# Diffusion under Geometric Constraint

## Quasi 1 dimensional effective diffusion equation

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# Motivation

Transport phenomena in

- Biological cell
- Zeolite
- Katalytic reactions in Porous media

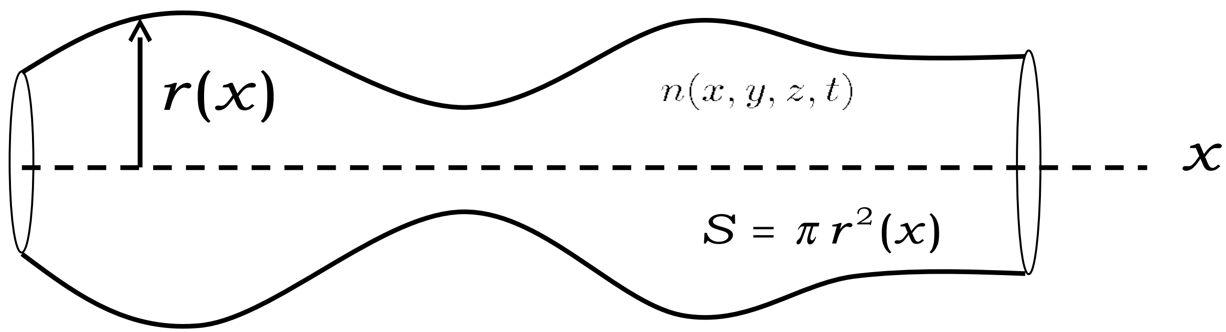
Quasi one-dimensional diffusion (Diffusion in Tube) with

Changing cross section (Channel model)

(P. Haenggi, P. Talkner et.al.)

non zero Curvature (Curved Tube)

