On Symmetry Reduction of Some P(1,4)-invariant Differential Equations

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Introduction

The development of theoretical and mathematical physics has required various extensions of the four-dimensional Minkowski space M(1,3) and, correspondingly, various extensions of the Poincaré group P(1,3).

The group P(1,4)

The natural extension of this group is the generalized Poincaré group P(1,4). The group P(1,4) is the group of rotations and translations of the five-dimensional Minkowski space M(1,4).

The group P(1,4)

The group P(1,4) has many applications in theoretical and mathematical physics.

See for example:

- Fushchich W.I., Krivsky I.Yu. // Nucl. Phys. B7. - 1968. - **17**, N 1. - 79-87.
- Fushchych W. // Theor. Math. Phys. – 1970. – Vol. 4. – N 3. – P. 360-367.
- Fushchich W.I. // Lett. Nuovo Cimento. - 1974. - **10**, N 4. - 163-168.
- Kadyshevsky V.G. // Fizika elementar. chastitz. i atomn. yadra. – 1980. – Vol. 11. – N 1. – P. 5-39.
- Fushchych W.I., Nikitin A.G., Symmetry of Equations of Quantum Mechanics, Allerton Press Inc., New York, 1994.

The Lie algebra of the group P(1,4) is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}, \quad \mu, \nu = 0, 1, \ldots, 4$ and $P'_\mu, \quad \mu = 0, 1, \ldots, 4$, satisfying the commutation relations

$$[P'_\mu, P'_\nu] = 0 \qquad [M'_{\mu\nu}, P'_\sigma] = g_{\mu\sigma} P'_\nu - g_{\nu\sigma} P'_\mu$$

$$[M'_{\mu\nu}, M'_{\rho\sigma}] = g_{\mu\rho} M'_{\nu\sigma} + g_{\nu\sigma} M'_{\mu\rho} - g_{\nu\rho} M'_{\mu\sigma} - g_{\mu\sigma} M'_{\nu\rho}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1, g_{\mu\nu} = 0, \; if \mu \neq \nu$. Here and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$

The Lie algebra of the group P(1,4)

Further, we will use the following basis elements:

$$G = M'_{40}, \quad L_1 = M'_{32}, \quad L_2 = -M'_{31}, \quad L_3 = M'_{21},$$

$$P_a = M'_{4a} - M'_{a0}, \quad C_a = M'_{4a} + M'_{a0}, \quad (a = 1, 2, 3),$$

$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \quad X_k = P'_k \quad (k = 1, 2, 3), \quad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

The group P(1,4)

Continuous subgroups of the group P(1,4) have been described in

- **V.M. Fedorchuk**, Ukr. Mat. Zh., **31**, No. 6, 717-722 (1979).
- **V.M. Fedorchuk**, Ukr. Mat. Zh., **33**, No. 5, 696-700 (1981).
- **W.I. Fushchich, A.F. Barannik, L.F. Barannik and V.M. Fedorchuk**, J. Phys. A: Math. Gen., **18**, No.14, 2893-2899 (1985).

The group P(1,4)

The group P(1,4) is the smallest group which contains, as subgroups:

- **the symmetry group of non-relativistic physics (extended Galilei group $\tilde{G}(1,3)$)**
- **The symmetry group of relativistic physics (Poincaré group P(1,3))**

W.I. Fushchych, A.G. Nikitin, *Symmetries of Equations of Quantum Mechanics*, (Allerton Press Inc., New York, 1994).

P(1,4)-invariant equations

Among the P(1,4)-invariant equations in the space M(1,4)×R(u) there is

P(1,4)-invariant equations

$$\Box_5 u = F(u),$$

where

$$u = u(x), \qquad x = (x_0, x_1, x_2, x_3, x_4) \in M(1,4),$$

$$\Box u = u_{00} - u_{11} - u_{22} - u_{33} - u_{44},$$

$$u_{nn} = \frac{\partial^2 u}{\partial x_n^2}, \qquad n = 0, \dots, 4.$$

P(1,4)-invariant equations

To perform the symmetry reduction of the above mentioned equation, we have used functional bases of invariants of nonconjugate subgroups of the group P(1,4).

P(1,4)-invariant equations

However, it turned out that the reduced equations, obtained with the help of nonconjugate subalgebras of the Lie algebra of the group P(1,4) of the given rank, were of different types.

P(1,4)-invariant equations

In my talk I plan to present new interesting facts arising during symmetry reduction of some P(1,4)-invariant differential equations.

1st interesting fact

Let us present the 1st interesting fact.

