# INTEGRABLE DIFFERENTIAL-DIFFERENCE EQUATIONS IN 2+1 DIMENSIONS

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# Dispersive/Dispersionless equations

Consider KP equation

$$(u_t - uu_x - u_{xxx})_x = u_{yy}$$

Change $\partial_x \to \epsilon\partial_x$, $\partial_y \to \epsilon\partial_y$, $\partial_t \to \epsilon\partial_t$

$$\left(u_t - uu_x - \epsilon^2 u_{xxx}\right)_x = u_{yy}$$

Set $\epsilon \to 0$ to obtain the so called dispersionless KP (dKP)

$$\left(u_t - uu_x\right)_x = u_{yy}$$

which can be written in the hydrodynamic form

$$u_t - uu_x = w_y$$

$$w_x = u_y$$

# Plan

- Differential equations in 2+1 dimensions
  - Method of hydrodynamic reductions. Example of dKP
  - Dispersive deformations of dispersionless integrable systems
  - Non-Degeneracy
- Differential-Difference equations in 2+1 dimensions
  - Method of hydrodynamic reductions. Dispersive deformations
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- Concluding Remarks

# The method of hydrodynamic reductions

Applies to quasilinear equations

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0,$$

$\mathbf{u} = (u^1, ..., u^n)^t$ and $A, B, C$ are $n \times n$ matrices. We seek n-phase solutions

$$\mathbf{u} = \mathbf{u}(R^1, ..., R^n)$$

where the phases $R^i(x, y, t)$ are required to satisfy a pair of commuting equations

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x$$

(*n-component reductions*)

**Definition** A 2+1D quasilinear system is said to be integrable if it possesses infinitely many $n$-component reductions parametrized by $n$ arbitrary functions of a single argument

# Example of dKP

One-component reductions will be enough for our purposes.

Consider

$$u_t - uu_x = w_y, \quad u_y = w_x$$

Seek for one-phase solutions or *planar simple waves*

$$u = R, \quad w = w(R)$$

where $R$ satisfies

$$R_y = \mu(R)R_x, \qquad R_t = \lambda(R)R_x$$

here

$w'(R) = \mu(R)$

$\lambda(R) = \mu^2(R) + R$ is the so called dispersion relation and

$\mu(R)$ is an arbitrary function

# Dispersive deformations of dispersionless integrable systems

Here is KP equation

$$u_t - uu_x - \epsilon^2 u_{xxx} = w_y, \qquad w_x = u_y$$

Look for one phase solutions

$$u = R, \quad w = w(R) + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3)$$

where

$$R_y = \mu(R)R_x + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3)$$

$$R_t = (\mu^2(R) + R)R_x + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3)$$

Here $(\dots)$ are required to be homogeneous polynomials in $x$- derivatives of $R$.

Substituting in the equation using $R_{yt} = R_{ty}$ we obtain

The deformed one-phase solutions

$$u = R, \quad w = w(R) + \epsilon^2(\mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3)R_x^2) + O(\epsilon^4)$$

and the deformed reductions

$$R_y = \mu R_x + \epsilon^2 \left(\mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3)R_x^2\right)_x + O(\epsilon^4)$$

$$R_t = (\mu^2 + R)R_x + \epsilon^2 \left((2\mu\mu' + 1)R_{xx} + (\mu\mu'' - \mu(\mu')^3 + (\mu')^2/2)R_x^2\right)_x + O(\epsilon^4)$$

-KP decoupled in infinite many ways to a pair of 1+1d equations ($\mu$ is arbitrary)

- Calculations are done up to $\epsilon^8$ (although $\epsilon^4$ is enough). Open to prove that reductions are inherited to any order $\epsilon$

**Conjecture** For any 2+1D integrable system, all hydrodynamic reductions of the dispersionless system can be deformed into reductions of its dispersive counterpart.

## Dispersive deformations of dispersionless integrable systems

Now suppose

$$u_t = uu_x + w_y + \epsilon(\dots) + \epsilon^2(\dots), \quad w_x = u_y$$

where $(\dots)$ denote differential polynomials of order two and three respectively in $x$- and $y$- derivatives of $u$ and $w$.

Require that one-phase solutions can be deformed as

$$u = R, \quad w = w(R) + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3)$$

where

$$R_y = \mu(R)R_x + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3)$$
$$R_t = (\mu^2(R) + R)R_x + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3)$$

And we will obtain KP equation.

- Reconstruction procedure does not always lead to one dispersive equation.

