*$$

for each $q, p \in P$.

(ii) The **maps** $\mathfrak{K} : P \to \mathcal{B}(V, \mathcal{H})$ satisfying the condition

$$\{\mathfrak{K}(p)v : p \in P \ \ and \ \ v \in V\}^{\perp} = \{0\}.$$

(iii) The **Hilbert spaces** $\mathcal{K} \subset V^P$ realized by the functions $f : P \to V$ such that evaluation functionals

$$E_p f := f(p)$$

are continuous maps of Hilbert spaces $E_p : \mathcal{K} \to V$ for every $p \in P$.

There exist functorial equivalences between the categories of the object defined above.

• Equivalence between **(ii)** and **(iii)** is given as follows. For $\mathfrak{K} : P \to \mathcal{B}(V, \mathcal{H})$ we define monomorphism of vector spaces $J : \mathcal{H} \to V^P$ by

$$J(\psi)(p) := \mathfrak{K}(p)^* \psi,$$

and

$$\mathfrak{K}(p) := E_p^*,$$

where $\psi \in \mathcal{H}$, $p \in P$.

• The passage from **(ii)** to **(i)** is given by

$$K(q, p) := \mathfrak{K}(q)^* \mathfrak{K}(p).$$

• In order to show the implication **(i)** $\Rightarrow$ **(iii)** let us take vector subspace $\mathcal{K}_0 \subset V^P$ consisting of the following functions

$$f(p) := \sum_{i=1}^{I} K(p, p_i) v_i,$$

defined for the finite sequences $p_1, \ldots, p_I \in P$ and $v_1, \ldots, v_I \in V$.

Due to positive definiteness of the kernel $K : P \times P \to \mathcal{B}(V)$ we define a scalar product between $g(\cdot) = \sum_{j=1}^{J} K(\cdot, q_j) w_j \in \mathcal{K}_0$ and $f \in \mathcal{K}_0$ as follows

$$\langle g|f \rangle := \sum_{i=1}^{I} \sum_{j=1}^{J} \langle K(p_i, q_j) w_j, v_i \rangle.$$

We obtain $\mathcal{K} \subset V^P$ as a closure of $\mathcal{K}_0$ with respect to the norm given by the above scalar product.

Proposition

Let P be a smooth manifold and V a finite dimensional complex Hilbert space. Then the following properities are equivalent:

(a) *The positive definite kernel $K : P \times P \to \mathcal{B}(V)$ is a smooth map.*

(b) *The map $\mathfrak{K} : P \to \mathcal{B}(V, \mathcal{H})$ is smooth.*

(c) *The Hilbert space $\mathcal{K} \subset V^P$ defined in (iii) consists of smooth functions, i.e. $\mathcal{K} \subset C^\infty(P, V)$.*

From now let us assume that P is a principal bundle

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

over the smooth manifold M with some Lie group G as the structural group. Additionally we introduce a faithful representation of G

$$T : G \longrightarrow \mathrm{Aut}(V)$$

in Hilbert space V and suppose that positive definite kernel $K : P \times P \to \mathcal{B}(V)$ has equivariance property

$$K(p, qg) = K(p, q)T(g)$$

where $p, q \in P$ and $g \in G$. This property is equivalent to each of the following two properties

$$\mathfrak{K}(pg) = \mathfrak{K}(p)T(g)$$

and

$$f(pg) = T(g^{-1})f(p)$$

for $f \in \mathcal{K}$.

Using the action of G on $P \times V$ defined by

$$P \times V \ni (p, v) \mapsto (pg, T(g^{-1})v) \in P \times V$$

one obtains the T-associated vector bundle

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{V} \\ & & \big\downarrow \widetilde{\pi} \\ & & M \end{array}$$

over M with the quotient manifold $\mathbb{V} := (P \times V)/G$ as its total space.

