# Relativistic Hyperbolic Geometry

Geometry, Integrability and Quantization

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Abraham A. Ungar
Department of Mathematics
North Dakota State University
Fargo, North Dakota

Email: abraham.ungar@ndsu.edu

The Einstein addition $\oplus$

in $(\mathbb{R}_c^n, \oplus)$

$$\boxed{\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\}} \tag{1}$$

for all $\mathbf{u}$ and $\mathbf{v}$ in the $c$-ball $\mathbb{R}_c^n$

$$\mathbb{R}_c^n = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\} \tag{2}$$

of the Euclidean $n$-space $\mathbb{R}^n$, $\gamma_{\mathbf{u}}$ being the Lorentz factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}}\,, \qquad \gamma_{\mathbf{u}}\ \textit{real} \ \Leftrightarrow\ \mathbf{u} \in \mathbb{R}_c^n \tag{3}$$

Einstein addition and the gamma factor are related by the *gamma identity*,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{4}$$

so that $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n \ \Rightarrow\ \mathbf{u}\oplus\mathbf{v} \in \mathbb{R}_c^n$.

**Reduction to the common vector addition:**

When $c \to \infty$, Einstein addition $\oplus$ in the ball $\mathbb{R}^n_c$ reduces to the common vector addition, $+$, in the Euclidean $n$-space $\mathbb{R}^n$:

$$\lim_{c\to\infty}(\mathbb{R}^n_c, \oplus) \;=\; (\mathbb{R}^n, +) \tag{5}$$

The groupoid $(\mathbb{R}^n, +)$ is regulated

1. algebraically, by the (associative-commutative) algebra of vector spaces; and
2. geometrically, by Euclidean geometry.

We will find that, in full analogy:

The Einstein groupoid $(\mathbb{R}^n_c, \oplus)$ is regulated

1. algebraically, by the nonassociative-noncommutative algebra of gyrovector spaces; and
2. geometrically, by the hyperbolic geometry of Bolyai and Lobachevsky.

Einstein's addition is noncommutative. In general

$$\boxed{\mathbf{u} \oplus \mathbf{v} \neq \mathbf{v} \oplus \mathbf{u}} \qquad (6)$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. Moreover, Einstein's addition is also nonassociative. In general

$$\boxed{(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} \neq \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})} \qquad (7)$$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$. Before the advent of gyrogroup theory in 1988 physicists have difficulty explaining why the Einstein velocity addition fell into such despair.

One might suppose that there is a price to pay in mathematical regularity when replacing ordinary vector addition with Einstein's addition. But we will now see that this is not the case.

$$a \oplus b = \mathrm{gyr}[a,b](b \oplus a) \qquad \text{Gyrocommutative Law}$$

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Coincidentally, the gyration that repairs the breakdown of commutativity in the Möbius addition repairs the breakdown of associativity as well, giving rise to identities that capture analogies,

$$\begin{array}{ll} a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c & \text{Left Gyroassoc. Law} \\ (a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b,a]c) & \text{Right Gyroassoc. Law} \\ \mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b] & \text{Left Reduction Prop.} \\ \mathrm{gyr}[a,b] = \mathrm{gyr}[a, b \oplus a] & \text{Right Red. Property} \end{array}$$

The emerging coincidences expose an algebraic structure that merits extension by abstraction, leading to the concept of the grouplike structure called a gyrogroup.

Gyrogroups are generalized groups that share remarkable analogies with groups. In full analogy with groups

(1) gyrogroups are classified into gyrocommutative gyrogroups and non-gyrocommutative gyrogroups; and

(2) some gyrocommutative gyrogroups admit scalar multiplication, turning them into gyrovector spaces.

(3) Gyrovector spaces, in turn, provide the setting for hyperbolic geometry just as vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified.

**Definition 1** (Gyrogroups). *The groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms. In* $G$ *there is at least one element,* $0$*, called a left identity, satisfying for all* $a, b, z \in G$

| | |
|---|---|
| $0 \oplus a = a$ | *Left Identity* |
| $\ominus a \oplus a = 0$ | *Left Inverse* |
| $a \oplus (b \oplus z) = (a \oplus b) \oplus \mathrm{gyr}[a, b]z$ | *Left Gyroassociative* |
| $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$ | *Gyroautomorphism* |
| $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$ | *Left Reduction Property* |

**Definition 2** (Gyrocommutative Gyrogroups). *The gyrogroup* $(G, \oplus)$ *is gyrocommutative if for all* $a, b \in G$

$a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$ *Gyrocommutative Law*

The Gyration
Expressed in Terms of
Einstein Addition

Solving the Left Gyroassociative Law

$$a\oplus(b\oplus z) = (a\oplus b)\oplus \mathrm{gyr}[a,b]z$$

we have

$$\boxed{\mathrm{gyr}[a,b]z = \ominus(a\oplus b)\oplus\{a\oplus(b\oplus z)\}}$$

and

$$\boxed{\|\mathrm{gyr}[a,b]z\| = \|z\|}$$

calling $\mathrm{gyr}[a,b]$ the gyration generated by $a$ and $b$.

Thus, the application of $\mathrm{gyr}[a,b]$ to $z$ gives a rotation of $z$, so that gyrations are, in fact, rotations.

Scalar Multiplication $\otimes$

for Einstein's gyrogroups $(\mathbb{R}^n_c, \oplus)$

$$n\otimes\mathbf{v} = \mathbf{v}\oplus\ldots\oplus\mathbf{v} \qquad \text{(n terms)} \tag{8}$$

$$n\otimes\mathbf{v} = c\frac{(1+\|\mathbf{v}\|/c)^n - (1-\|\mathbf{v}\|/c)^n}{(1+\|\mathbf{v}\|/c)^n + (1-\|\mathbf{v}\|/c)^n}\frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{9}$$

$$\boxed{\begin{aligned} r\otimes\mathbf{v} &= c\frac{(1+\|\mathbf{v}\|/c)^r - (1-\|\mathbf{v}\|/c)^r}{(1+\|\mathbf{v}\|/c)^r + (1-\|\mathbf{v}\|/c)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\,tanh\left(r\,tanh^{-1}\frac{\|\mathbf{v}\|}{c^2}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned}} \tag{10}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n_c$, $\mathbf{v} \neq \mathbf{0}$; and $r\otimes\mathbf{0} = \mathbf{0}$.

The scalar multiplication possesses the following properties. For any positive integer $n$ and for all $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}_c^n$,

$$
\begin{aligned}
n\otimes\mathbf{v} &= \mathbf{v}\oplus\ldots\oplus\mathbf{v} \qquad \text{n terms} \\
(r_1 + r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} \\
(r_1 r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) \\
r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) &= r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_2\otimes\mathbf{v}) \\
\|r\otimes\mathbf{v}\| &= |r|\otimes\|\mathbf{v}\| \\
\frac{|r|\otimes\mathbf{v}}{\|r\otimes\mathbf{v}\|} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} \\
\|\mathbf{u}\oplus\mathbf{v}\| &\leq \|\mathbf{u}\|\oplus\|\mathbf{v}\| \\
\mathrm{gyr}[a, b](r\otimes\mathbf{v}) &= r\otimes\mathrm{gyr}[a, b]\mathbf{v} \\
\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] &= I
\end{aligned}
\tag{11}
$$

Remarkably, we have a *monodistributive law* but no distributive law.