# Channel Model



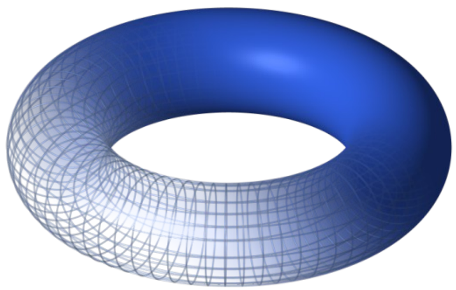

$$D(x) = \frac{1}{(1 + r'(x)^2)^{1/2}}$$

**Zwanzig ’92 J.Phys.Chem.,**

**Reguera, Rubi ’01 Phys.Rev.E,**

**Kalinay, Percus ’06, Phys.Rev. E**

# What about curved tube?



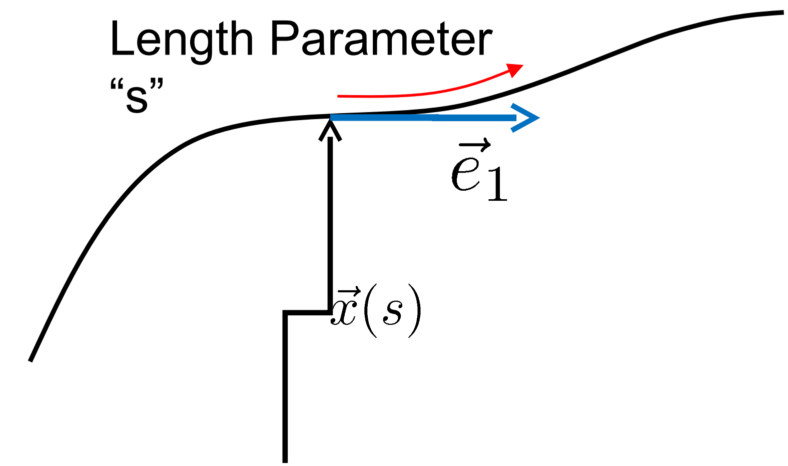
Torus

helix

## Or more general curved Tube ?

## How is the diffusion coefficient?

# Center line of Tube




$$\vec{e}_1 = \frac{d\vec{x}(s)}{ds}$$ : tangent

$$\frac{d\vec{e}_1}{ds} = \kappa\vec{e}_2, \qquad |\,\vec{e}_2\,| = 1.$$

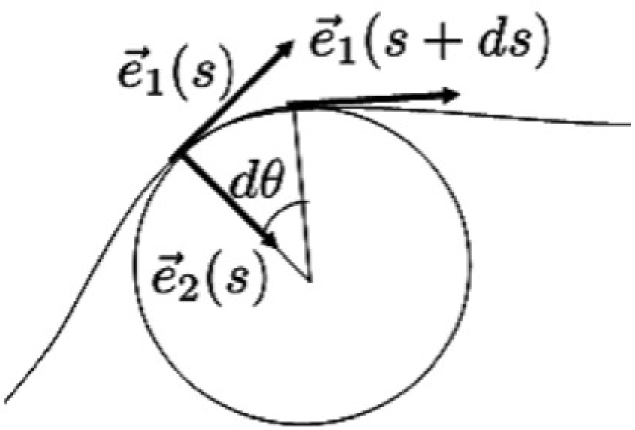
Curvature of line

$$\vec{e}_3 = \vec{e}_1 \times \vec{e}_2$$

$$\frac{d\vec{e}_2}{ds} = -\kappa\vec{e}_1 + \tau\vec{e}_3,$$

Torsion of line

$$\frac{d\vec{e}_3}{ds} = -\tau\vec{e}_2$$



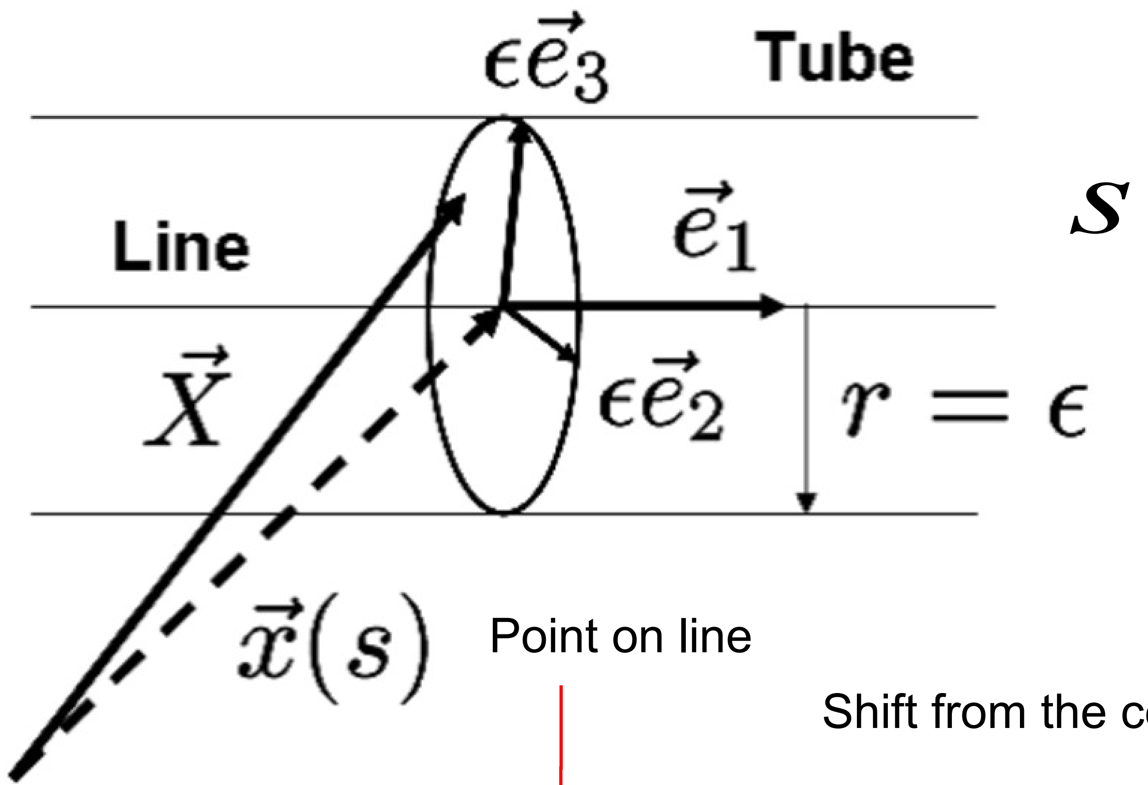

$\vec{e}_2$ is unit vector and has a direction to the center of curvature

# Frenet Seret equation

$$\frac{d\vec{e}_1}{ds} = \kappa \vec{e}_2, \quad \frac{d\vec{e}_2}{ds} = -\kappa \vec{e}_1 + \tau \vec{e}_3, \quad \frac{d\vec{e}_3}{ds} = -\tau \vec{e}_2$$

Forming triad on curved line in R3

# Curvilinear Coordinates in Tube




Point on line

Shift from the central line

$$\vec{X}(s, q^2, q^3) = \vec{x} + q^2\vec{e}_2 + q^3\vec{e}_3$$

Local coordinates

# Metric tensor

$$dS^2 = d\vec{X} \cdot d\vec{X} = \frac{\partial \vec{X}}{\partial q^i} \cdot \frac{\partial \vec{X}}{\partial q^j} dq^i dq^j \equiv G_{ij} dq^i dq^j \quad (q^1 = s)$$

$$G_{ij} = \begin{pmatrix} (1-\kappa q^2)^2 + \tau^2 r^2 & -\tau q^3 & \tau q^2 \\ -\tau q^3 & 1 & 0 \\ \tau q^2 & 0 & 1 \end{pmatrix}$$

$$r^2 = (q^2)^2 + (q^3)^2$$

$$G \equiv \det(G_{ij}) = (1 - \kappa q^2)^2.$$

# Diffusion Field in Tube

Diffusion equation in Tube :

$$\frac{\partial \phi^{(3)}}{\partial t} = D\Delta^{(3)}\phi^{(3)}$$

Normalization :

$$\int \phi^{(3)}(q^1, q^2, q^3)\sqrt{G}\, d^3q = N$$

$s = q^1$

$\epsilon \to 0$

Quasi-1D equation

$$\frac{\partial \phi^{(1)}}{\partial t} = D\Delta^{(eff)}\phi^{(1)}$$

Normalization

$$\int \phi^{(1)}(s)ds = N$$

From Two normalization conditions,

$$\int \phi^{(3)}(q^1, q^2, q^3)\sqrt{G}\; d^3q = \int \phi^{(1)} ds$$

$$\phi^{(1)} = \int \phi^{(3)}(s, q^2, q^3)\sqrt{G}\; dq^2 dq^3$$

Then we suppose,

Fluctuation on cross section

$$\phi^{(3)} = \frac{\phi^{(1)}(s,t)}{\sigma} + n(s, q^2, q^3, t)$$

$$0 = \int n\sqrt{G} d\sigma$$

: subsidiary condition

Mean value of φ(3) on cross section

Where,

$$d\sigma \equiv dq^2 dq^3; \quad \sigma = \pi\epsilon^2$$

: area of cross section (constant)

# Equation for $n$ field

$$\frac{\partial \phi^{(3)}}{\partial t} = D\hat{\Delta}\phi^{(3)}$$

$$\frac{\partial \phi^{(1)}}{\partial t} = D\int \sqrt{G}\hat{\Delta}\phi^{(3)} d\sigma$$

$$\frac{\partial n}{\partial t} = \frac{1}{\sigma}\{D\hat{\Delta} - \frac{\partial}{\partial s}D_{eff}\frac{\partial}{\partial s}\}\phi^{(1)} + \{D\hat{\Delta}n - \frac{D}{\sigma}\int \sqrt{G}\hat{\Delta}n\, d\sigma\}.$$

To be solved

$$\phi^{(3)} = \frac{\phi^{(1)}(s,t)}{\sigma} + n(s, q^2, q^3, t)$$

$$D_{eff} \equiv \frac{D}{\pi\epsilon^2}\int G^{1/2}G^{11}d\sigma = \frac{D}{\sigma}\int \frac{d\sigma}{1-\kappa q^2} = D\{1 + (\frac{\kappa\epsilon}{2})^2 + \mathcal{O}(\epsilon^4)\}.$$