Given $\pi(p) = m$, $\pi(q) = n$, we define by

$$K_T(m,n)([(p,v)],[(q,w)]) := \langle v, K(p,q)w \rangle,$$

the section

$$K_T : M \times M \longrightarrow pr_1^* \overline{\mathbb{V}}^* \otimes pr_2^* \mathbb{V}^*$$

of the bundle $pr_1^* \overline{\mathbb{V}}^* \otimes pr_2^* \mathbb{V}^* \to M \times M$.

The diagonal $K_{T|\Delta}$ of the kernel K_T determines positive semi-definite hermitian structure $H_K := K_{T|\Delta}$ on the bundle $\widetilde{\pi} : \mathbb{V} \to M$.

One has $I : \mathcal{H} \to C^\infty(M, \mathbb{V})$ a linear monomorphism of vector spaces defined by

$$I(\psi)(\pi(p)) := [(p, \mathfrak{K}(p)^*\psi)] = [(p, J(\psi)(p))].$$

Apart of hermitian structure H_K the positive hermitian kernel K defines on P a $\mathcal{B}(V)$-valued differential one-form

$$\vartheta(p) := (\mathfrak{K}(p)^*\mathfrak{K}(p))^{-1}\mathfrak{K}(p)^* d\mathfrak{K}(p) = K(p,p)^{-1} d_q K(p,q)_{|q=p},$$

which satisfy

$$\vartheta(pg) = T(g^{-1})\vartheta(p)T(g)$$

and

$$\langle v, K(p,p)\vartheta(p)w\rangle + \langle \vartheta(p)v, K(p,p)w\rangle = d\langle v, K(p,p)w\rangle.$$

Thus we conclude that $\vartheta \in C^\infty(P, T^*P \otimes \mathcal{B}(V))$ is the one-form of the metric connection ∇_K consistent with the hermitian structure H_K.

One-parameter groups of automorphisms and prequantization

Let $\xi \in C^\infty(P, TP)$ be the vector field tangent to the flow of authomorphisms $\tau : (\mathbb{R}, +) \to \mathrm{Aut}(P, \vartheta)$ of the principal bundle

$$\tau_t(pg) = \tau_t(p)g,$$

where $g \in G$ and $p \in P$, which preserve the connection form ϑ

$$\tau_t^* \vartheta = \vartheta.$$

Then one has

$$\xi(pg) = DR_g(p)\xi(p),$$

and

$$\mathcal{L}_\xi \vartheta = 0,$$

where $R_g(p) := pg$, $DR_g(p)$ is the derivative of R_g at p and $\mathcal{L}_\xi$ is Lie derivative with respect to ξ.

The space of vector fields preserving connection we denoted by $\mathcal{E}^0_G \subset C^\infty_G(P, TP)$.
For connection 1-form ϑ and the $DT(e)(\mathfrak{g})$-valued pseudotensorial 0-form, i.e. $DT(e)(\mathfrak{g})$-valued function such that

$$F(pg) = T(g^{-1})F(p)T(g),$$

one has

$$\Omega := \mathbf{D}\vartheta = d\vartheta + \frac{1}{2}[\vartheta, \vartheta],$$

$$\mathbf{D}F = dF + [\vartheta, F].$$

$C^\infty_G(P, DT(e)(\mathfrak{g}))$ — the space of $DT(e)(\mathfrak{g})$-valued functions satisfying equivariance condition
Now let us investigate the Lie algebra $\mathcal{P}_G$ which consists of pairs $(F, \xi) \in C^\infty_G(P, DT(e)(\mathfrak{g})) \times C^\infty_G(P, TP)$ such that

$$\xi \llcorner \Omega = \mathbf{D}F \qquad \Longleftrightarrow \qquad \mathcal{L}_\xi \vartheta = \mathbf{D}(F + \vartheta(\xi))$$

with the bracket $[\![\cdot,\cdot]\!] : \mathcal{P}_G \times \mathcal{P}_G \to \mathcal{P}_G$ defined for $(F, \xi), (G, \eta) \in \mathcal{P}_G$ by

$$[\![(F, \xi), (G, \eta)]\!] := (\{F, G\}, [\xi, \eta]),$$

where

$$\{F, G\} := 2\Omega(\xi, \eta) + \mathbf{D}G(\xi) - \mathbf{D}F(\eta) + [F, G] =$$
$$= -2\Omega(\xi, \eta) + [F, G] = \mathbf{D}G(\xi) + [F, G]$$

and $[\xi, \eta]$ is the commutator of vector fields.