P(1,4)-invariant equations

Let us consider an Ansatz

$$u(x) = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_4,$$

$$\omega_2 = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2},$$

where ω_1, ω_2 **are invariants of nonconjugate subalgebras of the Lie algebra of the group P(1,4)**

P(1,4)-invariant equations

Reduced equation

$$\varphi_{11} + \varphi_{22} + 2\varphi_2\omega_2^{-1} = -F(\varphi),$$

$$\varphi_i = \frac{\partial\varphi}{\partial\omega_i}, \qquad \varphi_{ik} = \frac{\partial^2\varphi}{\partial\omega_i\omega_k}, \qquad i,k = 1,2$$

is two-dimensional PDE.

P(1,4)-invariant equations

Let us consider an Ansatz

$$u(x) = \varphi(\omega_1, \omega_2),$$

$$\omega_1 = x_0 + x_4,$$

$$\omega_2 = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2},$$

where $\omega_1, \ \omega_2$ **are invariants of nonconjugate subalgebras of the Lie algebra of the group P(1,4)**

P(1,4)-invariant equations

Reduced equation

$$\varphi_{22} + 2\varphi_2 \omega_2^{-1} = -F(\varphi),$$

$$\varphi_i = \frac{\partial \varphi}{\partial \omega_i}, \qquad \varphi_{ik} = \frac{\partial^2 \varphi}{\partial \omega_i \omega_k}, \qquad i, k = 1, 2.$$

is ODE.

P(1,4)-invariant equations

Consequently, instead of the formula

$$\boldsymbol{\rho = n - R}$$

we obtain

$$\boldsymbol{\rho = n - R - 1}$$

$\boldsymbol{n}$ **denotes a number of independent variables of system** $\boldsymbol{(S)}$**,**

$\boldsymbol{\rho}$ **denotes a number of independent variables of system** $\boldsymbol{(S/H)}$**.**

P(1,4)-invariant equations

More details about it can be found in

Fedorchuk V.M., Ukr. Mat. Zh., 1996, **48**, N 4, 573-576.

P(1,n)-invariant equations

It should be noted that Grundland, Harnad, and Winternitz were the first to point out and to try to investigate this fact.

More details about it can be found in

A.M. Grundland, J. Harnad, P. Winternitz, J. Math. Phys., 1984, **25**, N 4, 791-806

P(1,4)-invariant equations

Among the P(1,4)-invariant equations in the space M(1,3)×R(u) there are

P(1,4)-invariant equations

$$\Box u(1 - u_\nu u^\nu) + u_{\mu\nu} u^\mu u^\nu = 0,$$

$$det\left\| u_{\mu\nu} \right\| = 0,$$

$$(u_0)^2 - (u_1)^2 - (u_2)^2 - (u_3)^2 = 1,$$

P(1,4)-invariant equations

where

$$u = u(x), \qquad x = (x_0, x_1, x_2, x_3) \in M(1,3),$$

$$u_\mu = \frac{\partial u}{\partial x_\mu}, \qquad u^\mu = g^{\mu\nu} u_\nu, \qquad u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu x_\nu}.$$

$$\mu, \nu = 0, 1, 2, 3.$$

$\Box$ **is the d'Alembertian.**

2^{nd} interesting fact

Let us present the 2^{nd} interesting fact.

P(1,4)-invariant equations

Let us consider an Ansatz

$$2x_0\omega - \left(x_1^2 + x_2^2 + x_3^2\right) = -\varphi(\omega),$$

$$\omega = x_0 + u,$$

where ω is an invariant of nonconjugate subalgebras of the Lie algebra of the group P(1,4).

P(1,4)-invariant equations

Reduced equations

$$\varphi''\omega^2 - 8\omega\varphi' + 8\varphi - 6\omega^2 = 0,$$

$$\frac{1}{2}\omega^2\varphi'' - \omega\varphi' + \varphi = 0,$$

$$\omega\varphi' - \varphi + \omega^2 = 0,$$

$$\varphi' = \frac{d\varphi}{d\omega}, \qquad \varphi'' = \frac{d^2\varphi}{d\omega^2},$$

respectively.

P(1,4)-invariant equations

More details about it can be found in

- **Fedorchuk V.,** J. Nonlinear Math. Phys., 1995, **2**, N 3-4, 329-333.

3rd interesting fact

Let us present the 3rd interesting fact.

P(1,4)-invariant equations

Let us consider so-called necessary conditions for the exist invariant solutions.

More details about it can be found in

- **L.V. Ovsiannikov**. Group Analysis of Differential Equations, Academic Press, New York, 1982.
- **P.J. Olver.** Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, 1986.

P(1,4)-invariant equations

Some nonconjugate subalgebras of the given rank of the Lie algebra of the group P(1,4) don’t satisfy so-called necessary conditions for the exist invariant solutions.