# Formal solution

$$n = e^{D\hat{\Delta}t} n(0) + \frac{D}{\sigma} \int_0^t e^{D\hat{\Delta}(t-t')} [\hat{F}\phi^{(1)}(t') - \int \sqrt{G}\hat{\Delta} n(t')\ d\sigma] dt'$$

$$\hat{F} \equiv \hat{\Delta} - \frac{\partial}{\partial s} \frac{D_{eff}}{D} \frac{\partial}{\partial s}$$

Iterative method with Initial condition $n(0) = 0$

$$n = n_0 + n_1 + n_2 + \cdots .$$

0th : $$n_0 = \frac{D}{\sigma} \int_0^t e^{D\hat{\Delta}(t-t')} \hat{F}\phi^{(1)}(t') dt'$$

# Markov approximation

At infinite t, S changes rapidly and M changes slowly, we obtain

$$\int_0^t S(t-t')\ M(t')\ dt' \sim \int_0^\infty S(\tau)\ d\tau\ M(t)$$

Then we have

$\mathcal{O}(\epsilon^{-2})$

$$n_0 = \frac{D}{\sigma}\int_0^t e^{D\hat{\Delta}(t-t')}\hat{F}\phi^{(1)}(t')dt'$$

To use Markov approximation,

It is important to hold

$$\Delta t >> \epsilon^2/D$$

time scale >> relaxation time on cross section

: equilibrium holds in cross section

$$n_0 \sim \frac{D}{\sigma}\int_0^\infty e^{D\hat{\Delta}\tau}d\tau\ \hat{F}\phi^{(1)}(t) = -\frac{1}{\sigma}\ \frac{1}{\hat{\Delta}}\hat{F}\phi^{(1)}(t)$$

# Higher orders in iterarti on

$$n_1 \simeq -\frac{D}{\sigma} \int_0^\infty e^{D\hat{\Delta}\tau} d\tau \int \sqrt{G}\hat{\Delta} n_0(t)\ d\sigma$$

$$= \frac{1}{\sigma} \frac{1}{\hat{\Delta}} \int \sqrt{G}\hat{\Delta} n_0(t)\ d\sigma$$

$$n_0 \simeq -\frac{1}{\sigma}\ \frac{1}{\hat{\Delta}} \hat{F}\phi^{(1)}(t)$$

$$= -\frac{1}{\sigma^2} \frac{1}{\hat{\Delta}} \int \sqrt{G}\hat{F}\phi^{(1)}(t)\ d\sigma = \xi_1, \qquad \hat{\Delta}\xi_1 = 0$$

From the definition of Deff, $\int \sqrt{G}\hat{F}\phi^{(1)} d\sigma = \int \sqrt{G}(\hat{\Delta} - \frac{\partial}{\partial s}\frac{D_{eff}}{D}\frac{\partial}{\partial s})\phi^{(1)} d\sigma = 0.$

In the same way

$$n_2 = -\frac{D}{\sigma} \int_0^\infty e^{D\hat{\Delta}\tau} d\tau \int \sqrt{G}\hat{\Delta}\xi_1(t) d\sigma$$

$$= \frac{1}{\sigma} \frac{1}{\hat{\Delta}} \int \sqrt{G}\hat{\Delta}\xi_1(t) d\sigma = \xi_2, \quad \hat{\Delta}\xi_2 = 0$$

# Solution

$$n = n_0 + \xi_1 + \xi_2 + \cdots$$

solutions of Laplace equation

$$\frac{\partial \phi^{(1)}}{\partial t} = D \int \sqrt{G}\hat{\Delta}\phi^{(3)} dq^2 dq^3 \qquad \phi^{(3)} = \frac{\phi^{(1)}(s,t)}{\sigma} + n(s,q^2,q^3,t)$$

$$\frac{\partial \phi^{(1)}}{\partial t} = \frac{\partial}{\partial s} D_{eff} \frac{\partial}{\partial s}\phi^{(1)} + D \int \sqrt{G}\hat{\Delta} n \, d\sigma$$

But we have

$$\int \sqrt{G}\hat{\Delta} n_0 \, d\sigma = -\frac{1}{\sigma}\int \sqrt{G}\hat{\Delta} \, \frac{1}{\hat{\Delta}}\hat{F}\phi^{(1)}(t) d\sigma = 0$$

1D effective equation does not suffered by fluctuation on cross section.

# Effective equation fo $\phi^{(1)}$ r

$$\frac{\partial \phi^{(1)}}{\partial t} = \frac{\partial}{\partial s} D_{eff} \frac{\partial}{\partial s} \phi^{(1)}$$

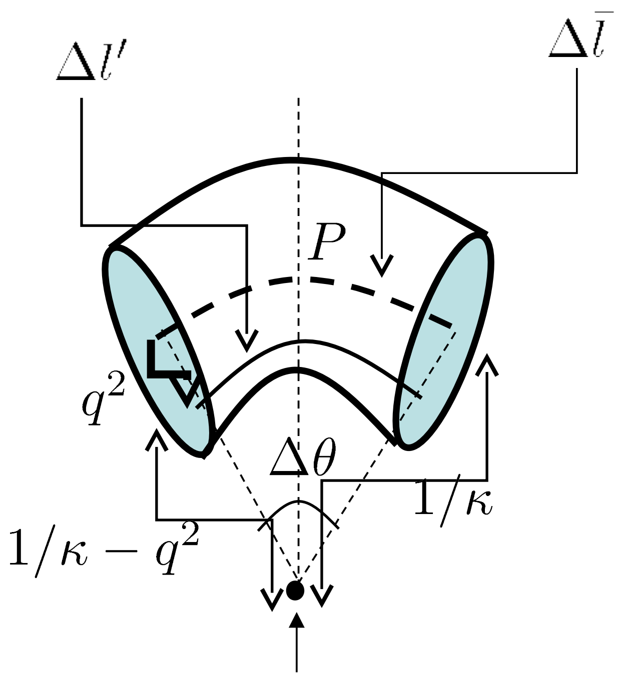
$$D_{eff} = \frac{D}{\pi \epsilon^2} \int \frac{d\sigma}{1 - \kappa q^2} = 2D \frac{1 - \sqrt{1 - (\kappa \epsilon)^2}}{(\kappa \epsilon)^2}$$