# Non-degeneracy conditions

All equations considered possess a dispersionless limit of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y,$$

here $\varphi, \psi, \eta$ are functions of $u, w$, which is supposed to be *non-degenerate* when

i) the dispersion relation $\lambda = \varphi + \mu\psi + \mu^2\eta$ defines an irreducible conic, i.e $\eta \neq 0$

ii) the system is not totally linearly degenerate, which is characterised by the equations

$$\eta_w = 0, \quad \psi_w + \eta_u = 0, \quad \phi_w + \psi_u = 0, \quad \phi_u = 0$$

Example: $u_t = w_y, \;\; w_x = u_y$

# Differential-Difference equations

We consider

$$u_t = F(u, w)$$

here $u(x, y, t)$ is a scalar field, $w(x, y, t)$ is the nonlocal variable, $F$ is a differential/difference operator in $x$ and $y$

**Notation**: The $\epsilon$-shift operators

$$T_x f(x, y) = f(x + \epsilon, y), \quad T_x^{-1} f(x, y) = f(x - \epsilon, y)$$

The forward/backward discrete derivatives

$$\triangle_x^+ = \frac{T_x - 1}{\epsilon}, \quad \triangle_x^- = \frac{1 - T_x^{-1}}{\epsilon}$$

same for $T_y,\ T_y^{-1},\ \triangle_y^+,\ \triangle_y^-$

# Hydrodynamic Reductions And Dispersive Deformations

**Example of Toda.**

Consider

$$u_t = u\triangle_y^- w, \quad w_x = \triangle_y^+ u$$

The corresponding dispersionless limit

$$u_t = uw_y, \qquad w_x = u_y$$

Seek solutions of the form $u = R, w = w(R)$ where

$$R_y = \mu(R)R_x, \qquad R_t = \mu^2(R)RR_x$$

and $w'(R) = \mu(R)$

Solutions and reductions of the dispersionless system can be deformed into solutions and reductions for the full Toda equation

$$
\begin{aligned}
u &= R, \\
w &= w(R) + \epsilon w_1 R_x + \epsilon^2 (w_2 R_{xx} + w_3 R_x^2) + O(\epsilon^3)
\end{aligned}
$$

and

$$
\begin{aligned}
R_y &= \mu R_x + \epsilon^2 (\alpha_1 R_{xxx} + \alpha_2 R_x R_{xx} + \alpha_3 R_x^3) + O(\epsilon^4) \\
R_t &= \mu^2 R R_x + \epsilon^2 (\beta_1 R_{xxx} + \beta_2 R_x R_{xx} + \beta_3 R_x^3) + O(\epsilon^4)
\end{aligned}
$$

with $w_i, \alpha_i, \beta_i$ functions of $R$.

After substituting in the equation we obtain

$$
\begin{aligned}
w_1 &= \frac{1}{2}\mu^2 \\
w_2 &= \frac{1}{12}\mu^2\left(2\mu + R\mu'\right) \\
w_3 &= \frac{1}{24}\left(R\left(\mu'\right)^2\left(2\mu - R\mu'\right) + \mu^2\left(11\mu' + R\mu''\right)\right) \\
\alpha_1 &= \frac{1}{12}R\mu^2\mu' \\
\alpha_2 &= \frac{1}{12}R\left(\left(\mu'\right)^2\left(4\mu - R\mu'\right) + 2\mu^2\mu''\right) \\
\alpha_3 &= \frac{1}{24}R\left(3\mu'\mu''\left(2\mu - R\mu'\right) + \mu^2\mu'''\right) \\
\beta_1 &= \frac{1}{12}R\mu^3\left(\mu + 2R\mu'\right) \\
\beta_2 &= \frac{1}{12}R\mu\left(R\left(\mu'\right)^2\left(11\mu - 2R\mu'\right) + 4\mu^2\left(3\mu' + R\mu''\right)\right) \\
\beta_3 &= \frac{R}{12}\left(R\left(\mu'\right)^3\left(2\mu - R\mu'\right) + 8R\mu^2\mu'\mu'' + \mu\left(\mu'\right)^2\left(11\mu - 3R^2\mu''\right) + \mu^3\left(4\mu'' + R\mu'''\right)\right)
\end{aligned}
$$

# Classification scheme

Suppose now that

$$u_t = f\triangle_y^- g, \quad w_x = \triangle_y^+ u.$$

The requirement that all one-phase solutions of the dispersionless system are inherited by the full dispersive equation leads to strong constraints on $f, g$

