ORDER OF TIME DERIVATIVES IN FUNDAMENTAL EQUATIONS OF PHYSICS

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1. Introduction

The problem is rather old, almost as quantum mechanics itself. The first, primary idea of Schrödinger was the relativistic one, with the d'Alembert operator on the left-hand side of quantum-mechanical equation, so, with the second-order time derivatives. Unfortunately, it turned out that the following results were in a rather clear contradiction with experimental data, although some kind of compatibility did exist. Schrödinger felt disappointed and at least temporarily he rejected his primary equation. Later on, basic on the idea of Lagrange-Hamilton optical-mechanical analogy and on certain de Broglie ideas, he in a sense derived his famous equation which seemed to remain in a beautiful agreement with spectroscopic data and was approximately compatible with the Bohr-Sommerfeld quantization rules. Nevertheless, it was of course drastically incompatible with the relativistic idea of Poincare symmetry. But, after the fall of the primary substantial interpretation by Schrödinger, it was compatible with the Born statistical interpretation of his formalism, and with the corresponding continuity equation for the probabilistic density (Veltman, 2003).
Later on history was rather complicated. Dirac formulated his relativistic quantum theory of electrons based on first-order space-time derivatives of multicomponent wave functions. The multicomponent character of waves had to do obviously with the particle spin. It was also understood that the relativistic velocity-dependence of the electron mass and the spin phenomena act in opposite directions, and because of this non-relativistic Schrödinger equation seemed to be better than his second order equation, rediscovered later on by Klein and Gordon. The formalism of quantum field theory rehabilitated the Klein-Gordon equation, i.e., primaevally the relativistic Schrödinger equation, as one describing some physics. And, let us also mention that the field-theoretic approach based on the Pauli exclusion principle removed certain problems with the quantum-mechanical Dirac equation for a single electron. And certain inconsistencies on one-particle relativistic theory were resolved. The only, and fundamental inadequacy which remained, was one connected with the essential non-linearity of the quantum field-theoretic equations for the field operators and the resulting interpretation difficulties. Nevertheless, they were in a sense solvable on the basis of renormalization procedure.

But in spite of everything said, the problem is still alive. There are certain not completely clear facts within the framework of field theory based on the Dirac-Clifford paradigm of first-order differential equations of quantum mechanics with $\hbar/2$-spin. One can show that they become more clear when we assume that in a sense some second-order equations are primary and the first-order ones are some approximations valid for slowly-varying fields. There are also some arguments from geometrodynamics and gauge theories, one of fundamental methods in modern fields theory. It is known that there are certain disadvantages in geometrodynamical gauge models based on the Poincare group as a gauge group, in spite of certain correct results following from that approach. It seems that the main reason is the fact that Poincare group is not semi-simple. The best way out seems to take the simplest semisimple extension of this group, namely the conformal group of Minkowski space. It must be stressed that this group does not act in the space-time manifold, which in geometrodynamics is a general non-flat manifold of a dynamical structure. Instead, it acts as a purely internal group operating with internal degrees of freedom of our matter fields. To be more precise, instead its universal conformal group $\mathrm{CO}(1,3)$, one should use its universal covering group $\mathrm{SU}(2,2)$ of pseudo-unitary mappings with the signature $(+,+,-,-)$, acting in the target spaces of matter fields and on the gauge connection components. The primary field equations are differential ones of the second-order in matter fields. After the careful rewriting in terms of the basic elements of Lie algebra $\mathrm{SU}(2,2)'$, the internal group rules both matter and geometry (gravitation). The use of conformal group is interesting in itself. It is the smallest semi-simple group containing Poincare group. It is also the largest group which in the geometrically Minkowskian formulation preserves the family of relativistic uniformly accelerated motions (described by the flat time-like hyperboles). It turns out that there are interesting aspects of this approach, having some correspondence with the usual gravitation theory and with generally-relativistic spinor fields. In the specially-relativistic limit, the theory seems to predict the existence of pairs of fundamental quarks and leptons, just as it is really in Nature.

We mean here the quark pairs (u,d), (c,s), (t,b) and those of leptons (ν_e, e), (ν_μ, μ), (ν_τ, τ). It is interesting that in this limit the fermion fields are described by the Klein-Gordon-Dirac equation combining the Klein-Gordon and Dirac operators, and that in this limit the Dirac behaviour of fields seems to be more remarkable. There are certain interesting facts concerning the spin-statistics problem. It seems that on the very fundamental level some fermion-boson mixing may appear, or that the two possibilities will be unified by some quite new approach. This framework seems to be related, in a rather unexpected way, to another aspect of the problem of the order of time derivatives in quantum mechanics. Namely, certain quite interesting aspects of the quantum-mechanical and quantum field-theoretic problems appears, when one temporarily forgets about the quantum nature of equations, and considers them simply as some Hamiltonian systems of mathematical physics. Certain primary ideas concerning this problem were formulated in our papers a few years ago (Sławianowski & Kovalchuk, 2002; 2008; 2010; Sławianowski et al., 2004; 2005). There are some arguments which seem to show that there is some so-to-speak inadequacy in the first-order Schrödinger equation. In any case, the second-order corrections seem to be just admissible if not desirable. There are also certain indications for that from the theory of stochastic processes. This approach has certain common points with the former gauge-theoretic one. Namely, once using the language of Hamiltonian dynamics (perhaps infinite-dimensional one) we do not feel any longer the usual reluctance of quantum people to the idea of non linearity. In particular, it turns out that the dynamical scalar product, i.e., one non-constant, but satisfying a closed system of equations with the wave amplitudes, is a natural constituent of the approach. The theory becomes then essentially nonlinear. Essentially, i.e., in such a way that nonlinearity is not an accidental term imposed onto some basic linear background. Everything is then nonlinear in the zeroth-order approximation. Nonlinearity is an essential feature, similar to one used in non-Abelian gauge theories and may be perhaps responsible for the decoherence and measurement paradoxes.

We have formulated some arguments in favour of $\mathrm{SU}(2,2)$ as a fundamental gauge group. Let us mention, incidentally, that this simply provokes the next question: Why the subgroup $\mathrm{SU}(2,2) \subset \mathrm{GL}(4,\mathbb{C})$ but not just the whole $\mathrm{GL}(4,\mathbb{C})$? The latter group appears in a natural way as the structure group of the principal fibre bundle of the complexification of the usual bundle of frames over the four-dimensional space-time manifold. Obviously, it preserves the signature of sesquilinear forms, nevertheless changing them otherwise. Therefore, the bispinor sesquilinear Hermitian form G of signature $(+,+,-,-)$ becomes an a priori free Hermitian form, the signature however being in a sense an integral of motion.

2. On the track of the scalar Klein-Gordon-Dirac formalism

The generally-relativistic Lagrangian of the Dirac field is given by the expression:

$$L = \frac{i}{2} e^{\mu}{}_{A} \gamma^{Ar}{}_{s} \left(\widetilde{\Psi}_r D_\mu \Psi^s - D_\mu \widetilde{\Psi}_r \Psi^s \right) \sqrt{|g|} - m \widetilde{\Psi}_r \Psi^r \sqrt{|g|} \tag{1}$$

with the following meaning of symbols:
(a) $\widetilde{\Psi}$ denote the Dirac-conjugation of Ψ,

$$\widetilde{\Psi}_r = \overline{\Psi}^{\bar{s}} G_{\bar{s}r}, \tag{2}$$

where G denotes the Dirac-conjugation form of mass, i.e., sesquilinear Hermitian form of the neutral signature $(+, +, -, -)$. If the Finkelstein-Penrose-Weizsäcker-van der Waerden point of view on the two-component spinors is accepted, then G is intrinsic, because the $\mathbb{C}^4$-space is then expressed as the Cartesian product of two mutually antidual copies of $\mathbb{C}^2$. Without this point of view, analytically Ψ is $\mathbb{C}^4$-valued.

(b) Dirac matrices γ^A satisfy the following anticommutation rules:

$$\left\{\gamma^A, \gamma^B\right\} = \gamma^A\gamma^B + \gamma^B\gamma^A = 2\eta^{AB} I_4, \tag{3}$$

$$\left[\eta^{AB}\right] = \mathrm{diag}\,(1, -1, -1, -1)\,. \tag{4}$$

Besides, γ^A are Hermitian with respect to G:

$$\Gamma^A{}_{\bar{r}s} = \overline{\Gamma^A{}_{s\bar{r}}} = \overline{\Gamma^A{}_{\bar{s}r}} = G_{\bar{r}z}\gamma^{Az}{}_s. \tag{5}$$

(c) The quantities $e^\mu{}_A$ are components of the tetrad field. Its dual cotetrad $e^A{}_\mu$ is analytically given by the reciprocal expression

$$e^A{}_\mu e^\mu{}_B = \delta^A{}_B. \tag{6}$$

(d) The metric tensor $g_{\mu\nu}$ is built of $e^A{}_\mu$ in a quadratic way:

$$g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu, \quad [\eta_{AB}] = \mathrm{diag}\,(1, -1, -1, -1)\,. \tag{7}$$

The Greek and capital Latin indices are shifted with the help of $g_{\mu\nu}$ and η_{AB}.