# Physical meaning of Diffusion Coefficient

$$D_{eff} = D < \frac{1}{1-\kappa q^2} >, \qquad < \cdots > = \frac{1}{\pi \epsilon^2} \int d\sigma \cdots .$$

average on cross section

Geometrical meaning




Center of curvature

$$\frac{\Delta l'}{\Delta \bar{l}} = \frac{1/\kappa - q^2}{1/\kappa} = 1 - \kappa q^2$$

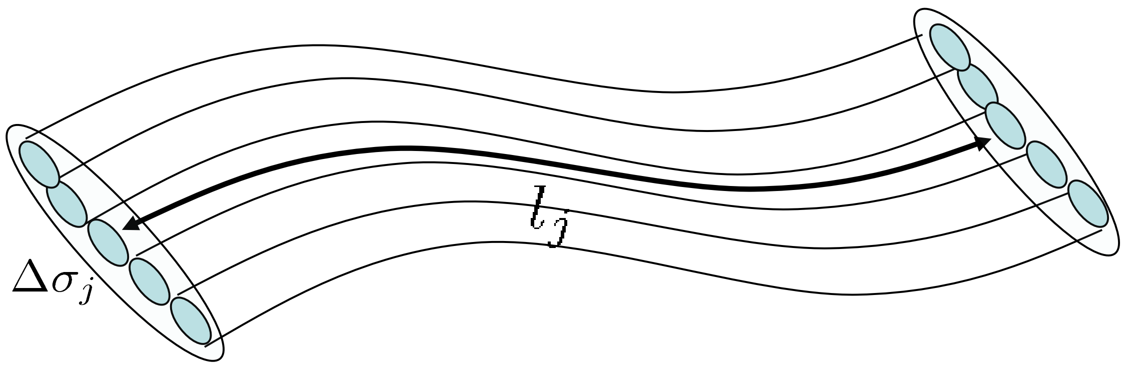
Rate of two point lengths along the tube

# Analogy to Ohm's law

( Ohm's law )

$$J = \frac{D}{l}\Delta N \quad \rightarrow \quad J\sigma = \frac{D\sigma}{l}\Delta N \sim I = \frac{1}{R}V$$

Diffusion flow ($J$) — Density difference ($\Delta N$) — Flow quantity ($J\sigma$)




Total conductivity $= \sum_j \frac{1}{R_j} = D \sum_j \frac{\Delta\sigma_j}{l_j} = D_{eff}\frac{\sigma}{\bar{l}}$

$$D_{eff} = \frac{D}{\sigma}\sum_j \frac{\Delta\sigma_j}{l_j/\bar{l}} = \frac{D}{\pi\epsilon^2}\int \frac{d\sigma}{1-\kappa q^2}$$

# Mean square displacement

For any function of position “s”,

$$< f(s) > \equiv \frac{\int f(s)\ \phi(s,t)\ ds}{\int \phi(s,t)\ ds}$$

:Avarage

$$\frac{\partial}{\partial t} < (\Delta s)^2 > = \frac{\partial}{\partial t}(< s^2 > - < s >^2)$$
$$= 2 < D_{eff}(s) > + 2 < (\Delta s) D'_{eff}(s) > \longrightarrow$$

“Usual case”:

$$< (\Delta s)^2 > = 2Dt$$

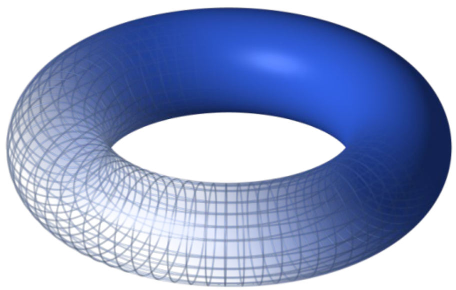
$$\begin{aligned} \frac{\partial^2}{\partial t^2} < (\Delta s)^2 > \ &= \ 6 < D''_{eff}(s) D_{eff}(s) > \\ &+ \ 2 < D'_{eff}(s)^2 > \\ &+ \ 2 < D'''_{eff}(s) D_{eff}(s)(\Delta s) > \\ &+ \ 2 < D''_{eff}(s) D'_{eff}(s)(\Delta s) > . \end{aligned}$$

# For constant curvature case, Diffusion coefficient is consta nt

$$\frac{\partial}{\partial t} < (\Delta s)^2 >= 2 < D_{eff} >= \mathrm{Const.}$$

$$< (\Delta s)^2 >= 2 < D_{eff} > t$$

# Constant curvature tubes (examples) $\frac{d\kappa}{ds} = 0$



Torus

helix

$$\kappa = \frac{1}{R}$$

$$\kappa = \frac{R\omega^2}{\mu^2 + R^2\omega^2}$$

$$\begin{aligned} x(u) &= R\cos\omega u, \\ y(u) &= R\sin\omega u, \\ z(u) &= \mu u \end{aligned}$$

$$< (\Delta s)^2 > \; = \; 4D\frac{1-\sqrt{1-(\kappa\epsilon)^2}}{(\kappa\epsilon)^2}\; t$$

# For general curved tube $\frac{d\kappa}{ds} \neq 0$

Higher Time derivatives of mean square displacement do not vanish. Only the short time expansion is possible.

$$\phi(t = 0, s) = \delta(s)$$

$$< (\Delta s)^2 >= a_1 t + a_2 t^2 + \cdots$$

$$\begin{cases} a_1 = \left.\frac{\partial < (\Delta s)^2 >}{\partial t}\right|_{t=0} = 2 < D_{eff}(s) >_{t=0} +2 < (\Delta s) D'_{eff}(s) >_{t=0} = 2D_{eff}(0). \\ a_2 = \left.\frac{1}{2}\frac{\partial^2 < (\Delta s)^2 >}{\partial t^2}\right|_{t=0} = 3D''_{eff}(0)D_{eff}(0) + D'_{eff}(0)^2. \end{cases}$$

