Visible Actions and Multiplicity-free Representations

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I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
II. INFINITE DIMENSIONAL EXAMPLES
III. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
IV. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

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Plan

I propose

a new method (based on "visible actions") to
prove/find/construct
multiplicity-free representations

for finite/infinite dimensional representations.

References (a) for Varna Lectures

- Overview
  Publ. Res. Inst. Math. Sci. (2005)

- Visible Actions — Classification Theory
  J. Math. Soc. Japan (2007) $\cdots$ GL_n case
  Transformation Groups (2007) $\cdots$ symmetric pairs
  A. Sasaki (IMRN (2009), IMRN (2011), Geometriae Dedicata (2010))
  Y. Tanaka (J. Algebra (2014), J. Math. Soc. Japan (2013), Tohoku J, B. Australian Math. Soc. (2013), etc)

- Multiplicity-free Theorem via Visible Actions
  Progr. Math. (2013) $\cdots$ general theory

References (b) for Varna Lectures

- Application to concrete examples
  Acta Appl. Math. (2004) $\cdots \otimes$ product, GL_n
  Progr. Math. (2008)

- Generalization of Kostant–Schmid formula
  Proc. Rep. Theory, Saga (1997)

- Multiplicity-free Theorems and Orbit Philosophy
  Adv. Math. Sci. (2003), AMS (with Nasrin)
  Nasrin (Geoemtriae Dedicata (2014))

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“Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake.”

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E.g. the Taylor series, the Fourier transform,
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Multiplicity-free property is 'rare' in general.

How to find such a structure systematically ?

New approach — “visible action”

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(Strongly) Visible Action (K– 2004)

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Propagation of MF property from fiber to sections

$\Uparrow$ (Progr. Math 2013)

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§1 Multiplicity-free representations

Ex.1 (Eigenspace decomposition)

$\mathcal{H}$: Vector sp./$\mathbb{C}$, $\dim < \infty$

$A \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$

s.t. all eigenvalues are distinct. ①

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{MF representations}

Taylor series (MF rep.)

Ex.2 (Taylor expansion, Laurent expansion)

$$f(z_1, \ldots, z_n) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}$$

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$^{\exists 1} a_\alpha \in \mathbb{C}$ for each α

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$\Uparrow$

$\dim \mathrm{Hom}_{(\mathbb{C}^\times)^n}(\tau, \mathcal{O}(\{0\}) \leq 1$

($^\forall \tau = \tau_\alpha$: irred. rep. of $(\mathbb{C}^\times)^n$)

i.e. $(\mathbb{C}^\times)^n \curvearrowright \mathcal{O}(\{0\})$ is MF

Fourier series

Ex.3 (Fourier series expansion)

$$L^2(S^1) \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}e^{inx}$$

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$S^1 \curvearrowright L^2(S^1)$ is MF (multiplicity-free)

Peter–Weyl (MF rep.)

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G: compact (Lie) group

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$\Longrightarrow$ $G \times G \curvearrowright L^2(G)$ is MF

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M: compact Riemannian manifold
Δ_M: Laplace–Beltrami operator on M
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(direct sum of eigenspaces)

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Ex.5 $M = S^{n-1}$ (unit sphere)

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<u>Ex.5</u> $M = S^{n-1}$ (unit sphere)

$\Delta_{S^{n-1}} f = \lambda f$, $f \neq 0$

$\Rightarrow$

$\lambda = -l(l + n - 2)$ for some $l \in \mathbb{N}$.

Fourier series $\Longrightarrow$ spherical harmonics

$O(n) \curvearrowright S^{n-1} \subset \mathbb{R}^n$

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$$\mathcal{H}_l := \left\{ f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} f = -l(l+n-2)f \right\}$$

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$O(n)$ acts on both sides; $\mathcal{H}_l$ irred.