Solving the Equation

$$\mathbf{a}\oplus\mathbf{x} = \mathbf{b} \tag{12}$$

for $x$ in a gyrogroup $(G, \oplus)$. If $\mathbf{x}$ is a solution, then by the right gyroassociative law and the identity $\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{a}] = id$ we have

$$\begin{aligned} \mathbf{x} &= \mathbf{0}\oplus\mathbf{x} \\ &= (\ominus\mathbf{a}\oplus\mathbf{a})\oplus\mathbf{x} \\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{a}]\mathbf{x}) \\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathbf{x}) \\ &= \ominus\mathbf{a}\oplus\mathbf{b} \end{aligned} \tag{13}$$

Thus if a solution exists, it must be given uniquely by

$$\mathbf{x} = \ominus\mathbf{a}\oplus\mathbf{b} \tag{14}$$

Solving the Equation

$$\mathbf{x}\oplus\mathbf{a} = \mathbf{b} \tag{15}$$

for $x$ in a gyrogroup $(G, \oplus)$. If $\mathbf{x}$ is a solution then, by the left gyroassociative law and the left loop property we have

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}\oplus\mathbf{0} \\
&= \mathbf{x}\oplus(\mathbf{a}\oplus(\ominus\mathbf{a})) \\
&= (\mathbf{x}\oplus\mathbf{a})\oplus\mathrm{gyr}[\mathbf{x},\mathbf{a}](\ominus\mathbf{a}) \\
&= (\mathbf{x}\oplus\mathbf{a})\oplus(\ominus\mathrm{gyr}[\mathbf{x},\mathbf{a}]\mathbf{a}) \\
&= (\mathbf{x}\oplus\mathbf{a})\ominus\mathrm{gyr}[\mathbf{x},\mathbf{a}]\mathbf{a} \\
&= \mathbf{b}\ominus\mathrm{gyr}[\mathbf{x},\mathbf{a}]\mathbf{a} \\
&= \mathbf{b}\ominus\mathrm{gyr}[\mathbf{x}\oplus\mathbf{a},\mathbf{a}]\mathbf{a} \\
&= \mathbf{b}\ominus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{a} \\
&= \mathbf{b}\boxminus\mathbf{a}
\end{aligned} \tag{16}$$

where we use the obvious notation:

$$\mathbf{a}\ominus\mathbf{b} = \mathbf{a}\oplus(\ominus\mathbf{b}).$$

**Definition 3** (Gyrogroup Dual Operation). *Let $(G, \oplus)$ be a gyrogroup. A secondary binary operation $\boxplus$ in $G$ is defined by the equation*

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \qquad \textit{Secondary Operation}$$

**Theorem 4.** *Let $(G, \oplus)$ be a gyrogroup and let $\boxplus$ be its dual binary operation given by Definition (3),*

$$\boxed{a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b} \tag{17}$$

*Then*

$$\boxed{a \oplus b = a \boxplus \mathrm{gyr}[a, b]b} \tag{18}$$

*and*

$$\mathrm{Aut}(G, \boxplus) = \mathrm{Aut}(G, \oplus) \tag{19}$$

**Theorem 5** (Gyrogroup Cancellation Laws). *Let $(G, \oplus)$ be a gyrogroup, and let $\boxplus$ be its dual operation. Then, for all $a, b, c \in G$,*

$$a \oplus (\ominus a \oplus b) = b \qquad \textit{Left Cancellation Law}$$
$$(b \boxminus a) \oplus a = b \qquad \textit{Right Cancellation Law}$$
$$(b \ominus a) \boxplus a = b \qquad \textit{Dual Right Cancellation Law}$$

Euclidean Geodesics

$$\begin{aligned} L_p &= \mathbf{a} + (-\mathbf{a} + \mathbf{b})t \\ L_s &= (\mathbf{b} - \mathbf{a})t + \mathbf{a} \end{aligned} \qquad (20)$$

Newtonian Mechanics: $\boxed{\mathbf{v}_0 + \mathbf{a}t \quad \text{or} \quad \mathbf{a}t + \mathbf{v}_0}$

Relative to the Euclidean Metric

$$\begin{aligned} d(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} - \mathbf{b}\| \\ d(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} - \mathbf{b}\| \end{aligned} \qquad (21)$$

Hyperbolic Geodesics

$$\begin{aligned} L_p &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t \\ L_s &= (\mathbf{b} \boxminus \mathbf{a})\otimes t\oplus\mathbf{a} \end{aligned} \qquad (22)$$

Einsteinian Mechanics: $\boxed{\mathbf{v}_0\oplus\mathbf{a}\otimes t \quad \text{or} \quad \mathbf{a}\otimes t\oplus\mathbf{v}_0}$

Relative to the Hyperbolic Dual Metrics

$$\begin{aligned} d_\oplus(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a}\ominus\mathbf{b}\| \\ d_\boxplus(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} \boxminus \mathbf{b}\| \end{aligned} \qquad (23)$$

Einstein gyrovector spaces
form the setting
for
the Beltrami ball model
of
hyperbolic geometry

just as

vector spaces
form the setting
for
the common model
of
Euclidean geometry.