(e) The operation D_μ symbolizes the covariant differentiation of bispinors. It is given by the following sequence of expressions:

$$\Gamma^{\alpha}{}_{\beta\mu} = e^{\alpha}{}_{A}\Gamma^{A}{}_{B\mu}e^{B}{}_{\beta} + e^{\alpha}{}_{A}e^{A}{}_{\beta,\mu}, \tag{8}$$

$$\Gamma^{A}{}_{B\mu} = \frac{1}{2}\mathrm{Tr}\left(\gamma^{A}\omega_{\mu}\gamma_{B}\right), \tag{9}$$

$$\omega_{\mu} = \frac{1}{2}\Gamma_{LK\mu}\Sigma^{LK} = \frac{1}{2}\eta_{LM}\Gamma^{M}{}_{K\mu}\Sigma^{LK}, \tag{10}$$

$$\Sigma^{LK} = \frac{1}{4}\left(\gamma^{L}\gamma^{K} - \gamma^{K}\gamma^{L}\right), \tag{11}$$

where, obviously, the following identities hold:

$$\eta_{AC}\Gamma^{C}{}_{B\mu} + \eta_{BC}\Gamma^{C}{}_{A\mu} = 0, \qquad \nabla^{\Gamma}{}_{\mu}g_{\alpha\beta} = 0. \tag{12}$$

This means that the following is satisfied:

$$\Gamma^{\alpha}{}_{\beta\mu} = \left\{ {\alpha \atop \beta\mu} \right\} + S^{\alpha}{}_{\beta\mu} + S_{\beta\mu}{}^{\alpha} - S_{\mu}{}^{\alpha}{}_{\beta}.$$

This means that $\left\{ {\alpha \atop \beta\mu} \right\}$ are coefficients of the Levi-Civita connection built of the metric g, and $S^{\alpha}{}_{\beta\mu} = \Gamma^{\alpha}{}_{[\beta\mu]}$ is the torsion tensor of $\Gamma^{\alpha}{}_{\beta\mu}$.

(c) Finally, there is a strange feature of Lagrangian (1), namely, the one that it is essentially based on the covariant vector density

$$J^{r}{}_{s\mu} := \left(D_{\mu}\widetilde{\Psi}_{s}\Psi^{r} - \widetilde{\Psi}_{s}D_{\mu}\Psi^{r} \right) \sqrt{|g|}, \tag{14}$$

or, to be more honest, on its contravariant upper-index version

$$J^{r}{}_{s}{}^{\mu} := g^{\mu\nu} \left(D_{\nu}\widetilde{\Psi}_{s}\Psi^{r} - \widetilde{\Psi}_{s}D_{\nu}\Psi^{r} \right) \sqrt{|g|}. \tag{15}$$

The idea is that $J^{r}{}_{s}{}^{\mu}$ looks as a typical bosonic current. What is the symmetry group responsible for it? The algebraic prescription for $J^{r}{}_{s}{}^{\mu}$ does suggest that it is the group $\mathrm{U}(2,2)$ and that the corresponding Lagrangian for Ψ should be just the complex-four-dimensional Lagrangian for the field Ψ, this time invariant under the total $\mathrm{U}(2,2)$, no longer by $\mathrm{SL}(2,\mathbb{C})$. Nevertheless, some fundamental question remains, namely one concerning the relationship between Klein-Gordon equation of order two and first-order Dirac equation. It is though clear that the structural properties of differential equations are in very malicious way sensitive to the removing highest-order derivative term.

It is interesting to begin the analysis from some rather academic example of the scalar complex field interacting in a minimal way with the gauge filed e_μ, i.e., dynamically ruled by the unitary group U(1). Namely, let us assume the primeval globally invariant by U(1) Lagrangian for $\Psi : M \to \mathbb{C}$,

$$L_m = \left(\frac{1}{2} g^{\mu\nu} \overline{\partial_\mu \Psi} \partial_\nu \Psi - \frac{c}{2} \overline{\Psi} \Psi \right) \sqrt{|g|}. \tag{16}$$

Now, as usually we introduce the covector gauge field e_μ and the covariant derivative of Ψ,

$$D_\mu \Psi := \partial_\mu \Psi - iq e_\mu \Psi. \tag{17}$$

Substituting D_μ instead of ∂_μ to (16) we obtain as usual the locally-invariant expression

$$L_m = \frac{1}{2} g^{\mu\nu} \overline{D_\mu \Psi} D_\nu \Psi \sqrt{|g|} - \frac{c}{2} \overline{\Psi} \Psi \sqrt{|g|}. \tag{18}$$

This is Lagrangian for Ψ. The corresponding term for

$$f_{\mu\nu} = \partial_\mu e_\nu - \partial_\nu e_\mu \tag{19}$$

is as usually given by

$$L_g = -\frac{1}{4} g^{\mu\varkappa} g^{\nu\lambda} f_{\mu\nu} f_{\varkappa\lambda} \sqrt{|g|}, \tag{20}$$

and the total Lagrangian for (Ψ, f) is given by the sum

$$L = L_m + L_g \tag{21}$$

(the subscripts m, g refer respectively to the matter and gauge field).
Let us rewrite the matter term in the following form:

$$\begin{aligned} L_m &= q g^{\mu\nu} e_\mu \frac{i}{2} \left(\overline{\Psi} \partial_\nu \Psi - \overline{\partial_\nu \Psi} \Psi \right) \sqrt{|g|} \\ &- \left(\frac{c}{2} - \frac{q^2}{2} g^{\mu\nu} e_\mu e_\nu \right) \overline{\Psi} \Psi \sqrt{|g|} + \frac{1}{2} g^{\mu\nu} \overline{\partial_\mu \Psi} \partial_\nu \Psi \sqrt{|g|}. \end{aligned} \tag{22}$$

The first term, built of the first derivatives and of the algebraic expressions of fields, leads to first-order differential equations with respect to Ψ. The last, Klein-Gordon term leads through variational principle to the second-order equations in Ψ. The rigorous field equations have the following form:

$$q i e^\mu \partial_\mu \Psi - \left(\frac{c}{2} - \frac{q^2}{2} e^\mu e_\mu - \frac{iq}{2} e^\mu{}_{;\mu} \right) - \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \Psi = 0, \tag{23}$$

$$\partial_\nu f^{\mu\nu} = \frac{qi}{2} \left(\overline{\Psi} \partial_\mu \Psi - (\partial_\mu \overline{\Psi}) \Psi \right) + q^2 e^\mu \overline{\Psi} \Psi. \tag{24}$$

Obviously, the semicolon symbol in (23) denotes the g-metric Levi-Civita affine connection, or rather divergence. It is interesting that the vector field e^{μ} plays a role similar to that of gravitational tetrad, in spite of all differences. And in general, the pair (Ψ, e^{μ}) is formally analogous to the pair $(\Psi^{r}, e^{\mu}{}_{A})$, or equivalently $(\Psi^{r}, e^{A}{}_{\mu})$, i.e., bispinor and tetrad/cotetrad. But there is no rigorous Clifford analogy. It is interesting that on the right hand side of (24) there is a combination of two terms: Dirac-like current and Schrödinger current.

It is interesting that the system of equations (23), (24) may be simplified by assuming that the system of first-order derivatives of Ψ is smaller than the system of quantities built of Ψ in an algebraic way, and similarly, the system of second derivatives of Ψ is smaller than the first- and zeroth-order derivatives of Ψ. But this means that the system (23), (24) may be approximated by the following one:

$$ie^{\mu}\partial_{\mu}\Psi - \left(\frac{c}{2q} - \frac{q}{2}e^{\mu}e_{\mu} - \frac{i}{2}e^{\mu}{}_{;\mu}\right)\Psi = 0, \tag{25}$$

$$\partial_{\nu} f^{\mu\nu} = q^{2}e^{\mu}\overline{\Psi}\Psi. \tag{26}$$

It is interesting that this system, except the Clifford analogy, is structurally similar to the Dirac system of equations. It is difficult to state a priori if the essentially nonlinear system (25), (26) may have anything to do with reality. Nevertheless, the point is that it is both nonlinear, and as a system imposed on the pair $\left(\Psi, e_{\mu}\right)$ it shows certain similarity to the Dirac-Maxwell system. And the bosonic current $i\left(\partial_{\mu}\overline{\Psi}\Psi - \overline{\Psi}\partial_{\mu}\Psi\right)\sqrt{|g|}$ is an obvious counterpart of the SU$(2,2)$ current given by $i\left(D_{\mu}\widetilde{\Psi}_{s}\Psi^{r} - \widetilde{\Psi}_{s}D_{\mu}\Psi^{r}\right)\sqrt{|g|}$. Obviously, the model (25), (26) is a bit non-physical and crazy, especially with its separation of terms. Nevertheless, it seems to follow from it that the above demands and objections concerning the U$(2,2)$-invariance and the particular role of the bosonic currents and tetrads may be easily answered on the basis of the spinor counterpart of L_m (22) and its first-order limit (23), (24).