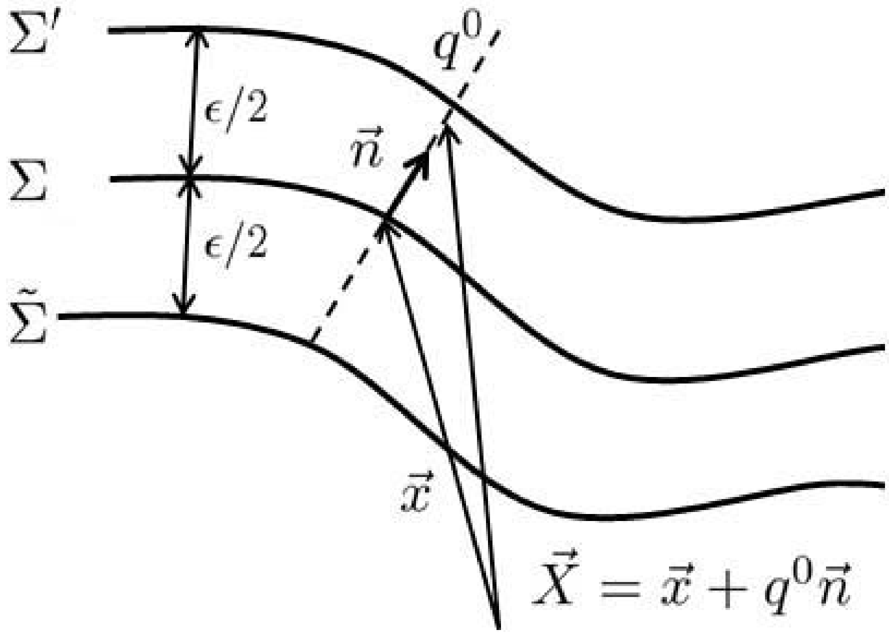
Where, $$D_{eff}(s) = 2D\frac{1 - \sqrt{1 - (\kappa(s)\epsilon)^2}}{(\kappa(s)\epsilon)^2}$$

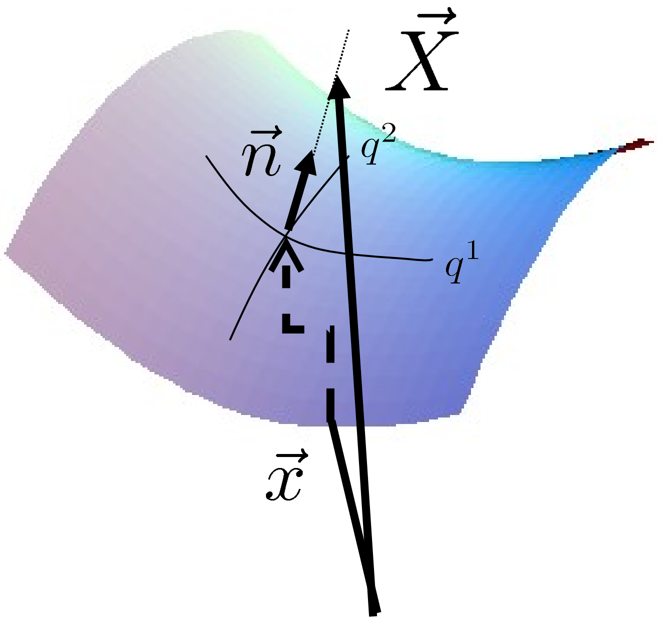
# Conclusion

○ Effective quasi one dimensional diffusion eq. is given by using Markov approximation.
Diffusion coefficient depends on local curvature.

○ Mean square displacement can be expanded by curvature and its derivatives.

# 曲面に挟まれた空間での拡散




$$\vec{X}(q^0, q^1, q^2) = \vec{x}(q^1, q^2) + q^0\vec{n}$$

$$-\epsilon/2 \leq q^0 \leq \epsilon/2$$

# Metric neighborhood of surface

$$G_{\mu\nu} = \frac{\partial \vec{X}}{\partial q^\mu} \cdot \frac{\partial \vec{X}}{\partial q^\nu}$$

$$G_{ij} = g_{ij} + q^0 \left( \frac{\partial \vec{x}}{\partial q^i} \cdot \frac{\partial \vec{n}}{\partial q^j} + \frac{\partial \vec{x}}{\partial q^j} \cdot \frac{\partial \vec{n}}{\partial q^i} \right) + (q^0)^2 \frac{\partial \vec{n}}{\partial q^i} \cdot \frac{\partial \vec{n}}{\partial q^j} \quad (i, j = 1, 2)$$

$$g_{ij} = \vec{B}_i \cdot \vec{B}_j .$$

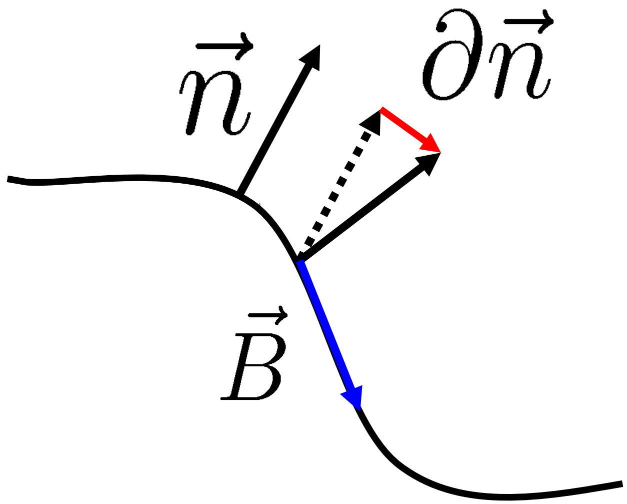
$$G_{0i} = G_{i0} = 0, \quad G_{00} = 1.$$

# 曲面の曲率

## 第２基本計量

$$\kappa_{ij} = \frac{\partial \vec{n}}{\partial q^i} \cdot \vec{B}_j .$$

$$g_{ij} = \vec{B}_i \cdot \vec{B}_j$$




$$\vec{B}_k = \frac{\partial \vec{x}}{\partial q^k}$$

Tangential vector

# Tools for geometrical study

**Gauss equation**

$$\frac{\partial \vec{B}_i}{\partial q^j} = -\kappa_{ij}\vec{n} + \Gamma^k_{ij}\vec{B}_k,$$