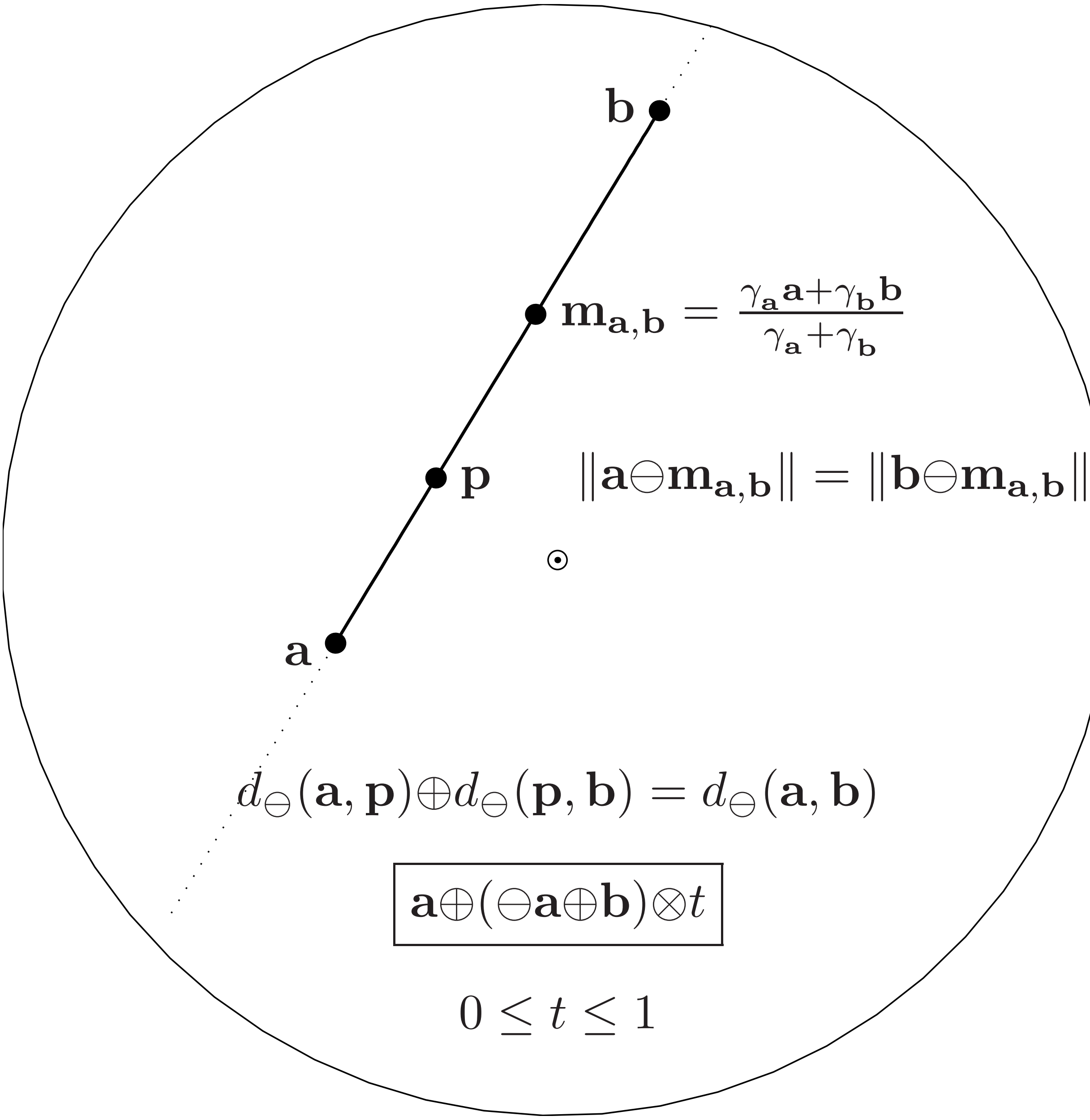


Figure 1: $\oplus = \oplus_{\rm E}$. The gyroline segment linking the two points **a** and **b** in an Einstein gyrovector space $(\mathbb{V}_c, \oplus_{\rm E}, \otimes)$. **p** is a generic point between **a** and **b** and $m_{\mathbf{a},\mathbf{b}}$ is the midpoint of **a** and **b**.

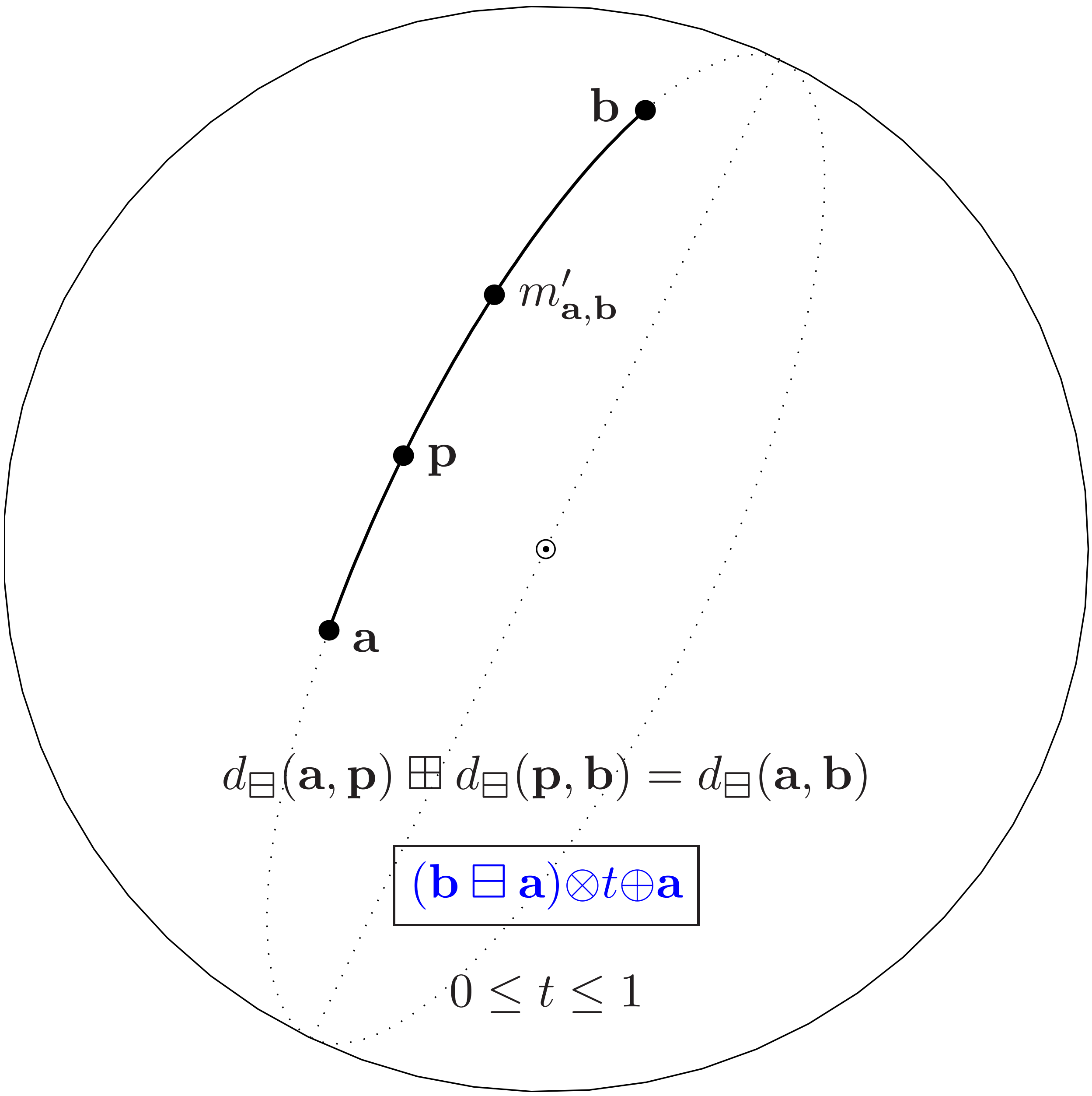


Figure 2: $\oplus = \oplus_{\mathrm{E}}$. The cogyroline segment linking the two points **a** and **b** in an Einstein gyrovector space $(\mathbb{V}_c, \oplus_{\mathrm{E}}, \otimes)$. **p** is a generaic point cobetween **a** and **b** and $m'_{\mathbf{a},\mathbf{b}}$ is the comidpoint of **a** and **b**.

The Hyperbolic Angle

and

The Hyperbolic Dual Angle

$$\boxed{\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|}} \qquad (24)$$

**Definition 6** (The Hyperbolic Dual Angle)**.** *The measure of the hyperbolic dual angle* $\alpha$ *between two geometric dual gyrovectors* $\mathbf{b}\boxminus\mathbf{a}$ *and* $\mathbf{d}\boxminus\mathbf{c}$ *is given by the equation*

$$\boxed{\cos\alpha = \frac{\boxminus\mathbf{a}\boxplus\mathbf{b}}{\|\boxminus\mathbf{a}\boxplus\mathbf{b}\|}\cdot\frac{\boxminus\mathbf{c}\boxplus\mathbf{d}}{\|\boxminus\mathbf{c}\boxplus\mathbf{d}\|}} \qquad (25)$$