3. Second order Klein-Gordon equation

We need a few things, for instance, affine connection in space-time manifold, spinor connection, $\mathrm{U}(2,2)$-gauge field, metric tensor, and in certain approaches some field of frames, e.g., generalization of the tetrad field. The space-time manifold M is assumed structure-less and nothing but the differential-geometric structure is assumed in it. Unlike this, in the target space $\mathbb{C}^4$, we assume some internal geometry based on the use of some sesquilinear Hermitian G form of signature $(+,+,-,-)$, as mentioned above. This form does belong to the internal structure of $\mathbb{C}^4$, and to be more rigorous, we can assume it to be a complex linear space of dimension four, endowed with the mentioned neutral signature. When this form is fixed, it distinguishes within the complex group $\mathrm{GL}(4,\mathbb{C})$, the pseudounitary group consisting of transformations preserving G, so that the following holds:

$$G_{\bar{r}s} = G_{\bar{z}t}\overline{U}^{\bar{z}}{}_{\bar{r}}U^{t}{}_{s}. \tag{27}$$

The Lie algebra of this group consists of linear mappings u which satisfy:

$$G_{\bar{r}z}u^{z}{}_{s} + \overline{G_{\bar{s}z}u^{z}{}_{r}} = 0. \tag{28}$$

So, roughly speaking, $\mathrm{U}(2,2)'$ consists of matrices (linear mapping of the target space) which are G-anti-Hermitian. Let us mention that any particular choice of G is only a matter of convenience. It is only its global signature that matters. As mentioned, in the Weyl, Penrose, Finkelstein and Weizsäcker procedure the typical choice is

$$[G_{\bar{r}s}] = \begin{bmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{bmatrix}. \tag{29}$$

In the Dirac procedure one prefers the choice:

$$[G_{\bar{r}s}] = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}. \tag{30}$$

Transition between these representations is described by the matrix

$$\frac{1}{\sqrt{2}} \begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix}. \tag{31}$$

Similarly, the Weyl, Penrose, Finkelstein and Weizsäcker procedure leads to the following expressions for the Dirac matrices:

$$\gamma^A = \eta^{AB}\gamma_B = \begin{bmatrix} 0 & \widetilde{\sigma}^A \\ \sigma^A & 0 \end{bmatrix} = \begin{bmatrix} 0 & \eta^{AB}\sigma_B \\ \sigma^A & 0 \end{bmatrix}. \tag{32}$$

In the Dirac representation we have that

$$\gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} = \begin{bmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{bmatrix}, \qquad \gamma^R = \begin{bmatrix} 0 & \sigma^R \\ -\sigma^R & 0 \end{bmatrix}, \qquad R = 1, 2, 3. \tag{33}$$

Obviously, those are two particular choices, we quote them only as the two most important ones.

The globally U(2,2)-invariant second-order Klein-Gordon Lagrangian for the $\mathbb{C}^4$-valued scalar field on M is given by

$$L_m(\Psi; g) = \frac{b}{2} g^{\mu\nu} \partial_\mu \overline{\Psi}^{\bar{r}} \partial_\nu \Psi^s G_{\bar{r}s} \sqrt{|g|} - \frac{c}{2} G_{\bar{r}s} \overline{\Psi}^{\bar{r}} \Psi^s \sqrt{|g|}. \tag{34}$$

Making use of the Dirac-conjugate field,

$$\widetilde{\Psi}_r := \overline{\Psi}^{\bar{s}} G_{\bar{s}r}, \tag{35}$$

we can rewrite (34) in the following form:

$$L_m(\Psi; g) = \frac{b}{2} g^{\mu\nu} \partial_\mu \widetilde{\Psi} \partial_\nu \Psi \sqrt{|g|} - \frac{c}{2} \widetilde{\Psi} \Psi \sqrt{|g|}. \tag{36}$$

First let us consider the problem of the local $\mathrm{U}(2,2) \simeq \mathrm{U}(4, G)$-invariance. To do that we must begin with introducing the connection form of the $\mathrm{U}(4, G)$-connection. This is a $\mathrm{u}(4, G)$-valued differential form

$$M \ni x \mapsto \vartheta_x \in L\left(T_x M, u(4, G)\right) \tag{37}$$

transforming under the local $\mathrm{U}(4, G)$-valued local transformations $U : M \to \mathrm{U}(4, G)$ as follows:

$$(U\vartheta)_x = U(x) \vartheta_x U(x)^{-1} - dU_x U(x)^{-1}. \tag{38}$$

This connection form is controlled by the two real parameters corresponding to $\mathrm{SU}(2,2) \simeq \mathrm{SU}(4,G)$ and to the one-parameter dilatation group. The corresponding covariant derivative of the four-component Klein-Gordon field has the following form:

$$\nabla_\mu \Psi = \partial_\mu \Psi + g\left(\vartheta_\mu - \frac{1}{4}\mathrm{Tr}\,\vartheta_\mu \mathrm{I}\right)\Psi + \frac{q}{4}\mathrm{Tr}\,\vartheta_\mu \Psi = \partial_\mu \Psi + g\vartheta_\mu \Psi + \frac{q-g}{4}\mathrm{Tr}\,\vartheta_\mu \Psi. \tag{39}$$

Similarly for the Dirac-conjugate field we have the following dual formula:

$$\nabla_\mu \widetilde{\Psi} = \partial_\mu \widetilde{\Psi} - g\widetilde{\Psi}\left(\vartheta_\mu - \frac{1}{4}\mathrm{Tr}\,\vartheta_\mu \mathrm{I}\right) - \frac{q}{4}\widetilde{\Psi}\mathrm{Tr}\,\vartheta_\mu = \partial_\mu \widetilde{\Psi} - g\widetilde{\Psi}\vartheta_\mu - \frac{q-g}{4}\widetilde{\Psi}\mathrm{Tr}\,\vartheta_\mu. \tag{40}$$

The curvature form $\Phi = D\vartheta$ is then expressed as follows:

$$\Phi_{\mu\nu} = d\vartheta_{\mu\nu} + g\left[\vartheta_\mu, \vartheta_\nu\right] = \partial_\mu \vartheta_\nu - \partial_\nu \vartheta_\mu + g\left[\vartheta_\mu, \vartheta_\nu\right]. \tag{41}$$

Let us observe that the conserved Noether current following from the Noether theorem applied to (34) is given by

$$j^r{}_{s\mu} = \frac{b}{2}\left(\Psi^r \partial_\mu \widetilde{\Psi}_s - \partial_\mu \Psi^r \widetilde{\Psi}_s\right)\sqrt{|g|}. \tag{42}$$

One can show that the gauge invariant Lagrangian for the Ψ-matter has the following form:

$$L_m(\Psi, \vartheta, g) = \frac{b}{2} g^{\mu\nu} \nabla_\mu \widetilde{\Psi} \nabla_\nu \Psi \sqrt{|g|} - \frac{c}{2}\widetilde{\Psi}\Psi\sqrt{|g|}. \tag{43}$$

The gauge-invariant current

$$J(\Psi,\vartheta,g)^r{}_{s\mu} = \frac{b}{2}\left(\Psi^r\nabla_\mu\widetilde{\Psi}_s - \nabla_\mu\Psi^r\widetilde{\Psi}_s\right)\sqrt{|g|} \tag{44}$$

may be obtained from the Lagrangian (43) by performing its differentiation with respect to the connection ϑ,

$$\frac{\partial L_m(\Psi,\vartheta,g)}{\partial\vartheta^r{}_{s\mu}} = gJ^s{}_r{}^\mu + \frac{q-g}{4}J^z{}_{z\mu}\delta^s{}_r. \tag{45}$$