**Weingarten equation**

$$\frac{\partial \vec{n}}{\partial q^j} = \kappa^m_j \vec{B}_m$$

**Second Fundamental Tensor**

$$\kappa_{ij} = \frac{\partial \vec{n}}{\partial q^i} \cdot \vec{B}_j.$$

**Mean Curvature**

$$\kappa = g^{ij}\kappa_{ij},$$

**Ricci Scalar, Gauss Curvature**

$$R/2 = det(g^{ik}\kappa_{kj}) = det(\kappa^i_j).$$

# Form of metric with Curvature

$$G_{ij} = g_{ij} + 2q^0 \kappa_{ij} + (q^0)^2 \kappa_{im} \kappa_j^m .$$

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & G_{ij} \end{pmatrix} .$$

$$G_{ij} = (g_i^m + q^0 \kappa_i^m)(g_{mj} + q^0 \kappa_{mj}) \sim (g + q^0 \kappa)^2$$

# Embedding of Diffusion Field

$$\frac{\partial \phi^{(3)}}{\partial t} = D\Delta^{(3)}\phi^{(3)}, \quad 1 = \int \phi^{(3)}(q^0, q^1, q^2)\sqrt{G}\ d^3q$$

$$\downarrow$$

$$\frac{\partial \phi^{(2)}}{\partial t} = D\tilde{\Delta}^{(2)}\phi^{(2)}, \quad 1 = \int \phi^{(2)}(q^1, q^2)\sqrt{g}\ d^2q$$

$$\begin{aligned} 1 &= \int \phi^{(3)}(q^0, q^1, q^2)\sqrt{G}\ d^3q \\ &= \int \Big[\int_{-\epsilon/2}^{\epsilon/2} dq^0 \big(\phi^{(3)}\sqrt{G/g}\big)\Big]\sqrt{g}\ d^2q \\ &= \int \phi^{(2)}(q^1, q^2)\sqrt{g}\ d^2q \end{aligned}$$

$$\phi^{(3)}\sqrt{G/g} \rightarrow \tilde{\phi}^{(3)}$$

$$\phi^{(2)}(q^1, q^2) = \int_{-\epsilon/2}^{\epsilon/2} \tilde{\phi}^{(3)} dq^0,$$

We multiply $\sqrt{G/g}$ and integrate by $q^0$ to equation $\frac{\partial \phi^{(3)}}{\partial t} = D\Delta^{(3)}\phi^{(3)}$

$$\frac{\partial \phi^{(2)}}{\partial t} = D \int_{-\epsilon/2}^{\epsilon/2} \tilde{\Delta}^{(3)} \tilde{\phi}^{(3)} dq^0$$

$$\tilde{\Delta}^{(3)} \equiv \sqrt{G/g}\ \ \Delta^{(3)} \sqrt{g/G}.$$

$$\tilde{\Delta}^{(3)} = g^{-1/2} \frac{\partial}{\partial q^\mu} G^{1/2} G^{\mu\nu} \frac{\partial}{\partial q^\nu} (g/G)^{1/2} = \tilde{\Delta}^{(2)} + \tilde{\Delta}^{(1)}.$$

$$\tilde{\Delta}^{(2)} \equiv g^{-1/2} \frac{\partial}{\partial q^i} G^{1/2} G^{ij} \frac{\partial}{\partial q^j} (g/G)^{1/2}, \quad \tilde{\Delta}^{(1)} \equiv \frac{\partial}{\partial q^0} G^{1/2} \frac{\partial}{\partial q^0} G^{-1/2}.$$

## Boundary Condition

$$\int_{-\epsilon/2}^{\epsilon/2} \tilde{\Delta}^{(1)} \tilde{\phi}^{(3)} dq^0 = g^{-1/2} \int_{-\epsilon/2}^{\epsilon/2} \frac{\partial}{\partial q^0} (G)^{1/2} \frac{\partial}{\partial q^0} \phi^{(3)} \; dq^0$$

$$= g^{-1/2} [\; (G)^{1/2} \frac{\partial \phi^{(3)}}{\partial q^0} ] \,|_{-\epsilon/2}^{\epsilon/2} = 0.$$

## Main Part

$$\tilde{\Delta}^{(2)} = \Delta^{(2)} + q^0 \hat{A} + (q^0)^2 \hat{B} + \mathcal{O}(\epsilon^3), \qquad (1)$$

where,

$$\hat{A} = -g^{-1/2} \frac{\partial}{\partial q^i} g^{1/2} (2\kappa^{ij} \frac{\partial}{\partial q^j} + g^{ij} \frac{\partial \kappa}{\partial q^j}), \qquad (2)$$

$$\hat{B} = g^{-1/2} \frac{\partial}{\partial q^i} g^{1/2} (3\kappa^{im} \kappa_m^j \frac{\partial}{\partial q^j} + \frac{1}{2} g^{ij} \frac{\partial (\kappa^2 - R)}{\partial q^j} + 2\kappa^{ij} \frac{\partial \kappa}{\partial q^j}), \qquad (3)$$

# Equation

$$\frac{\partial \phi^{(2)}}{\partial t} = D\Delta^{(2)}\phi^{(2)} + D\hat{A}\int_{-\epsilon/2}^{\epsilon/2} q^0\tilde{\phi}^{(3)}dq^0 + D\hat{B}\int_{-\epsilon/2}^{\epsilon/2}(q^0)^2\tilde{\phi}^{(3)}dq^0 + \mathcal{O}(\epsilon^3).$$

法線方向への
拡散時間は $\epsilon^2/D$
観測時間スケール
が十分に長ければ
、
この方向は常に平
衡と考えてよい。

# Assumption

$$\frac{\partial \phi^{(3)}}{\partial q^0} = g^{1/2}\frac{\partial G^{-1/2}\tilde{\phi}^{(3)}}{\partial q^0}.$$

$\phi^{(2)}(q^1,q^2) = \int_{-\epsilon/2}^{\epsilon/2}\tilde{\phi}^{(3)}dq^0,$

$$\tilde{\phi}^{(3)} = \frac{1}{N}(G/g)^{1/\ldots}\,{}^{(2)}(q^1,q^2),\quad N \equiv \int_{-\epsilon/2}^{\epsilon/2}(G/g)^{1/2}dq^0.$$