$\Downarrow$

$O(n) \curvearrowright L^2(S^{n-1})$ is <u>MF</u>

$G \curvearrowright L^2(G/K)$ is MF (E. Cartan ('29)–I. M. Gelfand ('50))

⊗-product rep.

$$SL_2(\mathbb{C}) \overset{\pi_k}{\curvearrowright} S^k(\mathbb{C}^2) \quad (k = 0, 1, 2, \dots)$$

irred.

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MF

Notation (finite dimensional reps)

$G = GL_n(\mathbb{C})$

Hightest weight

$$\lambda = (\lambda_1 \ldots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$$\Downarrow$$

Irreducible rep.

$$\pi_\lambda^{GL_n} \equiv \pi_\lambda$$

<u>Ex.8</u>

$$\lambda = (k, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright S^k(\mathbb{C}^n)$$

$$\lambda = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright \Lambda^k(\mathbb{C}^n)$$

$\otimes$-product rep. (GL_n-case)

Ex.9 (Pieri's rule)

$$\pi_{(\lambda_1,\ldots,\lambda_n)} \otimes \pi_{(k,0,\ldots,0)} \simeq \bigoplus_{\substack{\mu_1 \geq \lambda_1 \geq \cdots \geq \mu_n \geq \lambda_n \\ \sum(\mu_i - \lambda_i) = k}} \pi_{(\mu_1,\ldots,\mu_n)}$$

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MF as a GL_n-module.

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MF as a GL_n-module.

Ex.10 (counterexample)

$\pi_{(2,1,0)} \otimes \pi_{(2,1,0)}$ is NOT MF as a $GL_3(\mathbb{C})$-module.

$\otimes$-product for GL_3

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$$\begin{aligned} &\pi_{(2,1,0)} \otimes \pi_{(2,1,0)} \\ &\simeq \pi_{(4,2,0)} \oplus \pi_{(4,1,1)} \oplus \pi_{(2,2,0)} \\ &\quad \oplus \underline{2}\pi_{(3,2,1)} \oplus \pi_{(2,2,2)} \end{aligned}$$

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Lecture 2. Various examples of MF representations

MF = multiplicity-free

Plan of Today

- finite-dimensional examples (continued)
- infinite-dimensional examples

Notation (finite dimensional reps)

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When is $\pi_\lambda \otimes \pi_\nu$ MF?

$G = GL_n$

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(Necessary Condition)
If $\pi_\lambda \otimes \pi_\nu$ is MF

$\Rightarrow$?

When is $\pi_\lambda \otimes \pi_\nu$ MF?

$G = GL_n$

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(Necessary Condition)

If $\pi_\lambda \otimes \pi_\nu$ is MF

then at least one of λ or ν is of the form

$$(\underbrace{a, \ldots, a}_{p}, \underbrace{b, \ldots, b}_{n-p}),$$

for some $a \geq b$ and some p

$\otimes$-product rep. (continued)

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n,\ a \geq b,\ p + q = n$$

Ex.11 (Stembridge 2001)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module
iff one of the following holds

1) $\min(a-b, p, q) = 1$ (and ν is any),
2) $\min(a-b, p, q) = 2$ and
★ ν is of the form $\nu = (\underbrace{x \cdots x}_{n_1}\ \underbrace{y \cdots y}_{n_2}\ \underbrace{z \cdots z}_{n_3})$ $(x \geq y \geq z)$,
3) $\min(a-b, p, q) \geq 3$, ★ &
$\min(x-y,\ y-z, n_1, n_2, n_3) = 1$.