This was about the matter Lagrangian. What concerns the gauge Lagrangian, the simplest possibility of the gauge-invariant model is the following one:

$$L_{YM}(\vartheta,g) = \frac{a}{4}\mathrm{Tr}\left(\Phi_{\mu\nu}\Phi_{\varkappa\lambda}\right)g^{\mu\varkappa}g^{\nu\lambda}\sqrt{|g|} + \frac{a'}{4}\mathrm{Tr}\,\Phi_{\mu\nu}\mathrm{Tr}\,\Phi_{\varkappa\lambda}g^{\mu\varkappa}g^{\nu\lambda}\sqrt{|g|}, \tag{46}$$

where a, a' are constants. The first term, controlled by the parameter a is the main, Maxwell-like expressions. The second term is additional one, built of the traces of field strengths. It is an auxiliary expression, nevertheless it is geometrically admissible and it may be some reasonable, helpful correction to the first one. In any case, it is a merely supplementary expression, although it may be convenient and physically justified.
Let us observe that in spite of the non-homogeneous transformation rule (38), the curvature two-form (41) transforms according to the tensorial homogeneous rule under (37), (38):

$$(U\Phi)_x = U(x)\Phi(x)U(x)^{-1}. \tag{47}$$

Because of this the Yang-Mills Lagrangian (46) is invariant under the local $\mathrm{U}(2,2) \simeq \mathrm{U}(H,G)$ transformations (38). And similarly, the matter Lagrangian (43) is invariant.
This is the main, gauge constituent of the theory. Let us now mention only about the relationship of $\mathrm{SU}(2,2)$-matrices Φ to the representations $\mathrm{SL}(2,\mathbb{C}) \ni A \mapsto U[A] \in \mathrm{U}(2,2)$ corresponding to the Weyl-Penrose-Finkelstein-Weizsäcker and to the Dirac representation of $G_{\bar{r}s}$. In the first group of representation (W-P-F-W) we have the following realizations of $U[A]$:

$$U[A] = \begin{bmatrix} A & 0 \\ 0 & A^{-1+} \end{bmatrix}, \qquad u[a] = \begin{bmatrix} a & 0 \\ 0 & -a^{+} \end{bmatrix} \tag{48}$$

respectively for $\mathrm{SL}(2,\mathbb{C})$ and its Lie algebra. In Dirac representation

$$U[A] = \frac{1}{2}\begin{bmatrix} A + A^{-1+} & A - A^{-1+} \\ A - A^{-1+} & A + A^{-1+} \end{bmatrix}, \qquad u[a] = \frac{1}{2}\begin{bmatrix} a - a^{+} & a + a^{+} \\ a + a^{+} & a - a^{+} \end{bmatrix} \tag{49}$$

for the group and algebra. The right "plus" superscript denotes obviously the Hermitian matrix conjugate. Obviously, quite independently of the choice of any representation the following holds:

$$U[A]\gamma_K U[A]^{-1} = \gamma_L P[A]^L{}_K, \tag{50}$$

where $P : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)^{\uparrow}$ is the covering projection. The mappings $U : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{U}(2,2)$ and $P : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)^{\uparrow}$ generate the corresponding homomorphisms of Lie algebras, $u : \mathrm{SL}(2,\mathbb{C})' \to \mathrm{U}(2,2)'$ and $p : \mathrm{SL}(2,\mathbb{C})' \to \mathrm{SO}(1,3)'$. They are synchronized by

$$[u[a], (\gamma)_K] = \gamma_L p[a]^L{}_K. \tag{51}$$

It is also worth to note the following expressions:

$$P[A]^L{}_K = \frac{1}{4}\mathrm{Tr}\left(\gamma^L U[A]\gamma_K U[A]^{-1}\right), \tag{52}$$

$$p[a]^L{}_K = \frac{1}{2}\mathrm{Tr}\left(\gamma^L u[a]\gamma_K\right), \tag{53}$$

and

$$u[a] = \frac{1}{2}p[a]^L{}_K \Sigma_L{}^K, \tag{54}$$

where after the shift of indices we have that

$$\Sigma^{LK} = \frac{1}{4}\left(\gamma^L\gamma^K - \gamma^K\gamma^L\right) = \frac{1}{4}\left[\gamma^L, \gamma^K\right]. \tag{55}$$

4. What about the metric tensor?

In the gauge Lagrangians above, the metric tensor in a sense played the parameter role. Our idea was to construct the $\mathrm{U}(2,2) \simeq \mathrm{U}(H,G)$-invariant theory of gravitation. The main constituents of the theory were the four-component complex Klein-Gordon field and the corresponding Maxwell-like gauge field. We will show that there are interesting and very important points for which this is important, perhaps even just exciting. But there is some weak point which was not yet completely explained. It is just the role and physical status of the metric tensor, which is present in the Klein-Gordon and gauge Lagrangian, however its geometric and physical sense is not yet full understood. It is clear that it must occur there if we are to be able to construct Lagrangians. But what is its meaning and how to identify properly its physical role? As one of potentials of gravitation, or as some secondary variable? And if the second possibility is to be chosen, what are the primary variables the byproduct of which is the metric tensor? Situation in this respect was clear only in the standard Einstein General Relativity. There it was just the only gravitational potential (or perhaps a superpotential if the connection coefficients were interpreted as proper potentials). But within any gauge framework the metric tensor is a merely one of a few potentials.In this paper we concentrate on the theory aspects not very sensitive to this problem. Instead, we shall present a few possibilities.

First of all, let us notice that quite naively, one can assume the Hilbert-Einstein term for the metric tensor g,

$$L_{HE}(g) = -dR(g)\sqrt{|g|} + l\sqrt{|g|}. \tag{56}$$

where d, l are real constants. The special case $d = 0$ is not to be a priori rejected. Namely, if $d = 0$ and perhaps $l = 0$, then variation of the action functional with respect to $g_{\mu\nu}$ enables one to express $g_{\mu\nu}$ through the other variables. But of course, the choice (56) looks rather naive. In any case, the total Lagrangian of the form

$$L(\Psi, \vartheta, g) := L_m(\Psi, \vartheta, g) + L_{YM}(\vartheta, g) + L_{HE}(g) \tag{57}$$

leads, after the variational procedure for the action, to the following system of equations:

$$g^{\mu\nu}\overset{g}{\nabla}_{\mu}\overset{g}{\nabla}_{\nu}\Psi + \frac{c}{b}\Psi = 0, \tag{58}$$

$$\chi^{\mu\nu}{}_{;\nu} + g\left[\vartheta_{\nu}, \chi^{\mu\nu}\right] = gJ^{\mu} + \frac{q-g}{4}\mathrm{Tr}\, J_{\mu}\mathrm{I}, \tag{59}$$

$$d\left(R(g)^{\mu\nu} - \frac{1}{2}R(g)g^{\mu\nu}\right) = \frac{l}{2}g^{\mu\nu} + \frac{1}{2}T^{\mu\nu}, \tag{60}$$

with the meaning of symbols as below:

$$g^{\mu\nu}\overset{g}{\nabla}_{\mu}\overset{g}{\nabla}_{\nu}\Psi + \frac{c}{b}\Psi = 0, \tag{58}$$

$$\chi^{\mu\nu}{}_{;\nu} + g\left[\vartheta_{\nu}, \chi^{\mu\nu}\right] = gJ^{\mu} + \frac{q-g}{4}\mathrm{Tr}\, J_{\mu}\mathrm{I}, \tag{59}$$

$$d\left(R(g)^{\mu\nu} - \frac{1}{2}R(g)g^{\mu\nu}\right) = \frac{l}{2}g^{\mu\nu} + \frac{1}{2}T^{\mu\nu}, \tag{60}$$

with the meaning of symbols as below:

1. The semicolon ";" is the g-Levi-Civita covariant differentiation.
2. The symbol $\overset{g}{\nabla}_{\mu}$ denotes the complete covariant differentiation. The Levi-Civita covariant differentiation of the space-time indices is joined there with the $\mathrm{U}(2,2) \simeq \mathrm{U}(H,G)$ covariant differentials of internal indices. Let us quote a typical example:

$$\overset{g}{\nabla}_{\mu}Y^{r}{}_{\nu} + g\vartheta^{r}{}_{s\mu}Y^{s}{}_{\nu} + \frac{q-g}{4}\vartheta^{z}{}_{z\mu}Y^{r}{}_{\nu} - {}^{g}\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} Y^{r}{}_{\lambda}, \tag{61}$$

and similarly, i.e., dually, or in the Leibniz-multiplication sense, for other quantities.

3. χ is the field momentum conjugate to ϑ, so

$$\chi^{r}{}_{s}{}^{\mu\nu} = \frac{\partial L_{YM}}{\partial \vartheta^{s}{}_{r\mu,\nu}} = -a\Phi^{r}{}_{s\alpha\beta}g^{\alpha\mu}g^{\beta\nu}\sqrt{|g|} \tag{62}$$
$$-a'\delta^{r}{}_{s}\Phi^{z}{}_{z\alpha\beta}g^{\alpha\mu}g^{\beta\nu}\sqrt{|g|},$$

or in the shortened form with the g-shifting of indices,

$$\chi^{\mu\nu} = -a\Phi^{\mu\nu}\sqrt{|g|} - a'\mathrm{I}\,\mathrm{Tr}\,\Phi^{\mu\nu}\sqrt{|g|}. \tag{63}$$

4. $T^{\mu\nu}$ denotes the symmetric energy-momentum tensor of the fields ϑ, Ψ, so we have that

$$T^{\mu\nu} = T_m^{\mu\nu} + T_{YM}^{\mu\nu}, \tag{64}$$

where, obviously,

$$T_m^{\mu\nu} = -\frac{2}{\sqrt{|g|}} \left(\frac{\partial L_m}{\partial g_{\mu\nu}} - \left(\frac{\partial L_m}{\partial g_{\mu\nu,\alpha}} \right)_{,\alpha} \right), \tag{65}$$

$$T_{YM}^{\mu\nu} = -\frac{2}{\sqrt{|g|}} \left(\frac{\partial L_{YM}}{\partial g_{\mu\nu}} - \left(\frac{\partial L_{YM}}{\partial g_{\mu\nu,\alpha}} \right)_{,\alpha} \right). \tag{66}$$

We do not quote the explicit formulae.