P(1,4)-invariant equations

It means, that from the invariants of some subgroups of the group P(1,4) we cannot construct Ansatzes which provide the symmetry reduction.

P(1,4)-invariant equations

An example:

$$\langle L_3, X_0 + X_4, X_4 - X_0 \rangle$$

$$x_{3,} \quad (x_1^2 + x_2^2)^{1/2}$$

P(1,4)-invariant equations

It means that using only the rank of those nonconjugate subalgebras, we cannot explain differences in the properties of the reduced equations.

P(1,4)-invariant equations

It is known that the nonconjugate subalgebras of the Lie algebra of the group P(1,4) of the same rank may have different structural properties.

P(1,4)-invariant equations

Therefore, to explain above mentioned interesting facts, we suggest to try to investigate the connections between structural properties of nonconjugate subalgebras of the same rank of the Lie algebra of the group P(1,4) and the properties of the reduced equations corresponding to them.

P(1,4)-invariant equations

In order to realize above mentioned investigation we have to perform the following steps:

1. **Classify low-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4).**
2. **Classify functional bases of invariants those subalgebras.**
3. **Classify of the obtained reduced equations.**

Classification of real Lie algebras

The complete classification of real structures of Lie algebras of dimension less or equal five has been obtained by Mubarakzyanov

- **G. M. Mubarakzyanov,** Izv. Vyssh. Uchebn. Zaved., Ser. Mat., No. 1(32), 114–123 (1963).
- **G. M. Mubarakzyanov,** Izv. Vyssh. Uchebn. Zaved., Ser. Mat., No. 3(34), 99–106 (1963).

Some of the results obtained

Let us present some of the results obtained

Classification of low-dimensional subalgebras

By now, we have classified all low-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) into classes of isomorphic subalgebras.

Classification of low-dimensional subalgebras

The results of the classification can be found in:

- Fedorchuk V.M., Fedorchuk V.I. Proceedings of Institute of Mathematics of NAS of Ukraine, 2006, V.3, N2, 302-308.
- V. M. Fedorchuk, V. I. Fedorchuk, Journal of Mathematical Sciences, 2012, Vol. 181, No. 3, 305 - 319.
- Vasyl Fedorchuk and Volodymyr Fedorchuk, Abstract and Applied Analysis, vol. 2013, Article ID 560178, 16 pages, 2013. doi:10.1155/2013/560178.

Classification of low-dimensional subalgebras

There are 20 one-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4)

Classification of invariants

All of them belong to the one type A_1 (step 1).

Consequently, all invariants of these subalgebras belong to the same type (step 2).

Classification of invariants

Some examples:

$\langle P_3 \rangle$ (A_1)

$$x_1, x_2, x_0 + u,$$

$$(x_0^2 - x_3^2 - u^2)^{1/2}$$

Classification of invariants

Some examples:

$\langle G \rangle$ (A_1)

$$x_{1,}\, x_{2,}\, x_3, (x_0^2 - u^2)^{1/2}$$

Classification of low-dimensional subalgebras

There are 49 two-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4)

Classification of invariants

All of them belong to the two types: 2A_1 and A_2 (step 1).

Consequently, all invariants of these subalgebras belong to the two types, respectively (step 2).

Classification of invariants

Some examples:

$\langle P_1, P_2 \rangle$ $(2A_1)$

$$x_{3,}\, x_0 + u,$$

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2}$$

Classification of invariants

Some examples:

$\langle -G, P_3 \rangle$ (A_2)

$$x_1, x_2, (x_0^2 - x_3^2 - u^2)^{1/2}$$

Classification of low-dimensional subalgebras

There are 94 three-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4)

Classification of invariants

All of them belong to the 10 types: $3A_1$, $A_1 + A_2$, $A_{3,1}$,…, $A_{3,6}$ (step 1).

Consequently, all invariants of these subalgebras belong to 10 types, respectively (step 2).

Classification of invariants

Some examples:

$\langle P_1, P_2, P_3 \rangle$ $(3A_1)$

$$x_{3,}\, x_0 + u,$$

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2}$$

Classification of invariants

Some examples:

$$\langle -G, P_3 \rangle \oplus \langle L_3 \rangle \; (A_2 \oplus A_1)$$

$$x_{3,}, (x_1^2 + x_2^2)^{1/2}, (x_0^2 - u^2)^{1/2}$$

Classification of invariants

Until now, we have classified the functional bases of invariants in the space $M(1,3)\times R(u)$ of one, two, and three-dimensional non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ using the classification of these subalgebras.

Classification of invariants

In other words, we have established a connection between the classification of one, two, and three-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) and their invariants in the space M(1,3)×R(u).

Thank you for your attention!