$\otimes$-product rep. (continued)

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Geometric interpretation (K–, 2004)

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Geometric interpretation (K–, 2004) $\cdots$ visible action

Restriction $(GL_n \downarrow G_{n_1} \times GL_{n_2} \times GL_{n_3})$

Ex.12 $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$

$\pi^{GL_n}_{(\underbrace{x,\ldots,x}_{n_1},\underbrace{y,\ldots,y}_{n_2},\underbrace{z,\ldots,z}_{n_3})}|_{GL_p \times GL_q}$ is MF

if $\min(p, q) \leq 2$ or

if $\min(n_1, n_2, n_3, x - y, y - z) \leq 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

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(Kostant $n_3 = 0$; Krattenthaler 1998)

Ex.13 $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$ ([5])

$n = n_1 + n_2 + n_3$

$\pi^{GL_n}_{(\underbrace{a,\ldots,a}_{p},\underbrace{b,\ldots,b}_{q})}|_{GL_{n_1} \times GL_{n_2} \times GL_{n_3}}$ is MF

if $\min(n_1, n_2, n_3) \le 1$ or

if $\min(p, q, a - b) \le 2$

"Triunity"

MF results for

Ex.11 $\pi_\lambda \otimes \pi_\nu$

Ex.12 $GL_n \downarrow GL_p \times GL_q \ (p + q = n)$

Ex.13 $GL_n \downarrow GL_{n_1} \times GL_{n_2} \times GL_{n_3} \ (n_1 + n_2 + n_3 = n)$

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can be proved by combinatorial methods (e.g. Littlewood–Richardson rule) but
will be explained by “triunity” of visible actions on flag varieties:

$$\begin{cases} G \curvearrowright (G\times G)/(L\times H) & \text{(diagonal action)} \\ L \curvearrowright G/H \\ H \curvearrowright G/L \end{cases}$$

for $H \subset G \supset L$.

Restriction ($GL_n \downarrow GL_{n-1}$)

Finite dimensional rep.

Ex.14 ($GL_n \downarrow GL_{n-1}$)

$$\pi^{GL_n}_{(\lambda_1,\dots,\lambda_n)}|_{GL_{n-1}} \simeq \bigoplus_{\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq \lambda_n} \pi^{GL_{n-1}}_{(\mu_1,\dots,\mu_{n-1})}$$

$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

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$$GL_n \supset GL_{n-1} \supset GL_{n-2} \supset \cdots \supset GL_1$$

MF MF MF MF

$\Longrightarrow$ Gelfand–Tsetlin basis

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Infinite dimensional version

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Finite dimensional rep.

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$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

Infinite dimensional version

Ex.15 ♣ ref. ($U(p,q) \downarrow U(p-1,q)$)

$^\forall \pi$: irred. unitary rep. of $U(p,q)$ with higest weight

$\Longrightarrow$ restriction $\pi|_{U(p-1,q)}$ is MF as a $U(p-1,q)$-module

GL–GL duality

$$N = mn$$

Ex.16 (GL–GL duality à la R. Howe)

$\Longrightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \simeq S(M(m,n;\mathbb{C}))$

This rep. is MF

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$\Downarrow$ generalization 1

Hidden symmetry $\Longleftrightarrow$ Broken symmetry

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⇓ generalization 1

Hidden symmetry ⟺ Broken symmetry

Ex.17 (Progress in Math. 2008)

Branching law of holomorphic discrete series rep. with respect to symmectric pair

Hua–Kostant–Schmid, K–

finite dim, ∞ dim

compact subgp, non-compact subgp

$$U(m,n) \downarrow U(m) \times U(n)$$

$$U(m,n) \downarrow U(m_1,n_1) \times U(m_2,n_2)$$

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$\Downarrow$ generalization 2

MF space: $G \curvearrowright X \Longrightarrow G \curvearrowright \mathcal{O}(X)$

function on X

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Ex.18 (Kac's MF space '80)

$S(\mathbb{C}^N)$ is still MF as a $GL_{m-1} \times GL_n$ module

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Ex.18 (Kac's MF space '80)
$S(\mathbb{C}^N)$ is still MF as a $GL_{m-1} \times GL_n$ module

Ex.19 (counterexample)
$S(\mathbb{C}^N)$ is no more MF as a $GL_{m-1} \times GL_{n-1}$ module

MF for unitary rep (definition)

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<u>Observation</u>

$n \leq 1 \iff \mathrm{End}(\mathbb{C}^n)$ is commutative.