As mentioned, the Hilbert-Einstein term of Lagrangian in (57) looks rather naive, although perhaps it may be reasonable. Equations resulting from the version with vanishing coefficients (or vanishing "d" at least) also seem to be not bad, and in any case not to be a priori rejected. And, as mentioned, the field equations following from the first two terms of (57) seem promisible. But, as said above, the Hilbert-Einstein term seems to spoil the whole taste of the gauge approach. In Einstein-Cartan theory it was the tetrad field who saved the situation, nevertheless, also for some price (as mentioned, in no other gauge theory one explicitly uses the field of frames as a dynamical variable). What may be done in our formalism to replace in a reasonable way the role of tetrad? We would like to answer this question before the further development of our theory. There are a few, at least three natural ways. Certainly there is no possibility to build the metric tensor from the gauge field, in the sense:

$$g_{\mu\nu} := p\vartheta^{r}{}_{s\mu}\vartheta^{s}{}_{r\nu} + q\vartheta^{r}{}_{r\mu}\vartheta^{s}{}_{s\nu}, \tag{67}$$

what apparently might seem natural. The point is, however that (67) is only globally, but not locally $\mathrm{U}(2,2)$-invariant. But one can do it in a local way, by introducing some fields more elementary than the metric itself.

1. We may assume that besides the connection form ϑ, the geometrodynamical sector involves some additional $\mathbb{C}^4$-valued (H-valued, let us say) differential one-form W:

$$M \ni x \mapsto W_x \in \mathrm{L}(T_x M, \mathbb{C}^4).$$

Analytically we describe this object as $W^r{}_\mu$. And we assume that it is homogeneously transformable under the locally acting $\mathrm{U}(2,2)$,

$$W_x \mapsto U(x) W_x, \qquad \text{i.e.,} \qquad {}'W^r{}_\mu = U^r{}_s(x) W^s{}_\mu. \tag{68}$$

This form gives rise to the metric tensor field on M as follows:

$$g(W)_{\mu\nu} := Re\left(\widetilde{W}_{r\mu} W^r{}_\nu\right) = Re\left(\widetilde{W}_\mu W_\nu\right). \tag{69}$$

Therefore, this expression is the symmetric, thus real part of the Hermitian tensor $W^*_x G$. The quantity is locally $\mathrm{U}(2,2)$-invariant. The simplest gauge-invariant Lagrangian is given by

$$L(W,\vartheta) = a \nabla \widetilde{W}_{\mu\nu} \nabla W_{\varkappa\lambda} g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} + b\sqrt{|g|}. \tag{70}$$

In this expression a, b are some real constants, $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$, and ∇W denotes the exterior covariant differential of W, so that

$$\nabla W_{\mu\nu} = dW_{\mu\nu} + g\left(\vartheta_\mu W_\nu - \vartheta_\nu W_\mu\right) + \frac{{}'q - g}{4}\left(\mathrm{Tr}\,\vartheta_\mu W_\nu - \mathrm{Tr}\,\vartheta_\nu W_\mu\right). \tag{71}$$

Here ${}'q$, the coupling constant, is the kind of electric charge of W. Let us stress, the Lagrangian (70) is locally invariant under $\mathrm{U}(2,2)$. It is interesting that after the $\mathrm{SL}(2,\mathbb{C})$-reduction W is a 3/2-spin particle. This resembles the super-symmetric idea of gravitino.

2. Let us suppose that besides of ϑ, the geometric sector contains also another $\mathrm{U}(2,2)'$-valued differential form W, $M \ni x \mapsto W_x \in \mathrm{L}(T_x M, \mathrm{U}(2,2)')$. Analytically it is represented by the system of quantities $W^r{}_{s\mu}$. But unlike the connection form ϑ, just like in the previous idea, it suffers a homogeneous transformation rule under $\mathrm{U}(2,2)$,

$$W_x \mapsto U(x) W_x U(x)^{-1}, \qquad {}'W^r{}_{s\mu} = U^r{}_z W^z{}_{t\mu} U^{-1t}{}_s. \tag{72}$$

The corresponding metric field $g(W)$ on M is given by

$$g(W)_{\mu\nu} = a \mathrm{Tr}\left(W_\mu W_\nu\right) + b \mathrm{Tr}\, W_\mu \mathrm{Tr}\, W_\nu, \tag{73}$$

where a, b are constants and obviously $a \neq 0$; the a-term is dominant, whereas the b-term is a merely correction.

The exterior covariant differential of W is given by

$$\nabla W_{\mu\nu} = dW_{\mu\nu} + g\left[\vartheta_\mu, W_\nu\right] - g\left[W_\nu, \vartheta_\mu\right]. \tag{74}$$

The corresponding Maxwell Lagrangian for the form W is given by

$$\begin{aligned} L(W,\vartheta) = {} & a \mathrm{Tr}\left(\nabla W_{\mu\nu} \nabla W_{\varkappa\lambda}\right) g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} \\ & + b \mathrm{Tr}\left(\nabla W_{\mu\nu}\right) \mathrm{Tr}\left(\nabla W_{\varkappa\lambda}\right) g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} + c\sqrt{|g|} \end{aligned} \tag{75}$$

with constant coefficients a, b, c.

3. There is also a different model, maximally economic in the sense that its only dynamical variables are ϑ and Ψ. There is neither g nor any other geometric quantity used as Lagrangian argument. Instead, we use the metric-like tensor built in a locally-invariant gauge way from the basic field quantities:

$$g(\Psi,\vartheta)_{\mu\nu} = aRe\left(\nabla_{\mu}\widetilde{\Psi}\nabla_{\nu}\Psi\right) = aRe\left(G_{\bar{r}s}\nabla_{\mu}\overline{\Psi}^{\bar{r}}\nabla_{\nu}\Psi^{s}\right). \tag{76}$$

Obviously, it would be meaningless to substitute this metric to the usual Klein-Gordon Lagrangian, because the result would be trivial. However, there are modified Born-Infeld type schemes, in a sense very interesting ones, as usual Born-Infeld schemes are. The typical Born-Infeld scheme for (Ψ,ϑ) is the following one:

$$L(\Psi,\vartheta) = \sqrt{\left|\det\left[\frac{b}{2}g_{\mu\nu} + \frac{a}{4}\mathrm{Tr}\left(F_{\mu\varkappa}F_{\nu\lambda}\right)g^{\varkappa\lambda} + \frac{a'}{4}\mathrm{Tr}\,F_{\mu\varkappa}\mathrm{Tr}\,F_{\nu\lambda}g^{\varkappa\lambda}\right]\right|}. \tag{77}$$

This is the most natural Born-Infeld scheme.
Let us mention, there are also similar, but modified expressions with some "potential" terms, e.g.,

$$\begin{aligned} L(\Psi,\vartheta) &= \frac{a}{4}\mathrm{Tr}\left(F_{\mu\nu}F_{\varkappa\lambda}\right)g^{\mu\varkappa}g^{\nu\lambda}\sqrt{|g|} \\ &+ \frac{a'}{4}\mathrm{Tr}\,F_{\mu\nu}\mathrm{Tr}\,F_{\varkappa\lambda}g^{\mu\varkappa}g^{\nu\lambda}\sqrt{|g|} + b\sqrt{|g|}. \end{aligned} \tag{78}$$

In both expressions (77), (78) the space-time metric is given by (76), and a, a', b are some real constants.