# Expectation value of $q^0$

$$< f(q^0) > \equiv \frac{1}{N} \int_{-\epsilon/2}^{\epsilon/2} f(q^0)(G/g)^{1/2} dq^0.$$

$$\begin{aligned} N &= \epsilon + \frac{R}{24}\epsilon^3 + \mathcal{O}(\epsilon^5), \\ < q^0 > &= \frac{\kappa\epsilon^2}{12} + \mathcal{O}(\epsilon^4), \\ < (q^0)^2 > &= \frac{\epsilon^2}{12} + \mathcal{O}(\epsilon^4), \end{aligned}$$

# Anomalous Diffusion Equation

$$\frac{\partial \phi^{(2)}}{\partial t} = D\Delta^{(2)}\phi^{(2)} + \frac{\epsilon^2}{12}D(\hat{A}\kappa + \hat{B})\phi^{(2)}$$
$$= D\Delta^{(2)}\phi^{(2)} + \tilde{D}g^{-1/2}\frac{\partial}{\partial q^i}\ g^{1/2}\{(3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij})\frac{\partial}{\partial q^j} - \frac{1}{2}g^{ij}\frac{\partial R}{\partial q^j}\}\phi^{(2)}$$

$$-\frac{\partial \phi^{(2)}}{\partial t} = \nabla_i(J^i_N + J^i_A) = g^{-1/2}\frac{\partial}{\partial q^j}\ g^{1/2}(J^i_N + J^i_A) \qquad \tilde{D} = \frac{\epsilon^2}{12}D$$

Anomalous Diffusion flow

$$J^i_A = -\tilde{D}\{(3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij})\frac{\partial \phi^{(2)}}{\partial q^j} - \frac{1}{2}g^{ij}\frac{\partial R}{\partial q^j}\phi^{(2)}\}.$$

Diffusion vs

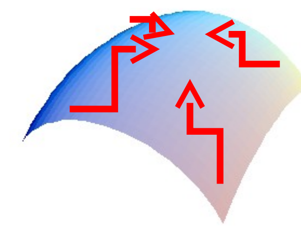
Concentration flow

curvature gradient

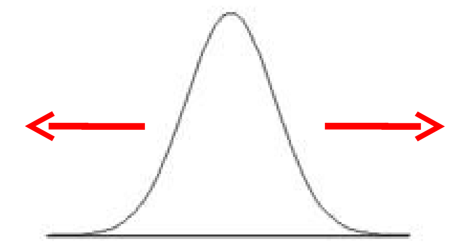
# Properties of Anomalous flow

$$J_A^i = -\tilde{D}\{(3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij})\frac{\partial \phi^{(2)}}{\partial q^j} - \frac{1}{2}g^{ij}\frac{\partial R}{\partial q^j}\phi^{(2)}\}.$$

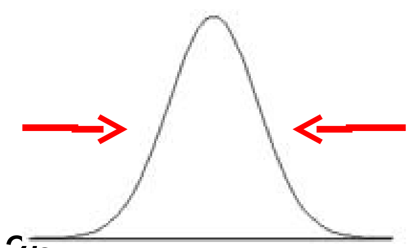
- Ricci Scalar gradient flow
  (From low to higher curvature)



- Diffusion ( $f^{ij} \equiv 3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij} > 0$ )



- Concentration( $f^{ij} \equiv 3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij} < 0$ )



# 拡散　V.S.　凝集

$$g_{ij} = \delta_{ij}, \quad \kappa^i_j = diag[1/r_1,\ 1/r_2]$$

$$f^{ij} = \delta^{ij}\left(\frac{1}{(r_i)^2} - \frac{2}{r_1 r_2}\right)$$

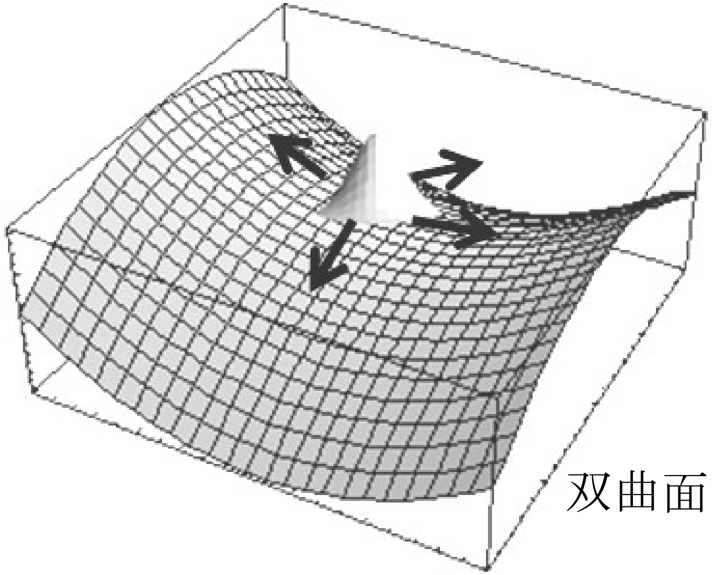
yperbolic Surface （双曲面）　R ＜　0　　$f^{11} > 0, \quad f^{22} > 0$　　拡散

vex-Concave R > 0　　$1/2 <| \frac{r_2}{r_1} |< 2. \quad f^{11} < 0, \quad f^{22} < 0$

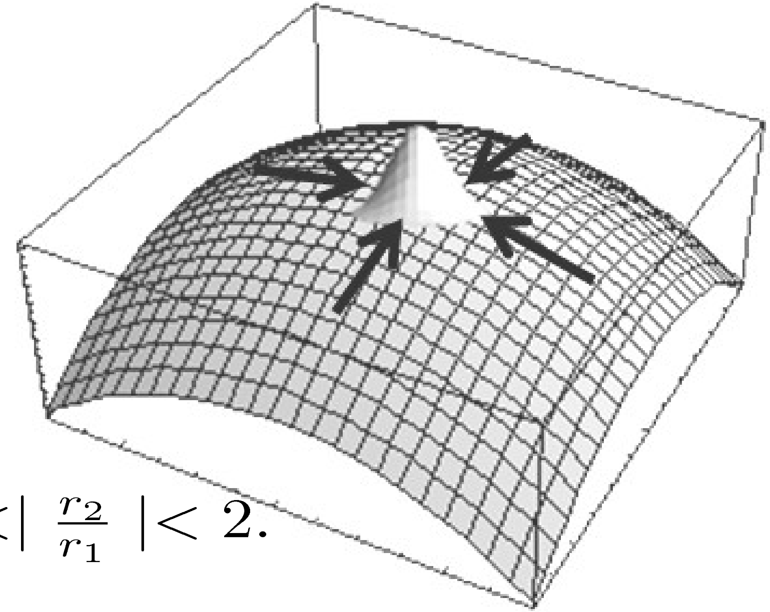
vex-Concave R > 0　　$| \frac{r_2}{r_1} |< 1/2, \quad \text{or} \quad | \frac{r_2}{r_1} |> 2. \quad f^{11} f^{22} < 0$