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$(\pi, \mathcal{H})$: unitary rep. of G $\Downarrow$ (Schur's lemma)

<u>Def.</u>

$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

<u>Def.</u> $\mathrm{End}_{\mathbb{C}}(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H}$ continuous linear maps$\}$

$\cup$

$\mathrm{End}_G(\mathcal{H}) := \{T \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}) : T \circ \pi(g) = \pi(g) \circ T, \ {}^{\forall} g \in G\}$

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Def.
$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

Recall:

Def. (naive) $(\pi, \mathcal{H})$ is MF
multiplicity-free
if $\dim \mathrm{Hom}_G(\tau, \pi) \leq 1$ ($\forall \tau$: irred. rep. of G).

MF for unitary rep (definition)

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Def.

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$\Downarrow$

Prop. The irreducible decomp. of π is unique, and $m_\pi(\tau) \leq 1$ for almost every τ with respect to $d\mu$.

In particular, multiplicity for any discrete spectrum ≤ 1

$$\pi \simeq \int_{\widehat{G}} m_\pi(\tau)\, \tau\, d\mu(\tau) \quad \text{(direct integral)}$$

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Def.
$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

$(\pi, \mathcal{U})$: continuous rep.

Def. We say $(\pi, \mathcal{U})$ is (unitarily) MF
if, for any unitary rep. $(\varpi, \mathcal{H})$ s.t.
there exists an injective continuous G-map $\mathcal{H} \hookrightarrow \mathcal{U}$,
$(\varpi, \mathcal{H})$ is MF.

Fourier transform (MF rep.)

<u>Ex.21</u> (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C} e^{i\zeta x} d\zeta$$

(direct integral of Hilbert spaces)

$$f(x) = \int_{\mathbb{R}} f(\zeta) e^{i\zeta x} d\zeta$$

Regular rep. of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $f(*) \to f(* - c)$

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$\mathrm{End}_{\mathbb{R}}(L^2(\mathbb{R})) \simeq L^\infty(\mathbb{R})$ (ring of multiplier operators)

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$\Longrightarrow$ unitary rep. $\mathbb{R} \curvearrowright L^2(\mathbb{R})$ is MF

continuous spectrum

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continuous rep. $\mathbb{R} \curvearrowright \mathcal{S}'(\mathbb{R})$ is also MF

I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
II. INFINITE DIMENSIONAL EXAMPLES
III. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
IV. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

Varna Lectures on visible actions and MF representations

I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
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MF = multiplicity-free (definition)

$$\pi : \underset{\text{group}}{G} \to GL_{\mathbb{C}}(\mathcal{H})$$

Def. (naive) $(\pi, \mathcal{H})$ is MF
multiplicity-free
if $\dim \operatorname{Hom}_G(\tau, \pi) \leq 1$ ($\forall \tau$: irred. rep. of G).

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$\Longrightarrow$ unitary rep. $\mathbb{R} \curvearrowright L^2(\mathbb{R})$ is MF
continuous spectrum

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \underbrace{\int^{\oplus}_{\sum \lambda_i = 0,\ \lambda_1 \geq \cdots \geq \lambda_n} \mathcal{H}_\lambda \quad \underbrace{d\lambda}_{\text{cont. spec.}}}_{\text{MF}}$$

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

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cont. spec.

MF

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

$$\begin{aligned} \mathrm{End}_G(L^2(G/K)) &\simeq L^\infty((\mathbb{R}^n/\mathbb{R}))^{\mathcal{S}_n} \\ &\simeq L^\infty(\mathbb{R}^{n-1}/\mathcal{S}_n) \end{aligned}$$

(ring of multiplier operators)

Plancherel formula for Riemannian symm. space G/K

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cont. spec.