• Let $\mathcal{E}_G$ be the Lie algebra of vector fields $\xi \in C^\infty_G(P, TP)$ for which exists $F \in C^\infty_G(P, DT(e)(\mathfrak{g}))$ such that $(F, \xi) \in \mathcal{P}_G$.

• Denote by $\mathcal{N}_G$ the set of $F \in C^\infty_G(P, DT(e)(\mathfrak{g}))$ such that $\mathbf{D}F = 0$.

• The subspace $\mathcal{P}^0_G \subset \mathcal{P}_G$ of such elements $(F, \xi) \in \mathcal{P}_G$ that $\xi \in \mathcal{E}^0_G$ and $F = F_0 - \vartheta(\xi)$, where $\mathbf{D}F_0 = 0$.

Summing up we have

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{N}_G & \xrightarrow{\iota_1} & \mathcal{P}_G & \xrightarrow{pr_2} & \mathcal{E}_G & \to & 0, \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{N}_G & \xrightarrow{\iota_1} & \mathcal{P}^0_G & \xrightarrow{pr_2} & \mathcal{E}^0_G & \to & 0, \end{array}$$

where horizontal arrows form the exact sequences of Lie algebras and vertical arrows are Lie algebra monomorphisms.

$$\iota_1(F) := (F, 0), \quad pr_2(F, \xi) := \xi.$$

From now on we will assume that M is a connected manifold and denote by $P(p)$ the set of elements of P which one can join with p by curves horizontal with respect to the connection ϑ. By $G(p)$ we denote the subgroup $G(p) \subset G$ consisting of those $g \in G$ for which $pg \in P(p)$, i.e. $G(p)$ is the holonomy group based at p. Let us recall that for connected base manifold M all holonomy groups $G(p)$ and their Lie algebras $\mathfrak{g}(p)$ are conjugated in G and $\mathfrak{g}$, respectively. Recall also that Lie algebra $\mathfrak{g}(p)$ is generated by $\Omega_{p'}(X(p'), Y(p'))$, where $p' \in P(p)$ and $X(p'), Y(p') \in T_{p'}P$. After these preliminary remarks we conclude that for $(F, \xi) \in \mathcal{P}_G$ the function F takes values $F(p')$ in $\mathfrak{g}(p)$ if $p' \in P(p)$. In the special case if $F \in \mathcal{N}_G$, i.e. when $\mathbf{D}F = 0$, function F is constant on $P(p)$ and $F(p) \in DT(e)(\mathfrak{g}(p)) \cap DT(e)(\mathfrak{g}'(p))$, where $\mathfrak{g}'(p)$ is the centralizer of the Lie subalgebra $\mathfrak{g}(p)$ in $\mathfrak{g}$.

In order to describe the Lie algebra $\mathcal{P}^0_G$ we define the linear monomorphism $\Phi : \mathcal{E}^0_G \to \mathcal{P}^0_G$ of Lie algebras by

$$\Phi(\xi) := (-\vartheta(\xi), \xi).$$

One has the decomposition

$$\mathcal{P}^0_G = \iota_1(\mathcal{N}_G) \oplus \Phi(\mathcal{E}^0_G)$$

of $\mathcal{P}^0_G$ into the direct sum of Lie subalgebra $\Phi(\mathcal{E}^0_G)$ and ideal $\iota_1(\mathcal{N}_G)$ of central elements of $\mathcal{P}^0_G$.