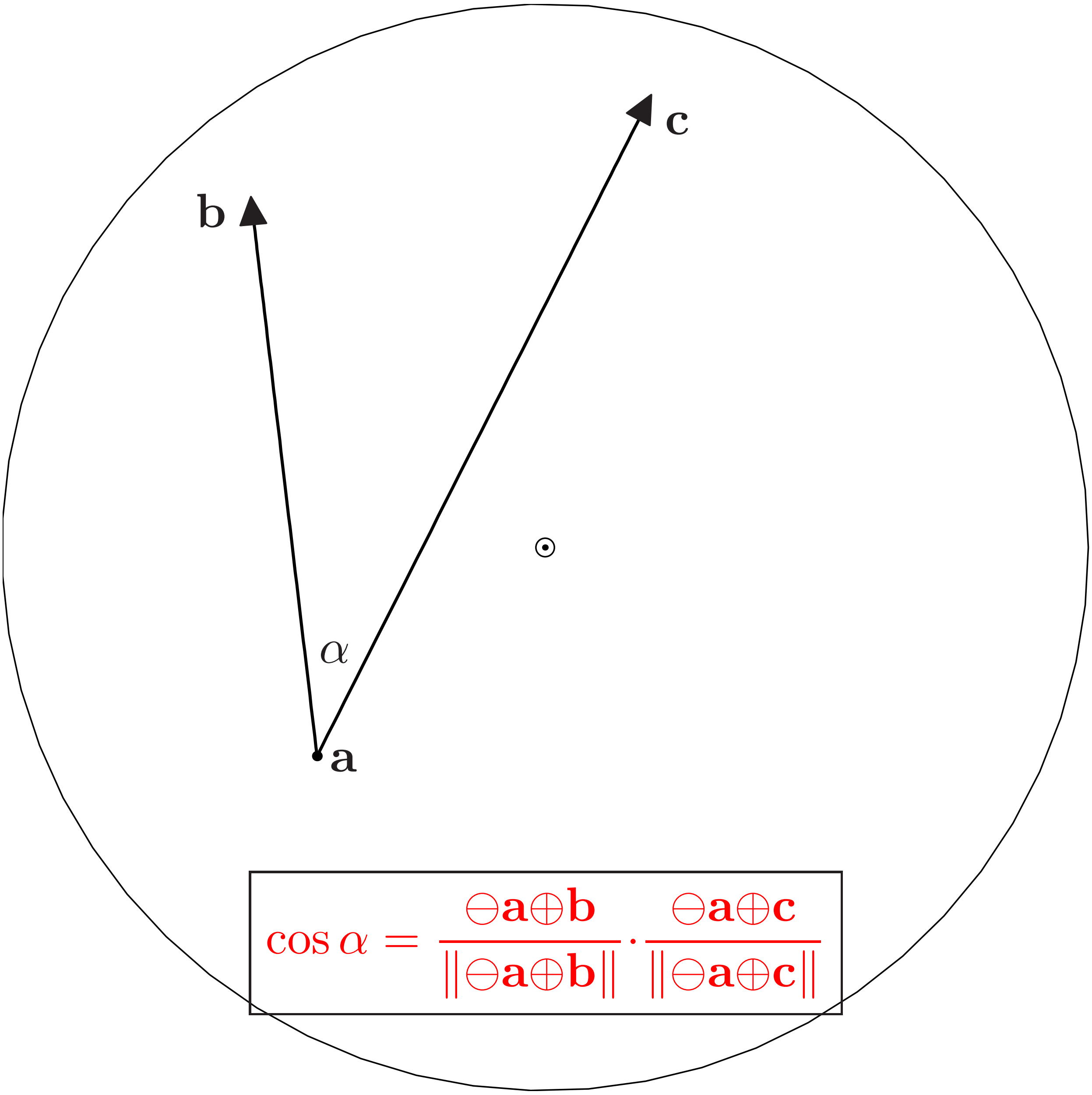


Figure 3: $\oplus = \oplus_{E}$. The Hyperbolic Angle in the Einstein gyrovector plane = The Beltrami disc model of hyperbolic geometry.

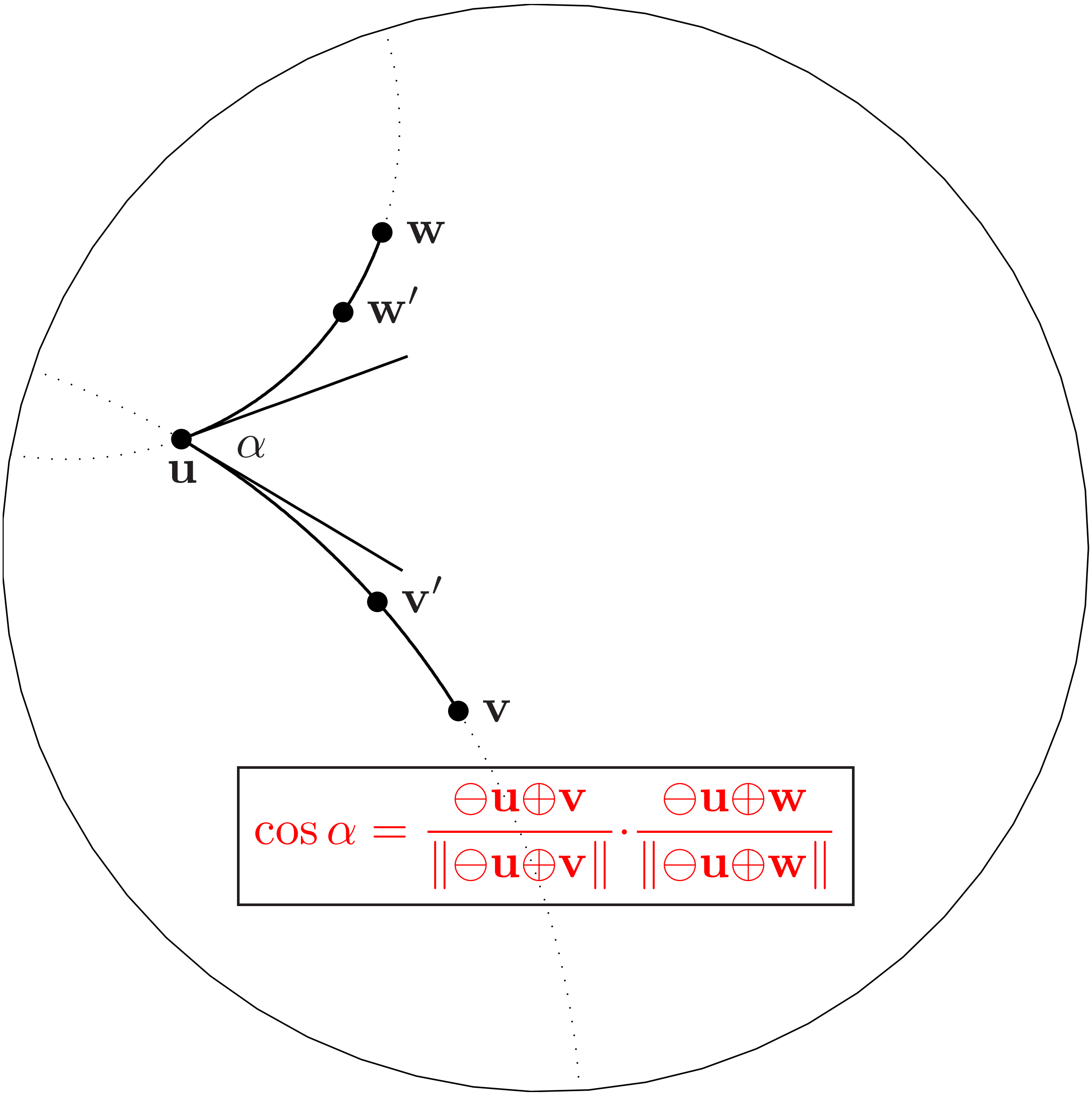


Figure 4: $\oplus = \oplus_{\mathrm{M}}$. A Möbius angle $\alpha$ generated by the two intersecting Möbius geodesic rays. It is coincident with the hyperbolic angle of the Poincaré disc model of hyperbolic geometry.