**At order** $\epsilon$:

$$g_u = 0, \quad f_u f_w = 0, \quad f_w\left(f g_{ww} + g_w f_w\right) = 0,$$

But $f_u = 0$ is linearly degenerate case. So: $g_u = 0, \quad f_w = 0$

**At order** $\epsilon^{\mathbf{2}}$: $f''(u) = 0, \quad g''(w)^2 - g'(w)g'''(w) = 0$

So already at this order we know

$$f(u) = \alpha u + \beta \quad and \quad g(w) = w \;\; or \;\; g(w) = e^w,$$

# Classes of equations

Named after their non-localities

**I. Intermediate Long Wave type 1**

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots)$$

$$\triangle_x^+ w = \frac{T_x + 1}{2} u_y$$

**II. Intermediate Long Wave type 2**

$$u_t = \psi u_y + \eta w_y + f \triangle_x^+ g + p \triangle_x^- q, \qquad \triangle_x^+ w = \frac{T_x + 1}{2} u_y$$

**III. Toda type**

$$u_t = \phi u_x + f \triangle_y^+ g + p \triangle_y^- q, \qquad w_x = \triangle_y^+ u$$

**IV. Fully discrete type**

$$u_t = f \triangle_x^+ g + h \triangle_x^- k + p \triangle_y^+ q + r \triangle_y^- s, \qquad \triangle_x^+ w = \triangle_y^+ u$$

## Classification Results: I. $\triangle_x^+ w = \frac{T_x+1}{2} u_y$

The following examples constitute a complete list of integrable equations of the form

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots)$$

with the non-locality of intermediate long wave type:

$$u_t = u u_y + w_y, \tag{1}$$

$$u_t = (w + \alpha e^u) u_y + w_y, \tag{2}$$

$$u_t = u^2 u_y + (uw)_y + \frac{\epsilon^2}{12} u_{yyy}, \tag{3}$$

$$u_t = u^2 u_y + (uw)_y + \frac{\epsilon^2}{12} \left( u_{yy} - \frac{3}{4} \frac{u_y^2}{u} \right)_y . \tag{4}$$

## Classification Results: I. Lax Pairs

| $Eq$ | $Lax\ pair$ | $Dis/less\ limit$ | $Dis/less\ Lax\ pair$ |
|---|---|---|---|
| (1) | $T_x\psi = \epsilon\psi_y - u\psi$<br>$\epsilon\psi_t = \frac{\epsilon^2}{2}\psi_{yy} + (w - \frac{\epsilon}{2}u_y)\psi$ | $u_t = uu_y + w_y$<br>$w_x = u_y$ | $e^{S_x} = S_y - u$<br>$S_t = \frac{1}{2}S_y^2 + w$ |
| (2) | $T_x\psi = \epsilon e^{-u}\psi_y - \alpha\psi$<br>$\psi_t = \frac{\epsilon}{2}\psi_{yy} + (w - \frac{\epsilon}{2}u_y)\psi_y$ | $u_t = (w + \alpha e^u)u_y + w_y$<br>$w_x = u_y$ | $e^{S_x} = e^{-u}S_y - \alpha$<br>$S_t = \frac{1}{2}S_y^2 + wS_y$ |
| (3) | $\epsilon(T_x - 1)\psi_y = -2u(T_x + 1)\psi$<br>$\psi_t = \frac{\epsilon^2}{12}\psi_{yyy} + (w - \frac{\epsilon}{2}u_y)\psi_y$ | $u_t = u^2u_y + (uw)_y$<br>$w_x = u_y$ | $\frac{e^{S_x}-1}{e^{S_x}+1}S_y = -2u$<br>$S_t = \frac{1}{12}S_y^3 + wS_y$ |
| (4) | $\epsilon(T_x - 1)\psi_y = \frac{\epsilon}{2}\frac{u_y}{u}(T_x - 1)\psi -$<br>$2u(T_x + 1)\psi$<br>$\psi_t = \frac{\epsilon^2}{12}\psi_{yyy} + (w - \frac{\epsilon}{2}u_y)\psi_y +$<br>$\frac{1}{2}(w_y - \frac{\epsilon}{2}u_{yy})\psi$ | $u_t = u^2u_y + (uw)_y$<br>$w_x = u_y$ | $\frac{e^{S_x}-1}{e^{S_x}+1}S_y = -2u$<br>$S_t = \frac{1}{12}S_y^3 + wS_y$ |