5. The main ideas of the Klein-Gordon $\mathrm{U}(2,2)$-ruled theory

Without judging the three presented models of the metric field, or rather the four of them if the Hilbert-Einstein possibility is admitted, we have nevertheless formulated them. And from the point of view of aesthetic criteria, they look quite reasonable. But now, let us discuss the main results of the very $\mathrm{U}(2,2)$-invariant Klein-Gordon gauge model of gravity as ruled by the first two terms of (57), or even by the total (57).
The field equations (58), (59), (60) become qualitatively readable when some special basis is chosen in the Lie algebra of $\mathrm{U}(2,2)$. This basis is somehow related to the twistor geometry, although literally it is something else than the conformal geometry in Minkowskian space. The basis elements are built algebraically of γ^A-matrices.
Let us introduce the following matrices built algebraically of γ^A-s:

$$\gamma^5 = -\gamma_5 = -\gamma^0\gamma^1\gamma^2\gamma^3, \tag{79}$$

$$^{A}\gamma = i\gamma^A\gamma^5 = -i\gamma^5\gamma^A, \tag{80}$$

$$\Sigma^{AB} = \frac{1}{4}\left(\gamma^A\gamma^B - \gamma^B\gamma^A\right). \tag{81}$$

It is clear that $^{A}\gamma$-s satisfy the opposite-sign anticommutation rules

$$\left\{^{A}\gamma, {}^{B}\gamma\right\} = -2\eta^{AB}I. \tag{82}$$

One can show that

$$ {}_{A}\gamma = {}^{B}\gamma\eta_{BA} = -\frac{i}{6}\varepsilon_{ABCD}\gamma^{B}\gamma^{C}\gamma^{D}, \tag{83}$$

where the convention $\varepsilon_{0123} = 1$ is used.
The Lie algebra $\mathrm{U}(2,2)' = \mathrm{U}(H,G)'$ may be spanned in the $\mathbb{R}$-sense on the matrices

$$ i\gamma^{A}, \qquad i^{A}\gamma, \qquad \Sigma^{AB}, \qquad i\gamma^{5}, \qquad iI_{4}. \tag{84}$$

Removing from this system the imaginary matrix iI_4, we obtain the basis of $\mathrm{SU}(2,2)' = \mathrm{SU}(H,G)'$. The matrices $i\gamma^A$, $i^A\gamma$ do not $\mathbb{R}$-span Lie algebras. But it is clear that their sum and difference are bases of Abelian Lie subalgebras:

$$ \tau_{A} := \frac{1}{2}\left(\gamma_{A} + {}_{A}\gamma\right), \qquad \chi^{A} := \frac{1}{2}\left(\gamma^{A} - {}^{A}\gamma\right), \tag{85}$$

$$ \left[\tau_{A}, \tau_{B}\right] = 0, \qquad \left[\chi^{A}, \chi^{B}\right] = 0. \tag{86}$$

In the twistor language the quantities τ_A generate Minkowskian translations, while χ^A are generators of the group of proper conformal mappings. Obviously, this interpretation is true only within the framework of Minkowskian-conformal geometry. The literal meaning of this interpretation is lost within the internal interpretation of those mappings. Nevertheless, the commutation rules of the conformal group are still valid. It becomes internal group just like the Lorentz group in Einstein-Cartan theory.
The connection from ϑ may be expanded as follows:

$$\vartheta_\mu = \frac{1}{2}\breve{\Omega}^{AB}{}_\mu \Sigma_{AB} + B_\mu \frac{1}{i}\gamma_5 + A_\mu i I + e^A{}_\mu i\tau_A + f_{A\mu} i\chi^A, \tag{87}$$

where, obviously, $\breve{\Omega}^{AB}{}_\mu = -\breve{\Omega}^{BA}{}_\mu$.
It may be convenient to introduce the object

$$\Omega^A{}_{B\mu} := \breve{\Omega}^A{}_{B\mu} + 2B_\mu \delta^A{}_B, \tag{88}$$

where the following holds:

$$B_\mu = \frac{1}{8}\Omega^A{}_{A\mu}, \qquad \breve{\Omega}^A{}_{B\mu} = \Omega^A{}_{B\mu} - \frac{1}{4}\Omega^C{}_{C\mu}\delta^A{}_B. \tag{89}$$

Therefore, $8B_\mu$ may be identified with the trace, and $\breve{\Omega}^A{}_{B\mu}$ — with the trace-less part of the object $\Omega^A{}_{B\mu}$. So, it may be natural to write ϑ_μ as follows:

$$\vartheta_\mu = \frac{1}{2}\Omega^{AB}{}_\mu \left(\Sigma_{AB} + \frac{1}{4} n_{AB} \frac{1}{i}\gamma^5\right) + e^A{}_\mu i\tau_A + f_{A\mu} i\chi^A + A_\mu iI, \tag{90}$$

or alternatively

$$\vartheta_\mu = \frac{1}{2g}\breve{\Gamma}^{AB}{}_\mu \Sigma_{AB} + \frac{1}{4g} Q_\mu \frac{1}{i}\gamma^5 + \frac{1}{g}\varepsilon^A{}_\mu i\tau_A + \frac{1}{g}\varphi_{A\mu} i\chi^A + A_\mu iI, \tag{91}$$

or just as

$$\vartheta_\mu = \frac{1}{2g}\Gamma^{AB}{}_\mu \left(\Sigma_{AB} + \frac{1}{4} n_{AB} \frac{1}{i}\gamma^5\right) + \frac{1}{g}\varepsilon^A{}_\mu i\tau_A + \frac{1}{g}\varphi_{A\mu} i\chi^A + A_\mu iI, \tag{92}$$

where the following auxiliary gauge symbols are used:

$$\Gamma^A{}_{B\mu} = g\Omega^A{}_{B\mu}, \qquad \breve{\Gamma}^A{}_{B\mu} = \Gamma^A{}_{B\mu} - \frac{1}{4}\Gamma^C{}_{C\mu}\delta^A{}_B, \tag{93}$$

$$Q_\mu = 4gB_\mu = \frac{g}{2}\Omega^A{}_{A\mu} = \frac{1}{2}\Gamma^A{}_{A\mu}, \tag{94}$$

$$\varepsilon^A{}_\mu = ge^A{}_\mu, \qquad \varphi_{A\mu} = gf_{A\mu}. \tag{95}$$

The systems of differential forms $[\Omega^{A}{}_{B} = (1/g)\Gamma^{A}{}_{B}]$, $[e^{A}]$, $[f_{A}]$ are parts of the connection form ϑ, and because of this, the action of x-dependent matrices U on them is inhomogeneous. But when we restrict ourselves to the U-injected group $SL(2, \mathbb{C})$, the transformation rule for $[e^{A}]$, $[f_{A}]$ becomes homogeneous. It is just the correspondence rule with the situation of Einstein-Cartan theory where $[e^{A}]$ was a gravitational cotetrad. Let us remind that in $GL(2, \mathbb{C})$-invariant spinor theory, $Q_{\mu} = (1/2)\Gamma^{A}{}_{A\mu}$ was the Weyl covector, and there was necessity to use an additional version of the cotetrad, transforming under dilatations in the inverse way. If we used the world metric $g_{\mu\nu}$, then Q_{μ} was the Killing covector, i.e.,

$$\nabla_{\lambda} g_{\mu\nu} = -Q_{\lambda} g_{\mu\nu}, \tag{96}$$

in the sense of the $GL(2, \mathbb{C})$-part of the $U(2,2)$-connection.

Let us stress that in certain formulae it is still more convenient to use the γ^A, ${}^A\gamma$-expansion than those based on τ_A, χ^A, namely,

$$\vartheta_\mu = \frac{1}{2}\Omega^{AB}{}_\mu \Sigma_{AB} + B_\mu \frac{1}{i}\gamma^5 + A_\mu iI + E_{A\mu} i\gamma^A + F_{A\mu} i^A\gamma, \tag{97}$$

where the following notation is used:

$$E^A = \frac{1}{2}\left(e^A + \eta^{AB} f_B\right) = \eta^{AB} E_B, \quad F^A = \frac{1}{2}\left(e^A - \eta^{AB} f_B\right) = \eta^{AB} F_B. \tag{98}$$

Let us now take the following expansion of the curvature vector-valued two-form:

$$\Phi = T(e)^A i\tau_A + T(f)^A i\chi_A + \frac{1}{2}\widetilde{R}^{AB}\Sigma_{AB} + G\frac{1}{i}\gamma^5 + FiI, \tag{99}$$

where the following partially clear symbols are used:

$$T(e)^A = de^A + g\Omega^A{}_B \wedge e^B = de^A + \Gamma^A{}_B \wedge e^B, \tag{100}$$

$$T(f)_A = df_A + gf_B \wedge \Omega^B{}_A = df_A + f_B \wedge \Gamma^B{}_A, \tag{101}$$

$$\begin{aligned}\widetilde{R}^A{}_B &= R(\Omega)^A{}_B - \frac{1}{4}R(\Omega)^C{}_C \delta^A{}_B - 2ge^A \wedge f_B \\ &\quad +2g\eta^{AC}\eta_{BD}e^D \wedge f_C = \frac{1}{g}\bigg(R(\Gamma)^A{}_B - \frac{1}{4}R(\Gamma)^C{}_C\delta^A{}_B \\ &\quad -2g^2 e^A \wedge f_B + 2g^2\eta^{AC}\eta_{BD}e^D \wedge f_C\bigg),\end{aligned} \tag{102}$$

$$G = \frac{1}{4g}dQ - ge^A \wedge f_A = \frac{1}{g}\left(\frac{1}{8}R(\Gamma)^A{}_A - g^2 e^A \wedge f_a\right), \tag{103}$$

$$F = dA, \tag{104}$$

where $R(\Gamma)$, $R(\Omega)$ denote the curvature two-form:

$$R(\Gamma)^A{}_B = d\Gamma^A{}_B + \Gamma^A{}_C \wedge \Gamma^C{}_B, \quad R(\Omega)^A{}_B = d\Omega^A{}_B + g\Omega^A{}_C \wedge \Omega^C{}_B. \tag{105}$$