MF

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

MF still holds for vector bundle case of 'small' fibers,

$\mathcal{V} := G \times_K \wedge^k(\mathbb{C}^n) \to G/K \quad (0 \leq k \leq n)$,

associated to the $SO(n)$-representation on the exterior power $\Lambda^k(\mathbb{C}^n)$, but no other cases (Deitmar, K– 2005)

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \int^{\oplus}_{\sum \lambda_i = 0,\ \lambda_1 \geq \cdots \geq \lambda_n} \mathcal{H}_\lambda \quad d\lambda$$

cont. spec.

MF

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

MF still holds under certain deformation of G-regular representation of $L^2(G/K)$

deformation coming from hidden symmetry.
E.g. Gelfand–Vershik's canonical rep of $SL_2(\mathbb{R})$.

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \int^{\oplus}_{\sum \lambda_i = 0,\ \lambda_1 \geq \cdots \geq \lambda_n} \mathcal{H}_\lambda \quad d\lambda$$

cont. spec.

MF

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

Other real forms of $SL(n,\mathbb{C})/SO(n,\mathbb{C})$:

Ex.23 (T. Oshima, Delorme) $G/H = SL(n,\mathbb{R})/SO(p, n-p)$

Multiplicity of most cont. spec. in $L^2(G/H)$

$$= \frac{n!}{p!(n-p)!} > 1 \text{ if } 0 < p < n.$$

$\Longrightarrow$ NOT MF

Broken symmetry and hidden symmetry

$$G \subset \widetilde{G} \quad \underset{\text{unitary rep}}{\xrightarrow{\pi}} \quad GL_{\mathbb{C}}(\mathcal{H})$$

Broken symmetry and hidden symmetry

Hidden symmetry

$$G \subset \widetilde{G} \quad \xrightarrow[\text{unitary rep}]{\pi} \quad GL_{\mathbb{C}}(\mathcal{H})$$

Broken symmetry and hidden symmetry

Hidden symmetry

$$G \subset \widetilde{G} \quad \underset{\text{unitary rep}}{\xrightarrow{\pi}} \quad GL_{\mathbb{C}}(\mathcal{H})$$

Broken symmetry

Broken symmetry and hidden symmetry

Hidden symmetry

$$G \subset \widetilde{G} \xrightarrow[\text{unitary rep}]{\pi} GL_{\mathbb{C}}(\mathcal{H})$$

Broken symmetry

Branching law
= description of broken symmetry

Deformation of $G \curvearrowright L^2(G/K)$

$$G \overset{\pi}{\curvearrowright} L^2(G/K)$$

Deformation of $G \curvearrowright L^2(G/K)$

$$G \overset{\pi}{\curvearrowright} L^2(G/K)$$

$$\cap$$

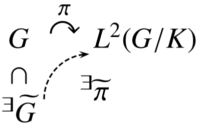
$${}^{\exists}\widetilde{G} \overset{{}^{\exists}\widetilde{\pi}}{\dashrightarrow} L^2(G/K)$$

Deformation of $G \curvearrowright L^2(G/K)$

$$\begin{array}{ccc} G & \overset{\pi}{\curvearrowright} & L^2(G/K) \\ \cap & \nearrow & {}^{\exists}\widetilde{\pi} \\ {}^{\exists}\widetilde{G} & & \end{array}$$

<u>Prop</u> For any classical reductive G, there exist
$\widetilde{G}$ ($\supsetneqq G$) and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

Deformation of $G \curvearrowright L^2(G/K)$



<u>Prop</u> For any classical reductive G, there exist $\widetilde{G}$ ($\supsetneqq G$) and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

<u>E.g.</u> $G = GL(n,\mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n,\mathbb{R})$

Deformation of $G \curvearrowright L^2(G/K)$

$$\begin{array}{ccc} G & \overset{\pi}{\curvearrowright} & L^2(G/K) \\ \cap & \nearrow & {}^{\exists}\widetilde{\pi} \\ {}^{\exists}\widetilde{G} & & \end{array}$$

> Prop For any classical reductive G, there exist
> $\widetilde{G}$ ($\supsetneqq G$) and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

> E.g. $G = GL(n, \mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n, \mathbb{R})$