## Classification Results: II. $\triangle_x^+ w = \frac{T_x+1}{2} u_y$

The following examples constitute a complete list of integrable equations of the form

$$u_t = \psi u_y + \eta w_y + f \triangle_x^+ g + p \triangle_x^- q$$

with the non-locality of intermediate long wave type:

$$\begin{aligned} u_t &= u u_y + w_y, \\ u_t &= (w + \alpha e^u) u_y + w_y, \\ u_t &= w u_y + w_y + \frac{\triangle_x^+ + \triangle_x^-}{2} e^{2u}, & (5) \\ u_t &= w u_y + w_y + e^u (\triangle_x^+ + \triangle_x^-) e^u. & (6) \end{aligned}$$

## Classification Results: II. Lax Pairs

| $Eq$ | $Lax\ pair$ | $Dis/less\ limit$ | $Dis/less\ Lax\ pair$ |
|---|---|---|---|
| (5) | $\epsilon\psi_y = (T_x e^u)T_x\psi + e^u T_x^{-1}\psi$<br>$\epsilon\psi_t = \frac{1}{2}e^{T_x(1+T_x)u}T_x^2\psi - \frac{1}{2}e^{(1+T_x^{-1})u}T_x^{-2}\psi$<br>$+ T_x(we^u)T_x\psi + we^uT_x^{-1}\psi$ | $u_t = 2e^{2u}u_x +$<br>$wu_y + w_y$<br>$w_x = u_y$ | $S_y = 2e^u \cosh S_x$<br>$S_t = e^{2u}\sinh 2S_x$<br>$+ 2we^u \cosh S_x$ |
| (6) | $\epsilon\psi_y = e^u(T_x\psi + T_x^{-1}\psi)$<br>$\epsilon\psi_t = \frac{1}{2}e^{(1+T_x)u}T_x^2\psi - \frac{1}{2}e^{(1+T_x^{-1})u}T_x^{-2}\psi +$<br>$we^u(T_x\psi + T_x^{-1}\psi) + \frac{\epsilon}{2}e^u[(\triangle_x^+ + \triangle_x^-)e^u]\psi$ | $u_t = 2e^{2u}u_x +$<br>$wu_y + w_y$<br>$w_x = u_y$ | $S_y = 2e^u \cosh S_x$<br>$S_t = e^{2u}\sinh 2S_x$<br>$+ 2we^u \cosh S_x$ |

## Classification Results: III. $w_x = \triangle_y^+ u$

The following examples constitute a complete list of integrable equations of the form

$$u_t = \phi u_x + f \triangle_y^+ g + p \triangle_y^- q$$

with the non-locality of Toda type:

$$u_t = u \triangle_y^- w, \tag{7}$$

$$u_t = (\alpha u + \beta) \triangle_y^- e^w, \tag{8}$$

$$u_t = e^w \sqrt{u} \triangle_y^+ \sqrt{u} + \sqrt{u} \triangle_y^- (e^w \sqrt{u}), \tag{9}$$

# Classification Results: III. Lax Pairs

| $Eq$ | $Lax\ pair$ | $Dis/less\ limit$ | $Dis/less\ Lax\ pair$ |
|---|---|---|---|
| (7) | $\epsilon T_y \psi_x = u\psi$<br>$\epsilon \psi_t = -T_y\psi + (T_y^{-1} w)\psi$ | $u_t = u w_y$<br>$w_x = u_y$ | $e^{S_y} S_x = u$<br>$S_t = -e^{S_y} + w$ |
| (8) | $\epsilon T_y \psi_x = (\alpha T_y u + \beta)\psi - (T_y u) T_y \psi$<br>$\epsilon \psi_t = -e^w T_y \psi + \alpha e^w \psi$ | $u_t = (\alpha u + \beta) e^w w_y$<br>$w_x = u_y$ | $e^{S_y} S_x = \alpha u + \beta - u e^{S_y}$<br>$S_t = -e^w e^{S_y} + \alpha e^w$ |
| (9) | $\epsilon T_y \psi_x = \epsilon \sqrt{\frac{T_y u}{u}} \psi_x - (T_y u) T_y \psi$<br>$- \sqrt{u T_y u}\, \psi$<br>$\epsilon \psi_t = \frac{1}{2} e^w T_y \psi - \frac{1}{2} (T_y^{-1} e^w) T_y^{-1} \psi$ | $u_t = e^w (u_y + u w_y)$<br>$w_x = u_y$ | $e^{S_y} S_x = S_x - u e^{S_y} - u$<br>$S_t = e^w \sinh S_y$ |