Let us remember that the torsion of a linear connection may be interpreted as a contribution to affine connection. The corresponding space-time objects, i.e., connections, torsions and curvatures are given by

$$\Gamma(e)^k{}_{ij} = e^k{}_A \Gamma^A{}_{Bj} e^B{}_i + e^k{}_A e^A{}_{i,j}, \tag{106}$$

$$\Gamma(f)^k{}_{ij} = -f_{Ai}\Gamma^A{}_{Bj} f^{kB} + f^{kA} f_{Ai,j}, \tag{107}$$

$$S(e)^k{}_{ij} = \Gamma(e)^k{}_{[ij]} = -\frac{1}{2} e^k{}_A T(e)^A{}_{ij}, \tag{108}$$

$$S(f)^k{}_{ij} = \Gamma(f)^k{}_{[ij]} = -\frac{1}{2} f^{kA} T(f)_{Aij}, \tag{109}$$

$$R(e)^m{}_{kij} = e^m{}_A e^B{}_k R^A{}_{Bij}, \tag{110}$$

$$R(f)^m{}_{kij} = -f_{Ak} f^{mB} R^A{}_{Bij}. \tag{111}$$

We do not quote the total explicit form of field equations. Being generally-covariant, they are over-determined, therefore, as usual suspected to be inconsistent. Nevertheless, substituting to the field equations the non-excited matter $\Psi = 0$, and the above equations (106)–(111), one can show that there are certain non-trivial solutions. Namely, let us take the following Einstein-Dirac metrics:

$$
\begin{aligned}
h(e,\eta)_{\mu\nu} &= \eta_{AB} e^{A}{}_{\mu} e^{B}{}_{\nu}, \\
h(f,\eta)_{\mu\nu} &= \eta^{AB} f_{A\mu} f_{B\nu}.
\end{aligned} \tag{112}
$$

And now, let us substitute the following conditions to the field equations (58), (59), (60):

$$\Psi = 0, \qquad f_{A\mu} = k\eta_{AB} e^{B}{}_{\mu}, \qquad g_{\mu\nu} = p h(e,\eta)_{\mu\nu}, \tag{113}$$

$$Q_{\mu} = 0, \qquad A_{\mu} = 0, \qquad S(e)^{\lambda}{}_{\mu\nu} = S(f)^{\lambda}{}_{\mu\nu} = 0. \tag{114}$$

It is simply marvellous that the very complicated system of equations following from (58), (59), (60) after substituting (106)–(114) is solvable, moreover, it is reducible to something very simple. Namely, the very complicated system of equations for the geometric fields reduces step by step to

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -12 \frac{g^2 k}{p} g^{\mu\nu}, \tag{115}$$

where $R^{\mu\nu}$ denotes the twice contravariant Ricci tensor built of $g_{\alpha\beta}$, and R is the curvature scalar. Substituting there $k = 1$, $p = 1$, one obtains simply the following equation

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -12 g^2 g^{\mu\nu}. \tag{116}$$

In any case one deals here with the Einstein equation with the cosmological constant. It is remarkable that this coupling constant is proportional to the gauge coupling constant. Such a coupling between microphysical model and macroscopic or even just cosmic scale physics is marvellous and philosophically fascinating. When the Einstein-Hilbert dynamics of $g_{\mu\nu}$ is used, we obtain also the condition

$$T^{\mu\nu} = 0. \tag{117}$$

But there is no contradiction between (117) and (115)/(116). There exist some common solutions, namely, ones corresponding to the constant curvature spaces:

$$R_{\alpha\beta\mu\nu} = \frac{4g^2k}{p}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right). \tag{118}$$

It is again the fascinating idea that the conformal flatness of space-time, expressed by (118) is so nicely compatible with the assumption that the theory is invariant under the covering group of the conformal group.
The field sector of the model corresponds smoothly to the gauge Poincare gravitation. When the field Lagrangian is expressed in terms of the above quantities e, f, $S(e)$, $S(f)$, $R(e)$, $R(f)$, one obtains expression quadratic in field variables just like in Poincare gauge models. However, the use of the semisimple $\mathrm{SU}(2,2)$ results in well-defined, rigorous ratio of constant coefficients, not accidental one like in Poincare model.

What concerns material sector, the use of the second-order Klein-Gordon equation and the presence of second-order derivatives of the matter field, is a drastic difference in comparison with the first-order Dirac equation. However, the situation is not very bad, on the contrary, it may seem promising and desirable. If we substitute to the matter equation the Dirac-Einstein assumption, e.g., with $p = 1$, $k = 1$, then we obtain the following Dirac-Klein-Gordon equation:

$$e^{\mu}{}_{A} i\gamma^{A} \left(\nabla_{\mu} + S^{\nu}{}_{\nu\mu} I_{4}\right) \Psi - \frac{4bg^{2} - c}{2bg}\Psi + \frac{1}{2g}\overset{g}{\nabla}_{\mu}\overset{g}{\nabla}_{\nu}\Psi = 0, \tag{119}$$

where ∇_{μ} is the SL$(2, \mathbb{C})$-part of the U$(2, 2)$-covariant derivative and $\overset{g}{\nabla}_{\mu}$ joints that differentiation with $\Gamma^{A}{}_{B}$-differentiation of objects with the capital Lorentz indices and with the g-Levi-Civita differentiation. It is interesting that the first two terms of (119) correspond exactly with the Dirac theory in Einstein-Cartan space-time. However, there is quite a natural question if the third, second-order d'Alembert term does not destroy completely this similarity.

The simplest way to answer this question is to consider the specially-relativistic situation, when $g_{\mu\nu} = \eta_{\mu\nu}$, $\Gamma^{A}{}_{B\mu} = 0$, $e^{A}{}_{\mu} = \delta^{A}{}_{\mu}$. It is clear that under this substitution one obtains the Dirac-Klein-Gordon differential equation

$$i\gamma^{\mu}\partial_{\mu}\Psi - \frac{4bg^{2} - c}{2bg}\Psi + \frac{1}{2g}\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\Psi = 0. \tag{120}$$

It is obvious that the general solution of this equation is a combination of two Dirac waves with two possible masses, namely, $m_{\pm}$, where

$$m_{\pm}^2 = \frac{c}{b} - 2g^2 \left(1 \pm \sqrt{\frac{c}{bg^2} - 3}\right). \tag{121}$$

This result is obtained when the amplitudes $\mathrm{U}(p) \exp\left(ip_\mu x^\mu\right)$ are substituted to (120). This means that there is a range of "Dirac" behaviour, $c/b > 3g^2$. The primeval mass parameter must be sufficiently large for that. Then the solution of (120) will be a superposition of two Dirac fields. If $c/b = 3g^2$, then one obtains the exactly Dirac behaviour. This means that there is no splitting of mass and that $m = |g|$. Below this threshold we are dealing with tachyonic or decay phenomena. It is also very interesting that if $c/b = 4g^2$, then one of the partner states is massless, namely, $m_- = 0$, $m_+ = 2|g|$. Obviously, the doubling of mass within the Dirac behaviour needs some explanation in terms of experimental data. Let us quote three possibilities (Sławianowski & Kovalchuk, 2008):

- If the energy gap $m_+ - m_-$ is very small (i.e., $|g|$ is small enough), then perhaps it is below the present accuracy of our experimental abilities.
- If the energy gap $m_+ - m_-$ is so large that perhaps it is too difficult to create/excite the state of higher mass.
- And finally, the most important and promising explanation. Perhaps the mass-state-doubling does exist and is just observed. This would be just the explanation of the mysterious relationship between fundamental quarks and fermions in the standard model of weak interactions. We mean their occurance in pairs (u, d), (c, s), (t, b) (quarks) and (ν_e, e), (ν_μ, μ), (ν_τ, τ) (leptons) (Veltman, 2003). For example, the situation $c/b = 4g^2$ might be a naive explanation of the pairing between heavy leptons and their nuetrinos.

6. Some additional interpretation problems

In one of our earlier papers we have discussed the idea of higher-order derivatives just from the point of view of the order of time derivatives (Sławianowski & Kovalchuk, 2002; 2010; Sławianowski et al., 2004; 2005), because the other continuous variables were absent. The Schrödinger equation was then interpreted as a Hamiltonian system of mathematical physics. And it was just then where the second-order time derivatives seemed not only admissible, but just necessary. In the field problems, this concerns, of course, the occurrence of all second-order space-time derivatives of the field quantities/wave functions. And, as shown above, those second-order space-time derivatives just seem desirable, not only admissible.