Now let us define the following Lie subalgebra

$$\mathcal{H}_G^0 := D\pi(\mathcal{E}_G^0),$$

of $C^\infty(M, TM)$, where $D\pi : TP \to TM$ is the tangent map of the bundle map $\pi : P \to M$.
We define the vector subspace $\mathcal{F}_G^0 \subset C_G^\infty(P, DT(e)(\mathfrak{g})) \times \mathcal{H}_G^0$ consisting of such elements $(F, X) \in C_G^\infty(P, DT(e)(\mathfrak{g})) \times \mathcal{H}_G^0$ which satisfy the condition (Hamilton equation)

$$X^* \llcorner \Omega = \mathbf{D}F,$$

where X^* is the horizontal lift of X with respect to ϑ.
One has

$$\xi = X^* - F^* \in \mathcal{E}_G^0,$$

where F^* is a vertical field defined by the function $F \in C_G^\infty(P, DT(e)(\mathfrak{g}))$

Proposition

One has the Lie algebras isomorphism between $(\mathcal{E}^0_G, [\cdot,\cdot])$ *and* $(\mathcal{F}^0_G, \{\!\{\cdot,\cdot\}\!\})$*, where the Lie bracket of* $(F,X),(G,Y) \in \mathcal{F}^0_G$ *is defined by*

$$\{\!\{(F,X),(G,Y)\}\!\} := (-2\Omega(X^*,Y^*) + [F,G],[X,Y]).$$

The following exact sequence of Lie algebras has place

$$0 \to \mathcal{N}_G \xrightarrow{\iota_1} \mathcal{F}^0_G \xrightarrow{pr_2} \mathcal{H}^0_G \to 0,$$

where $\iota_1(F) := (F,0)$ *and* $pr_2(F,X) := X$*.*

The integration of the horizontal part $\xi^h = X^*$ of $\xi \in \mathcal{E}^0_G$ gives the flow $\{\tau^h_t\}_{t\in\mathbb{R}}$ being the horizontal lift of the flow

$$\sigma : (\mathbb{R}, +) \longrightarrow \mathrm{Diff}(M)$$

defined by the projection of $\{\tau_t\}_{t\in\mathbb{R}}$ on the base M of the principal bundle P. The vector field $X \in \mathcal{H}^0_G$ is the velocity vector field of $\{\sigma_t\}_{t\in\mathbb{R}}$.

The flow

$$\widetilde{\tau}_t[(p,v)] := [(\tau_t(p),v)]$$

defines

$$(\widetilde{\Sigma}_t\psi)(\pi(p)) := \widetilde{\tau}_t\psi(\sigma_{-t}\circ\pi(p)) = \widetilde{\tau}_t\psi(\pi(\tau_{-t}(p))) = \widetilde{\tau}_t\psi(\pi(\tau^h_{-t}(p))),$$

where $\psi \in C^\infty(M,\mathbb{V})$.
The generator $Q_{(F,X)}$ of the one parameter group $\widetilde{\Sigma}_t$ is G-version of Kostant–Souriau prequantization operator

$$Q_{(F,X)} := -(\nabla_X + \widetilde{F}),$$

where $(F,X) \in \mathcal{F}^0_G$ and

$$\widetilde{F}([(p,v)]) := [(p,F(p)v)].$$

For

$$Q : \mathcal{F}_G^0 \longrightarrow \mathrm{End}(C^\infty(M, \mathbb{V}))$$

one has prequantization property

$$[Q_{(F,X)}, Q_{(G,Y)}] = Q_{\{\!\{(F,X),(G,Y)\}\!\}}.$$

In the non-degenerate case, i.e. when (F, X) is defined by F we have

$$[Q_F, Q_G] = Q_{\{F,G\}},$$

where $Q_F := Q_{(F,X_F)}$ and the bracket $\{F, G\}$ is defined by

$$\{F, G\} := -2\Omega(X_F^*, Y_G^*) + [F, G].$$

Quantization

We will quantize those flows which preserve $\mathcal{B}(V)$-valued positive definite kernel K

$$K(\tau_t(p), \tau_t(q)) = K(p, q), \qquad \text{for } p, q \in P \text{ and } t \in \mathbb{R}$$

i.e. $\{\tau_t\}_{t\in\mathbb{R}} \subset \mathrm{Aut}(P, K) \subset \mathrm{Aut}(P, \vartheta)$