**Theorem 7** (The Hyperbolic $\pi$ Theorem). *Let* $\Delta\mathbf{abc}$ *be the hyperbolic dual triangle in an Einstein gyrovector space* $(\mathbb{R}_c^n, \oplus, \otimes)$ *whose three vertices are the points* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}_c^n$. *Then, its three hyperbolic dual angles* $\alpha$, $\beta$ *and* $\gamma$, *given by*

$$\begin{aligned} \cos\alpha &= \frac{\mathbf{b} \boxminus \mathbf{a}}{\|\mathbf{b} \boxminus \mathbf{a}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{a}}{\|\mathbf{c} \boxminus \mathbf{a}\|} \\ \cos\beta &= \frac{\mathbf{a} \boxminus \mathbf{b}}{\|\mathbf{a} \boxminus \mathbf{b}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{b}}{\|\mathbf{c} \boxminus \mathbf{b}\|} \\ \cos\gamma &= \frac{\mathbf{a} \boxminus \mathbf{c}}{\|\mathbf{a} \boxminus \mathbf{c}\|} \cdot \frac{\mathbf{b} \boxminus \mathbf{c}}{\|\mathbf{b} \boxminus \mathbf{c}\|} \end{aligned} \tag{26}$$

*satisfy the identity*

$$\boxed{\alpha + \beta + \gamma = \pi} \tag{27}$$

Gyrovector spaces
and
Riemannian Geometry

The Riemannian line element in a gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$ is

$$\boxed{ds^2 = \|(\mathbf{x} + d\mathbf{x}) \ominus \mathbf{x}\|^2} \tag{28}$$

**Example I:** When $\oplus = \oplus_{\mathrm{M}}$, $ds^2$ turns out to be the Riemannian line element

$$\boxed{\begin{aligned} ds^2 &= \|(\mathbf{x} + d\mathbf{x}) \ominus_{\mathrm{M}} \mathbf{x}\|^2 \\ &= \frac{\sum_{i=1}^n dx_i^2}{\left(1 + \frac{1}{4} K \sum_{i=1}^n x_i^2\right)^2} \end{aligned}} \tag{29}$$

with curvature $K = -4/c^2$. For the special case when $n = 2$ this line element turns out to be the one of the Poincaré ball model of hyperbolic geometry.

**Example II:** When $\oplus = \oplus$, $ds^2$ turns out to be the Riemannian line element

$$ds^2 = \|(\mathbf{x} + d\mathbf{x}) \ominus_{\mathrm{E}} \mathbf{x}\|^2 =$$

$$c^2 \frac{\sum_{i=1}^{n} \{c^2 - (r^2 - x_i^2)\} dx_i^2 + 2 \sum_{\substack{i,j=1 \\ i<j}}^{n} x_i x_j dx_i dx_j}{(c^2 - r^2)^2} \tag{30}$$

where $r^2 = \sum_{i=1}^{n} x_i^2$. For the special case when $n = 2$ this line element turns out to be the one of the Beltrami–Klein ball model of hyperbolic geometry.

Summarizing the Structure of
Einstein Addition $\oplus$:

(1) It gives rise to gyrations:

$$\boxed{\mathrm{gyr}[a, b]z = \ominus(a\oplus b)\oplus \{a\oplus(b\oplus z)\}}$$

(2) It is gyrocommutative:

$$\boxed{a \oplus b = \mathrm{gyr}[a, b](b \oplus a)}$$

(3) It is gyroassociative:

$$\boxed{a\oplus(b\oplus z) = (a\oplus b)\oplus \mathrm{gyr}[a, b]z}$$

(4) It gives rise to a commutative coaddition:

$$\boxed{a \boxplus b = a\oplus \mathrm{gyr}[a, \ominus b]b}$$

(5) It gives rise to scalar multiplication $\otimes$.

(6) It gives rise to hyperbolic lines and angles, called "gyrolines" and "gyroangles".

(7) We will now see that it gives rise to hyperbolic vectors as well, called "gyrovectors".

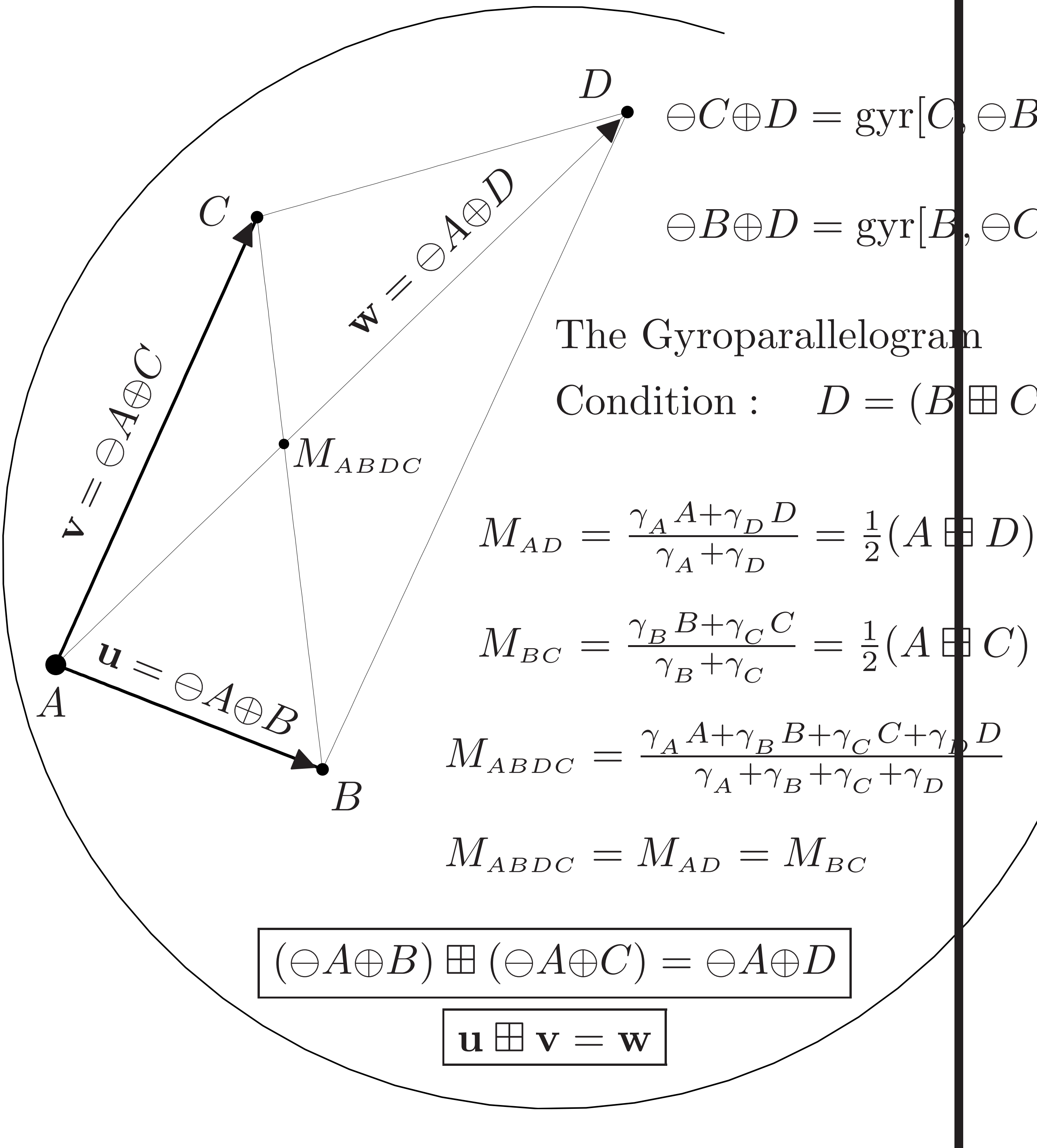


Figure 5: The Einstein Gyroparallelogram Law.