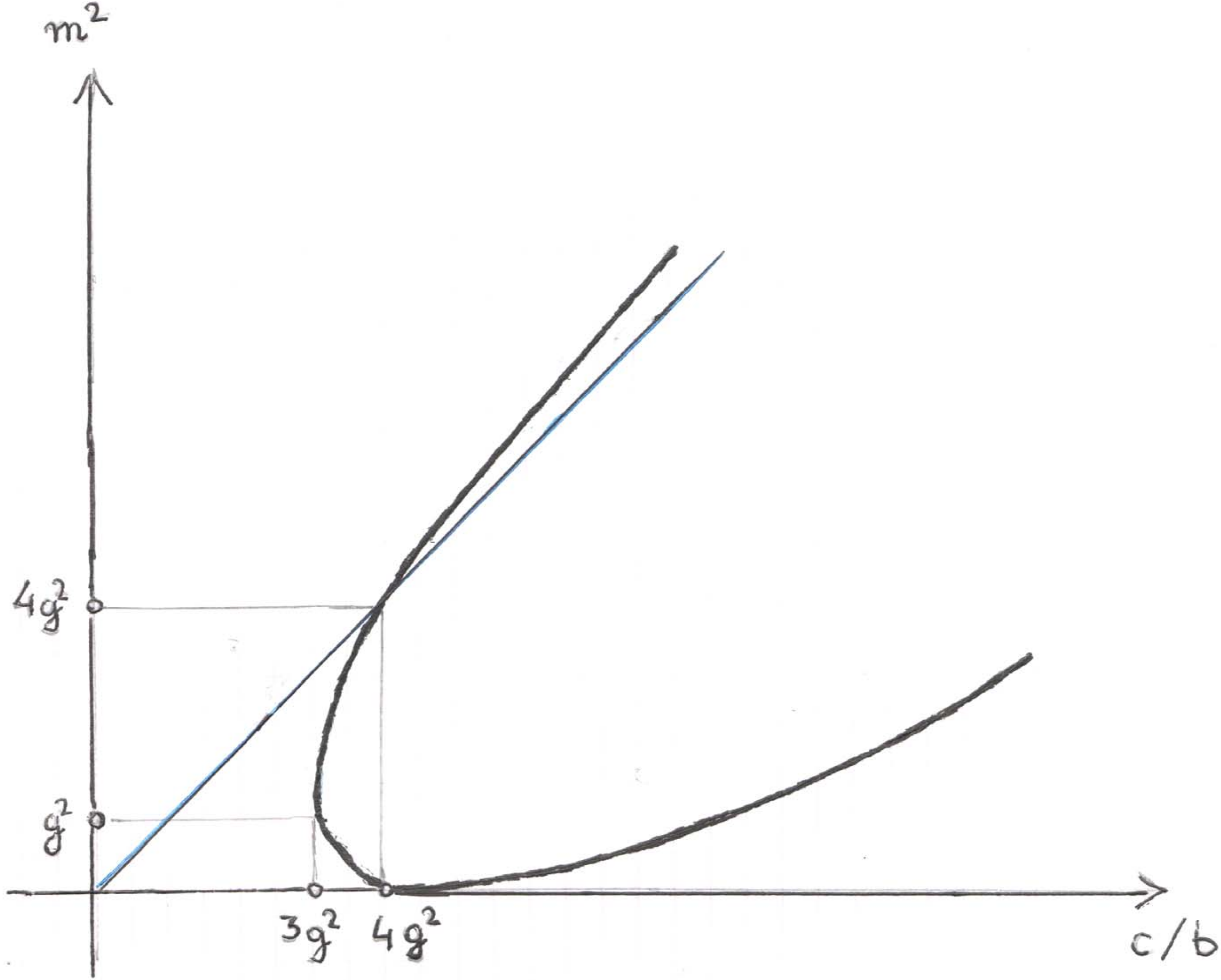
The next important question is: why just $U(2,2) \subset GL(4,\mathbb{C})$, not the total $GL(4,\mathbb{C})$? But it is seen that it is only signature $(+,+,-,-)$ of Hermitian G-forms, not the group $U(2,2)$ itself that matters. Namely, the mass form $G_{\overline{r}s}$ must by present in Lagrangian, however not as a fixed constant Hermitian form but as a dynamical, x-dependent form of signature $(+,+,-,-)$. The corresponding Lagrangian term would be proportional to

$$G^{s\overline{z}} G^{t\overline{r}} \frac{\partial G_{\overline{r}s}}{\partial x^{\mu}} \frac{\partial G_{\overline{z}t}}{\partial x^{\nu}} g^{\mu\nu}, \tag{122}$$

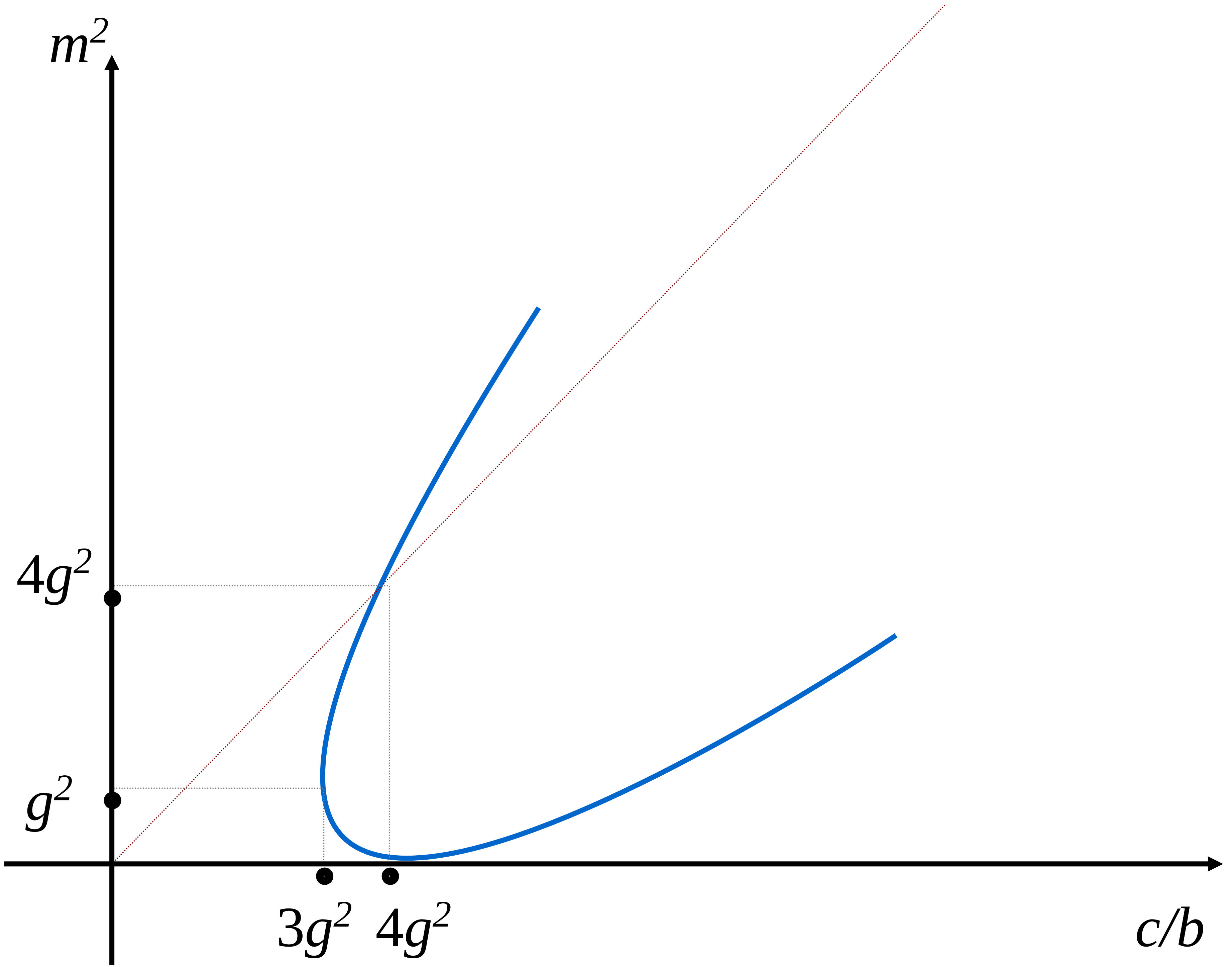
or perhaps to the Born-Infeld term

$$\sqrt{\det\left[G^{s\overline{z}} G^{t\overline{r}} G_{\overline{r}s,\mu} G_{\overline{z}t,\nu}\right]} \tag{123}$$

independent on the mentioned choice of the space-time metric $g_{\mu\nu}$. This term is evidently $GL(4,\mathbb{C})$-invariant.



m²
4g²
g²
3g²
4g²
c/b



m^2
$4g^2$
g^2
$3g^2$
$4g^2$
c/b

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