Theorem

The flow $\{\tau_t\}_{t\in\mathbb{R}} \subset Aut(P, K)$ if and only if there exists an unitary flow $U_t : \mathcal{H} \to \mathcal{H}$ on the Hilbert space $\mathcal{H}$ such that

$$\mathfrak{K}(\tau_t(p)) = U_t \mathfrak{K}(p),$$

where the map $\mathfrak{K} : P \to \mathcal{B}(V, \mathcal{H})$ satisfies conditions of the definition **(ii)** *and factorizes the kernel $K(p, q) = \mathfrak{K}(p)^* \mathfrak{K}(q)$. The unitary flow $\{U_t\}_{t\in\mathbb{R}}$ is defined by $\{\tau_t\}_{t\in\mathbb{R}}$ in a unique way.*

Theorem

The vector space $\mathcal{H}_0 := span\{\mathfrak{K}(p)(v), p \in P, v \in V\}$ *is the essential domain of the generator* $\hat{F}$*, where* $\hat{F}$ *is generator of* $U_t = e^{it\hat{F}}$*.*
One has the filtration

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \ldots \subset \mathcal{U}_\infty \subset \mathcal{D}(\widehat{F})$$

of the domain $\mathcal{D}(\widehat{F})$ *of the operator* $\widehat{F}$ *onto its essential domains, where*

$$\mathcal{U}_l := \mathcal{U}_{l-1} + \widehat{F}(\mathcal{U}_{l-1}), \quad \mathcal{U}_0 := \mathcal{H}_0.$$

This filtration is preserved by the flow $\{U_t\}_{t\in\mathbb{R}}$*. Moreover*

$$\widehat{F}\mathcal{U}_l \subset \mathcal{U}_{l+1}$$

and

$$\mathcal{U}_\infty \subset \mathcal{D}(\widehat{F}^l),$$

for $l \in \mathbb{N} \cup \{0\}$*.*

The following relations are valid

$$U_t = I^{-1} \circ \widetilde{\Sigma}_t \circ I$$

and

$$\widehat{F} = iI^{-1} \circ Q_{(F,X)} \circ I.$$

One also has

$$F(p) = i(\mathfrak{K}(p)^* \mathfrak{K}(p)^{-1} \mathfrak{K}^*(p) \hat{F} \mathfrak{K}(p).$$

For the further investigation of $\widehat{F}$ we will describe its representation in a trivialization

$$s_\alpha : \Omega_\alpha \to P, \quad \pi \circ s_\alpha = id_{\Omega_\alpha}$$

of $\pi : P \to M$, where $\bigcup_{\alpha \in A} \Omega_\alpha = M$ is a covering of M by the open subsets.
We note that on $\pi^{-1}(\Omega_\alpha)$ one has

$$\Omega(p) = T(h^{-1}) \left(d\vartheta_\alpha(m) + \frac{1}{2}[\vartheta_\alpha(m), \vartheta_\alpha(m)] \right) T(h),$$

$$\mathbf{D}F(p) = T(h^{-1}) \left(dF_\alpha(m) + [\vartheta_\alpha(m), F_\alpha(m)] \right) T(h),$$

for $p = s_\alpha(m)h$, where

$$\vartheta_\alpha := s_\alpha^* \vartheta \quad \textit{and} \quad F_\alpha := F \circ s_\alpha.$$