**Definition:** A set $S$ of $N$ points $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}^n$, $n \geq 2$, is *barycentrically independent* if the $N-1$ vectors $-A_1 + A_k$, $k = 2, \ldots, N$, are linearly independent.

**Examples:** The three vertices of a non-degenerate triangle in $\mathbb{R}^n$, $n \geq 2$, are barycentrically independent.

The four vertices of a non-degenerate tetrahedron in $\mathbb{R}^n$, $n \geq 3$, are barycentrically independent.

**Definition:** Let

$$S = \{A_1, \ldots, A_N\} \tag{31}$$

be a barycentrically independent set of $N$ points in $\mathbb{R}^n$. The real numbers $m_1, \ldots, m_N$, satisfying

$$\sum_{k=1}^{N} m_k \neq 0 \tag{32}$$

are **barycentric coordinates** of a point $P \in \mathbb{R}^n$ with respect to the set $S$ if

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{33}$$

Equation (33) is the unique barycentric coordinate representation of $P$ w.r.t the set $S$.

**Mechanical Interpretation:** $P$ is the center of mass (momentum) of a particle system in which the $k$th particle $((k = 1, 2, ..., N))$ has position (velocity) $A_k$ and mass $m_k > 0$.

**Covariance of Barycentric Coordinate Representations:** Let

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \qquad (34)$$

be the barycentric coordinate representation of a point $P \in \mathbb{R}^n$ in a Euclidean $n$-space $\mathbb{R}^n$ with respect to a barycentrically independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}^n$. The barycentric coordinate representation (34) is covariant, that is,

$$X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k} \qquad (35)$$

for all $X \in \mathbb{R}^n$, and

$$RP = \frac{\sum_{k=1}^{N} m_k R A_k}{\sum_{k=1}^{N} m_k} \qquad (36)$$

for all $R \in SO(n)$.

Thus:

Barycentric coordinate representations are covariant w.r.t.

1. translations; and
2. rotations.

Indeed, translations and rotations of $\mathbb{R}^n$ are known:

1. In geometry ($n \geq 2$), as the Euclidean motions; and
2. In mechanics ($n = 3$), as the rigid motions.

**Definition:** Let

$$S = \{A_1, \dots, A_N\} \tag{37}$$

be a barycentrically independent set of $N$ points in the ball $\mathbb{R}^n_s$. The real numbers $m_1, \dots, m_N$, satisfying

$$\sum_{k=1}^{N} m_k \gamma_{A_k} > 0 \tag{38}$$

are **gyrobarycentric coordinates** of a point $P \in \mathbb{R}^n_s$ with respect to the set $S$ if

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{39}$$

**Relativistic Mechanical Interpretation:** $P$ is the center of momentum of a particle system in which the $k$th particle ($k = 1, 2, ..., N$) has velocity $A_k$ and relativistic mass $m_k \gamma_{A_k}$, ($m_k > 0$). **Note that in Relativistic Mechanics mass is velocity dependent.**

**Covariance of Gyrobarycentric Coordinate Representations:**

Let

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{40}$$

be a gyrobarycentric coordinate representation of a point $P \in \mathbb{R}_s^n$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ with respect to a barycentrically independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}_s^n$.

Then

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \tag{41}$$

and

$$RP = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k} RA_k}{\sum_{k=1}^{N} m_k \gamma_{RA_k}} \tag{42}$$

Thus:

Gyrobarycentric coordinate representations are covariant w.r.t.

1. left gyrotranslations; and
2. rotations.

Indeed, left gyrotranslations and rotations of the ball $\mathbb{R}_c^n$ about its origin are known:

1. In geometry ($n \geq 2$), as the motions of hyperbolic geometry; and
2. In relativistic mechanics ($n = 3$), as the rigid motions of relativistically admissible velocities.

The notion of Euclidean barycentric coordinates dates back to Möbius, 1827, when he published his book *Der Barycentrische Calcul* (*The Barycentric Calculus*). The word *barycentric* is derived from the Greek word *barys* (heavy), and refers to center of gravity. Barycentric calculus is a method of treating geometry by considering a point as the center of gravity of certain other points to which weights are ascribed. Hence, in particular, barycentric calculus provides excellent insight into triangle and tetrahedron centers.

In full analogy, gyrobarycentric calculus provides excellent insight into gyrotriangle and gyrotetrahedron centers.