t　R ＝　0　　$r_2 = \infty, \quad f^{22} = 0, \quad f^{11} > 0$　　一方向拡散

# 拡散と凝集

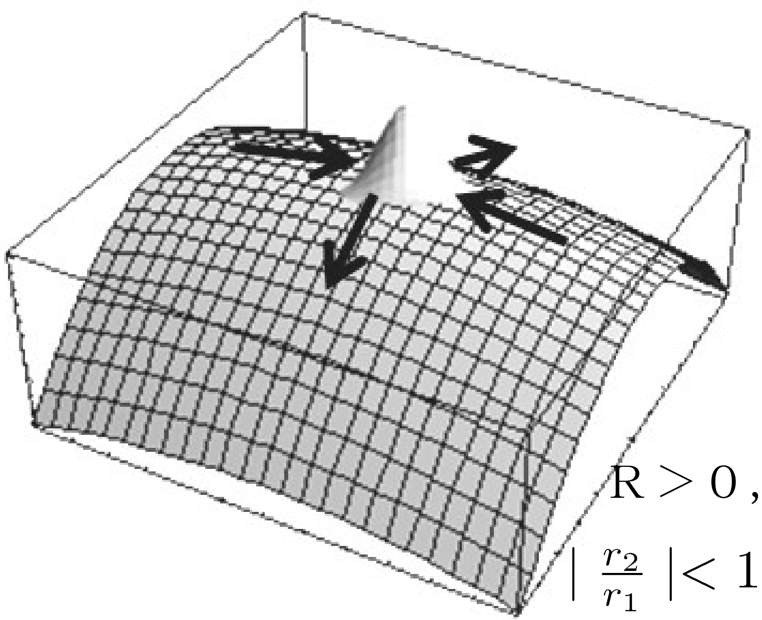


双曲面



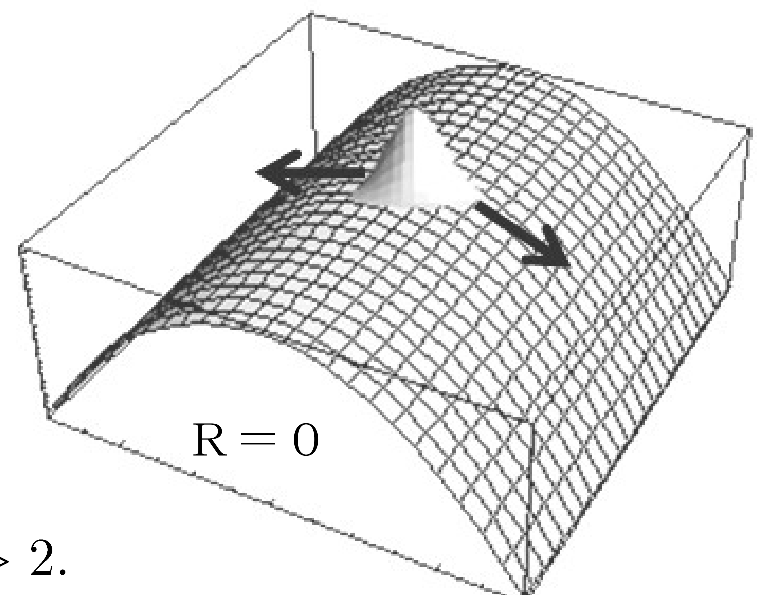
$R > 0$,

$1/2 < \left| \frac{r_2}{r_1} \right| < 2.$



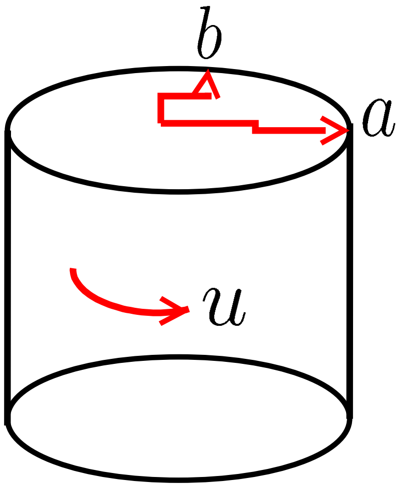
$R > 0$,

$\left| \frac{r_2}{r_1} \right| < 1/2$, or $\left| \frac{r_2}{r_1} \right| > 2.$



$R = 0$

# Case of Elliptic Cylinder




$$(\tfrac{x}{a})^2 + (\tfrac{y}{b})^2 = 1$$

$$x = a\cos\theta, \quad y = b\sin\theta.$$

$$f(\theta) \equiv \sqrt{a^2 \sin^2\theta + b^2\cos^2\theta}$$

$$du = \sqrt{dx^2 + dy^2} = f(\theta)\ d\theta$$

$$g_{\theta\theta} = f^2, \quad g_{zz} = 1, \quad g_{\theta z} = 0,$$

$$\kappa_{\theta\theta} = \frac{ab}{f}, \quad \kappa_{zz} = \kappa_{\theta z} = 0, \quad \kappa = \frac{ab}{f^3}$$

$$\frac{\partial\phi^{(2)}}{\partial t} = (\frac{1}{f}\frac{\partial}{\partial\theta})D_\theta(\frac{1}{f}\frac{\partial}{\partial\theta})\phi^{(2)} + D\frac{\partial^2}{\partial z^2}\phi^{(2)}$$

Effective Diffusion Coefficient depends on Curvature

$$D_\theta = D(1 + \frac{\epsilon^2\kappa^2}{12}).$$