$\widetilde{\pi}$ lies in a continuous family $\{\widetilde{\pi}_\lambda\}$ of irred unitary reps of $\widetilde{G}$

$\Longrightarrow$ $\pi_\lambda := \widetilde{\pi}_\lambda|_G$ is a continuous family of (non-irreducible) representations of G

$\Longrightarrow$ deformation of $G \curvearrowright L^2(G/K)$

Deformation of $G \curvearrowright L^2(G/K)$

$$G \overset{\pi}{\curvearrowright} L^2(G/K)$$
$$\cap \qquad {}^{\exists}\widetilde{\pi}$$
$${}^{\exists}\widetilde{G}$$

Prop For any classical reductive G, there exist $\widetilde{G}$ ($\supsetneqq G$) and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

E.g. $G = GL(n, \mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n, \mathbb{R})$

$\widetilde{\pi}$ lies in a continuous family $\{\widetilde{\pi}_\lambda\}$ of irred unitary reps of $\widetilde{G}$
$\Longrightarrow$ $\pi_\lambda := \widetilde{\pi}_\lambda|_G$ is a continuous family of (non-irreducible) representations of G
$\Longrightarrow$ deformation of $G \curvearrowright L^2(G/K)$ (still MF)
Sometimes discrete spectrum may appear!

Plan

Lectures 1 and 2
Various examples of
MF representations

MF = multiplicity-free

Existing methods to prove MF

Various techniques have been used in proving various MF results, in particular, for finite dim'l reps

For example, one may

1. look for an open orbit of a Borel subgroup.
2. apply Littlewood–Richardson rules and variants.
3. use computational combinatorics.
4. employ the Gelfand trick (the commutativity of the Hecke algebra).
5. apply Schur–Weyl duality and Howe duality.

New approach to prove/construct MF reps

Plan:

To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

Plan of Lecture 4

Lectures 1 and 2
Various examples of
MF representations

Plan of Lecture 4

Lectures 1 and 2	Lecture 3
Various examples of MF representations	Visible actions on complex manifolds

Plan of Lecture 4

<u>Lectures 1 and 2</u>
Various examples of
MF representations

<u>Lecture 3</u>
Visible actions on
complex manifolds

⇘ ⇙

<u>Lecture 4</u>
Propagation of MF

Representation on holomorphic sections

$\mathcal{L}$ holomorphic line bundle

$\downarrow p$

D complex manifold

Representation on holomorphic sections

$\mathcal{L}$ holomorphic line bundle
$\downarrow p$
D complex manifold

$$\mathcal{O}(D, \mathcal{L}) := \{s : D \to \mathcal{L} \quad \text{holomorphic} : p \circ s = \mathrm{id}_D\}.$$

Representation on holomorphic sections

$H \curvearrowright \begin{array}{c} \mathcal{L} \\ \downarrow p \\ D \end{array}$ equivariant holomorphic line bundle

$$\mathcal{O}(D, \mathcal{L}) := \{s : D \to \mathcal{L} \quad \text{holomorphic} : p \circ s = \mathrm{id}_D\}.$$

Example $\quad \mathcal{L} = D \times \mathbb{C} \quad$ (trivial line bundle)

$\mathcal{O}(D, \mathcal{L}) \simeq \mathcal{O}(D) \quad$ (= {holomorphic functions})

Representation on holomorphic sections

$H \curvearrowright \boxed{\begin{array}{c} \mathcal{L} \\ \downarrow p \\ D \end{array}}$ equivariant holomorphic line bundle

$$\mathcal{O}(D, \mathcal{L}) := \{s : D \to \mathcal{L} \quad \text{holomorphic} : p \circ s = \mathrm{id}_D\}.$$

If $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle, then we get a rep of H on $\mathcal{O}(D, \mathcal{L})$ by

$$s \mapsto g \cdot s(g^{-1} \cdot) \quad (g \in H)$$

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \rightarrow D$ is an H-equivariant holomorphic line bundle.