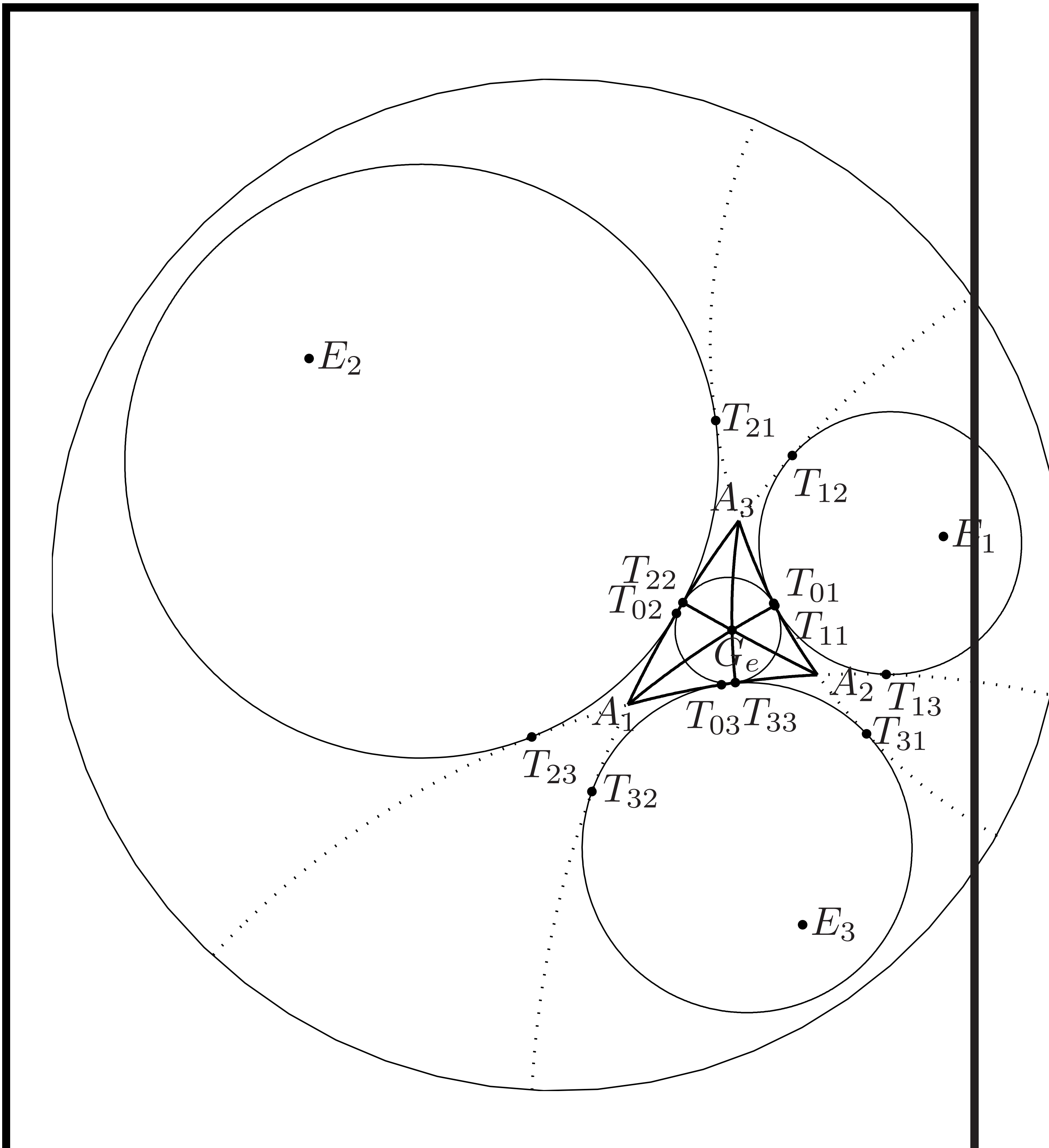


Figure 6: Gergonne Gyropoint

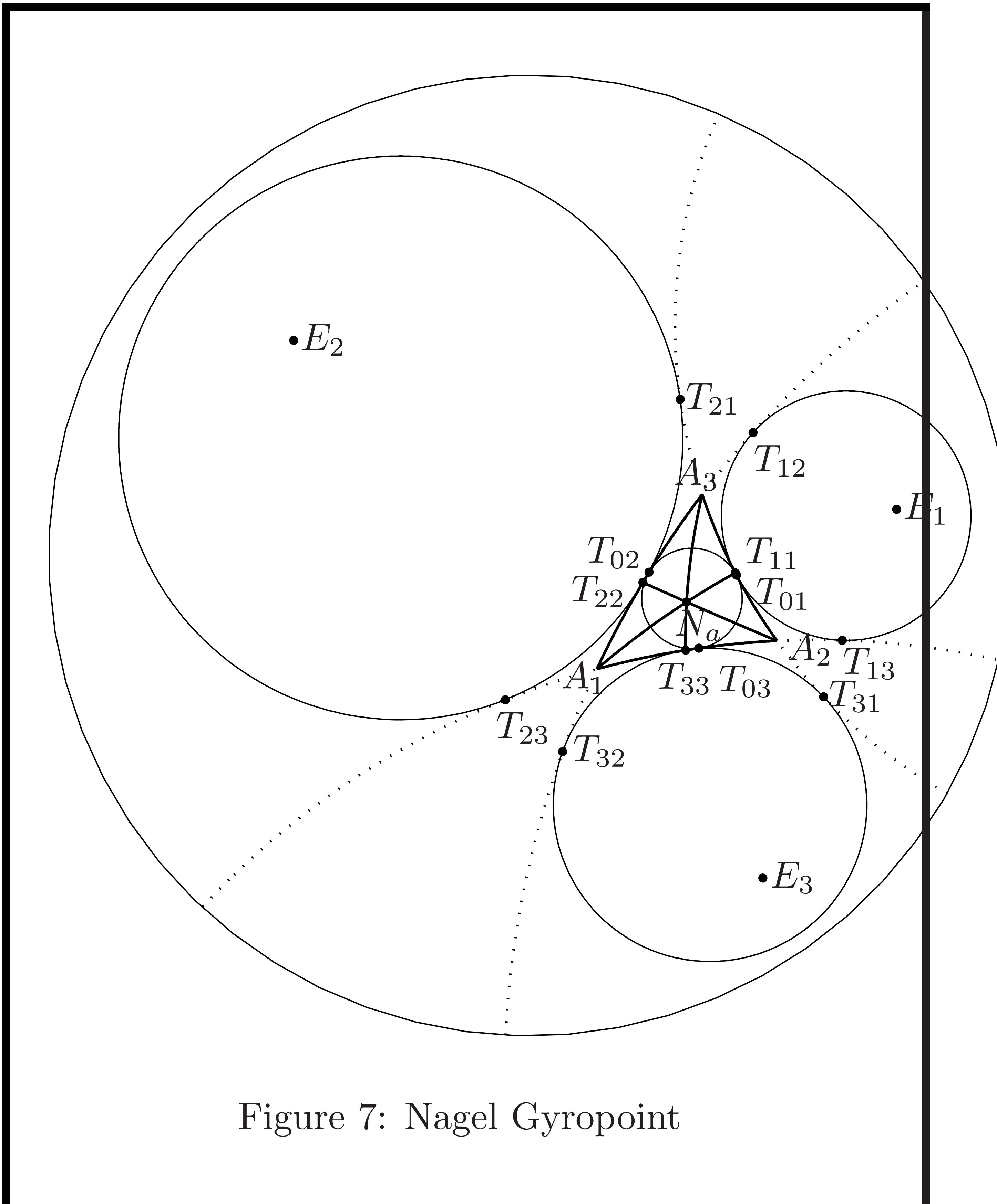


Figure 7: Nagel Gyropoint

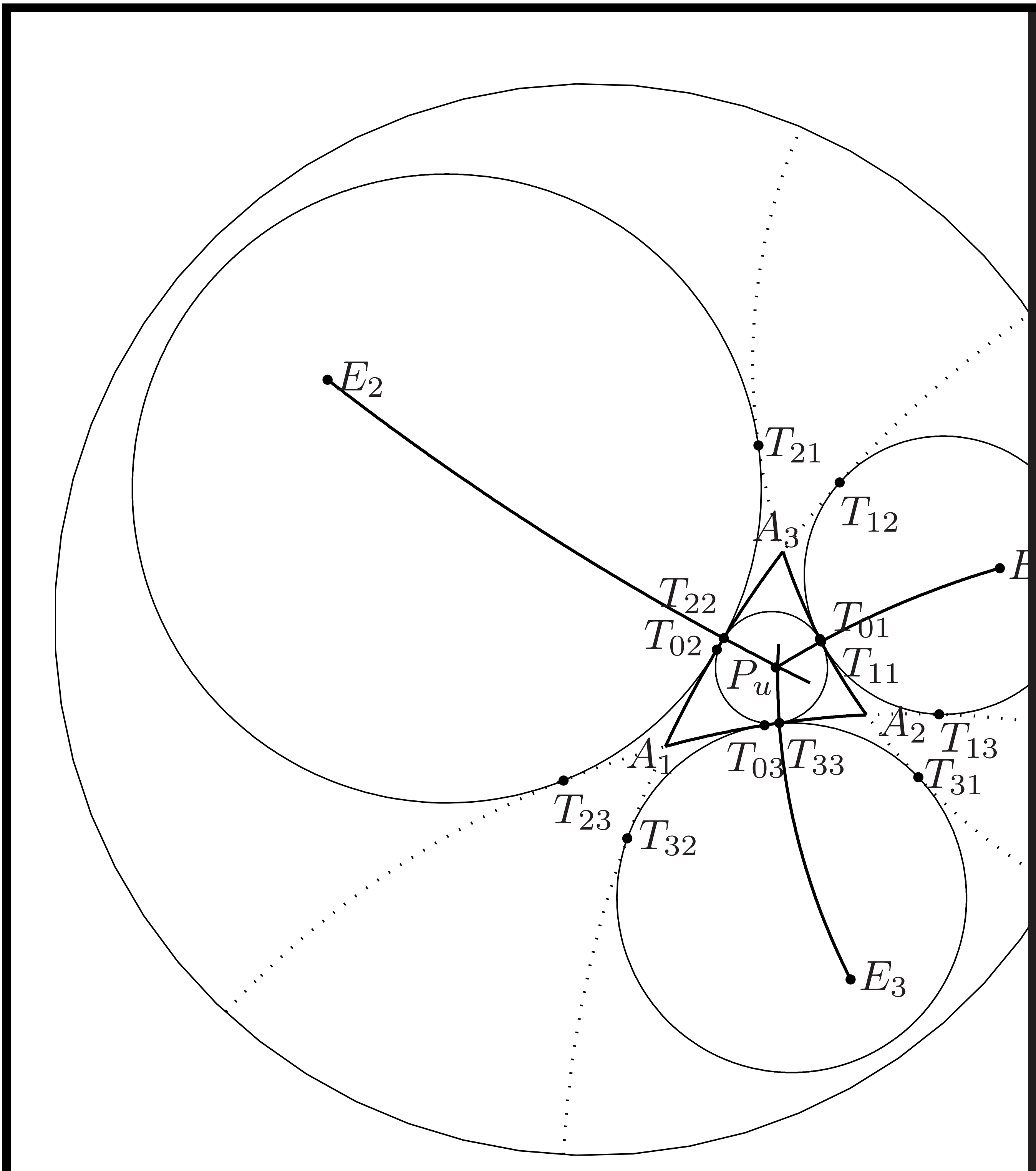


Figure 8: $P_u$ Gyropoint

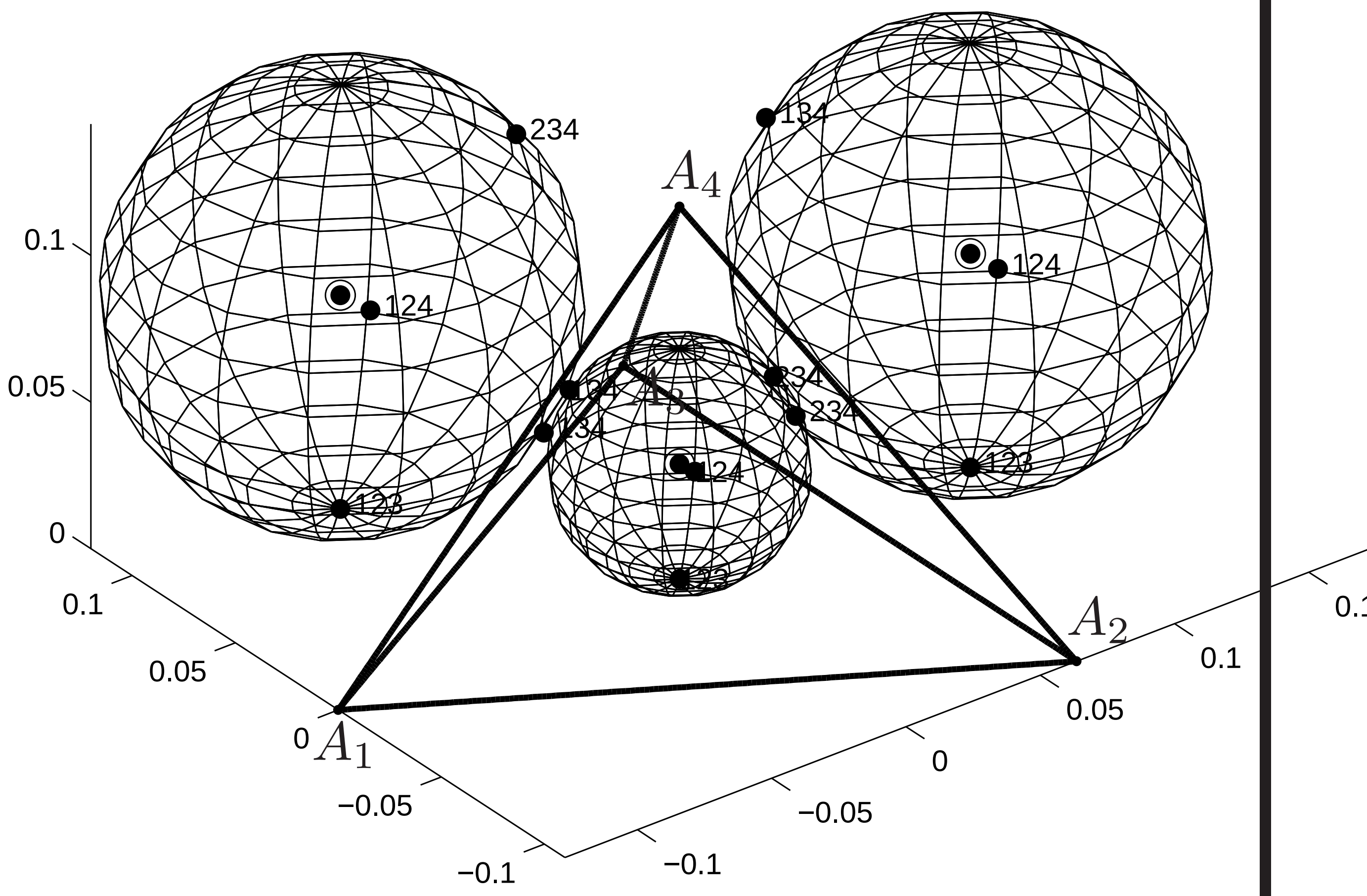


Figure 9: A gyrotetrahedron and its ingyrosphere and two of its seven exgyrospheres in an Einstein gyrovector space.

Euclidean barycentric coordinates
are useful in the
geometry of Quantum states
where
barycentric coordinates are
interpreted as
probabilities

as shown in the book:
Geometry of Quantum States
by I. Bengtsson and K. Zyczkowski
Cambridge (2006).

However,

Euclidean barycentric coordinates
are insensitive to the
Geometric Phases
in Quantum Mechanics.

Conjecture:

Relativistic barycentric coordinates
are useful in the
geometry of relativistic Quantum states
where
relativistic barycentric coordinates are
interpreted as
relativistically corrected probabilities

just as

Euclidean barycentric coordinates
are useful in the
geometry of non-relativistic Quantum states
where
Euclidean barycentric coordinates are
interpreted as
probabilities

Indeed,

Péter Lévay has shown that the notion of
mixed state geometric phase
in quantum mechanics
coincides with the notion of
gyration (= Thomas rotation = Thomas precession)
in relativistic hyperbolic geometry;
see

Péter Lévay, Thomas rotation and the mixed state geometric phase. J. Phys. A. **37**(16), 4593-4605, 2004.

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