# Classification Results: IV. $\triangle_x^+ w = \triangle_y^+ u$

The following examples provide a complete list of integrable equations of the form

$$u_t = f\triangle_x^+ g + h\triangle_x^- k + p\triangle_y^+ q + r\triangle_y^- s$$

with the fully discrete non-locality:

$$u_t = u\triangle_y^-(u - w) \quad (10)$$

$$u_t = u(\triangle_x^+ + \triangle_y^-)w \quad (11)$$

$$u_t = (\alpha e^{-u} + \beta)\triangle_y^- e^{u-w}, \quad (12)$$

$$u_t = (\alpha e^u + \beta)(\triangle_x^+ + \triangle_y^-)e^w \quad (13)$$

$$u_t = \sqrt{\alpha - \beta e^{2u}}\left(e^{w-u}\triangle_y^+\sqrt{\alpha - \beta e^{2u}} + \triangle_y^-(e^{w-u}\sqrt{\alpha - \beta e^{2u}})\right) \quad (14)$$

| $Eq$ | $Lax\ pair$ | $Dis/less\ limit$ | $Dis/less\ Lax\ pair$ |
|---|---|---|---|
| (10) | $T_xT_y\psi = -T_y\psi + (T_yu)T_x\psi$<br>$\epsilon\psi_t = T_y\psi - w\psi$ | $u_t = u(u_y - w_y)$<br>$w_x = u_y$ | $e^{S_x+S_y} = -e^{S_y} + ue^{S_x}$<br>$S_t = e^{S_y} - w$ |
| (11) | $T_xT_y\psi = T_y\psi - u\psi$<br>$\epsilon\psi_t = T_y\psi + (T_y^{-1}w)\psi$ | $u_t = u(u_y + w_y)$<br>$w_x = u_y$ | $e^{S_x+S_y} = e^{S_y} - u$<br>$S_t = e^{S_y} + w$ |
| (12) | $T_y^{-1}\psi = \frac{e^u}{\alpha+\beta e^u}T_x^{-1}\psi + \frac{1}{\alpha+\beta e^u}\psi$<br>$\epsilon T_x^{-1}\psi_t = -\epsilon e^{-u}\psi_t -$<br>$\alpha e^{-w}T_x^{-1}\psi + \beta e^{-w}\psi$ | $u_t = (\alpha + \beta e^u)e^{-w}\times$<br>$(u_y - w_y)$<br>$w_x = u_y$ | $e^{-S_y} = \frac{e^u e^{-S_x}+1}{\alpha+\beta e^u}$<br>$e^{-S_x}S_t = -e^{-u}S_t -$<br>$\alpha e^{-w}e^{-S_x} + \beta e^{-w}$ |
| (13) | $T_y^{-1}\psi = -\frac{e^u}{\alpha e^u+\beta}T_x\psi + \frac{1}{\alpha e^u+\beta}\psi$<br>$\epsilon T_x\psi_t = \epsilon e^{-u}\psi_t - \beta(T_xe^w)T_x\psi -$<br>$\alpha(T_xe^w)\psi$ | $u_t = (\alpha e^u + \beta)e^w\times$<br>$(u_y + w_y)$<br>$w_x = u_y$ | $e^{-S_y} = \frac{-e^u e^{S_x}+1}{\alpha e^u+\beta}$<br>$e^{S_x}S_t = e^{-u}S_t -$<br>$\beta e^w e^{S_x} - \alpha e^w$ |
| (14) | $T_xT_y\psi = \frac{\alpha}{\beta}(T_ye^{-u})T_y\psi +$<br>$\frac{T_y(e^{-u}\sqrt{\alpha-\beta e^{2u}})}{\sqrt{\alpha-\beta e^{2u}}}(T_x\psi - e^u\psi)$<br>$\epsilon\psi_t = -\alpha e^wT_y\psi + \beta(T_y^{-1}e^w)T_y^{-1}\psi$ | $u_t = \alpha(e^{w-u})_y -$<br>$\beta(e^{w+u})_y$<br>$w_x = u_y$ | $e^{S_x+S_y} = \frac{\alpha}{\beta}e^{-u}e^{S_y} +$<br>$e^{-u}e^{S_x} - 1$<br>$S_t = -\alpha e^w e^{S_y} +$<br>$\beta e^w e^{-S_y}$ |

## Concluding Remarks

- Method for classifying integrable differential and differential-difference equations with non-degenerate dispersionless limit
- No differential-delay equations passed the integrability test
- The method can be extended to fully discrete 3D equations (work in progress)

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# Thank you!