We find that for $\xi = X^* - F^* \in \mathcal{E}_G^0$ and for $\varphi_\alpha := F_\alpha + \vartheta_\alpha(X)$ we have

$$\mathcal{L}_X \vartheta_\alpha = d\varphi_\alpha + [\vartheta_\alpha, \varphi_\alpha].$$

The positive definite kernel $K : P \times P \to \mathcal{B}(V)$ in the trivialization is described by

$$\mathfrak{K}_\alpha(m) := \mathfrak{K} \circ s_\alpha(m),$$

$$K_{\overline{\alpha}\beta}(m, n) := \mathfrak{K}_\alpha^*(m) \mathfrak{K}_\beta(n),$$

for $m \in \Omega_\alpha$ and $n \in \Omega_\beta$ and connection form by

$$\vartheta_\alpha(m) = \left(\mathfrak{K}_\alpha(m)^* \mathfrak{K}_\alpha(m)\right)^{-1} \mathfrak{K}_\alpha(m)^* d\mathfrak{K}_\alpha(m).$$

We find that

$$i\widehat{F}\mathfrak{K}_\alpha(m)v = (X\mathfrak{K}_\alpha)(m)v + \mathfrak{K}_\alpha(m)\varphi_\alpha(m)v, \qquad (1)$$

where $v \in V$, $m \in \Omega_\alpha$.

The selfadjointess of $\widehat{F}$ implies the following relation

$$\mathfrak{K}_\beta(n)^*(X\mathfrak{K}_\alpha)(m) + (X\mathfrak{K}_\beta)(n)^*\mathfrak{K}_\alpha(m) + \mathfrak{K}_\beta(n)^*\mathfrak{K}_\alpha(m)\varphi_\alpha(m) + \varphi_\beta(n)^*\mathfrak{K}_\beta(n)$$

between the kernel map $\mathfrak{K}_\alpha : \Omega_\alpha \to \mathcal{B}(V, \mathcal{H})$ and $(F, X) \in \mathcal{F}^0_G$.

In the s_α-gauge section $I(\psi) \in C^\infty(M, \mathbb{V})$ and $Q_{(F,X)} I(\psi)$ are given by

$$I(\psi)(m) = [(s_\alpha(m), \mathfrak{K}_\alpha^*(m)\mathfrak{K}_\beta(n)v)]$$

and by

$$(Q_{(F,X)} I(\psi))(m) = iI(\hat{F}\psi)(m) = [(s_\alpha(m), \mathfrak{K}_\alpha^*(m)\hat{F}\mathfrak{K}_\beta(n)v)]$$

respectively, $m \in \Omega_\alpha$. Hence we obtain the expression on $Q_{(F,X)}$ in terms of the kernel $K_{\bar{\alpha}\beta}(m, n)$:

$$Q_{(F,X)}(K_{\bar{\alpha}\beta}(\cdot, n))(m)v = -(XK_{\bar{\alpha}\beta})(\cdot, n)(m)v - \phi_\alpha(m)^* K_{\bar{\alpha}\beta}(m, n)v.$$

- A.O., M. Horowski, "Positive kernels and quantization", J. Geom. Phys. **63**, (2013), 80-98
- A.O., "On reproducing kernels and quantization of states", Commun. Math. Phys. **114**, (1988), 577–597
- A.O., "Coherent states and geometric quantization", Commun. Math. Phys. **150**, (1992), 385-413.
- A.O., M. Świętochowski, "Coherent states map for MIC-Kepler system", J. Math. Phys. 38(10), 1997
- M. Horowski, A.O., "Geometry of the Kepler System in Coherent States Approach", Ann. Inst. Henri Poincaré, Vol. **59**, No. 1, 1993, p. 69-89.
- M. Horowski, A.O., A. Tereszkiewicz, "Some integrable systems in nonlinear quantum optics", J. Math. Phys. **44** (2003) 480-506.