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then
any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Definition "unitary subrep"
$\iff \underset{\text{Hilbert space}}{\mathcal{H}} \subset \mathcal{O}(D, \mathcal{L})$ such that every $g \in H$ leaves $\mathcal{H}$ invariant and acts as a unitary operator of $\mathcal{H}$.

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Example (Borel–Weil)
D = flag variety H/T (T : maximal torus of compact H)
$\rightsquigarrow$ Any irred rep of H is given as $H \curvearrowright D$ on $\mathcal{O}(D, \mathcal{L})$ for some H-equivariant line bundle $\mathcal{L} \to D$.

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then
any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Theorem 2 If $H \curvearrowright D$ strongly visibly, then
any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is MF.

G/K Hermitian symm. space

<u>Ex.22</u>

$$G = SL(2,\mathbb{R})$$
$$K = SO(2)$$
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$
$$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$$

G/K Hermitian symm. space

<u>Ex.22</u>

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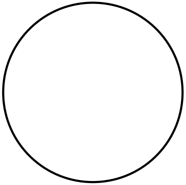
$$G/K \simeq \{ z \in \mathbb{C} : |z| < 1 \}$$

$K \curvearrowright G/K$



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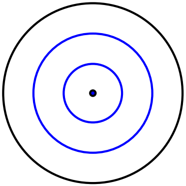

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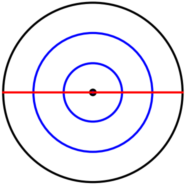

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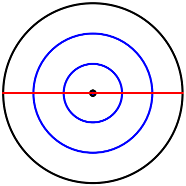

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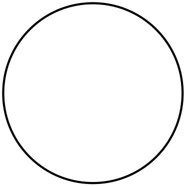

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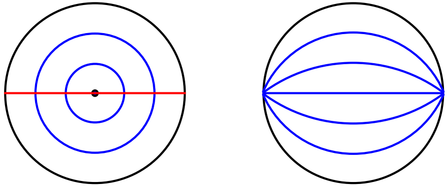

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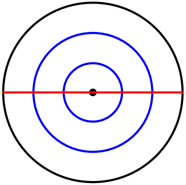

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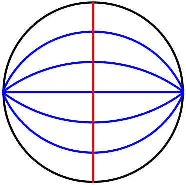

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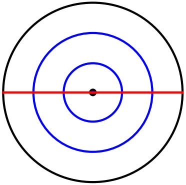

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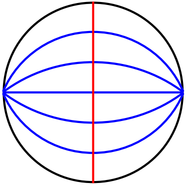

H-orbits

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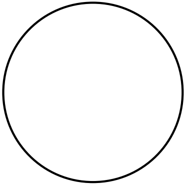

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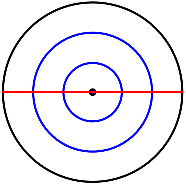

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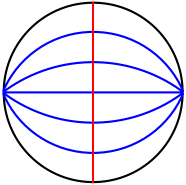

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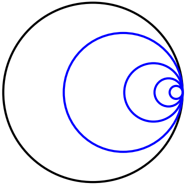

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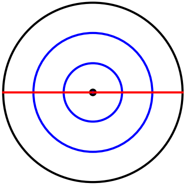

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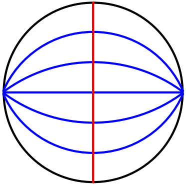

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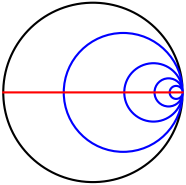

N-orbits

Visible actions on symmetric spaces

Theorem (Transf. Groups (2007))

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

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$\Downarrow$ Theorem 2

Ex.19 π_λ, π_μ highest wt. modules of scalar type

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$\Longrightarrow \pi_\lambda|_H$ is MF

Finite dimensional case

Also, for finite dimensional case

$\Downarrow$ Theorem 2

Eg.23 (Okada, '98, rectangular shaped rep)

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{n-p}) \in \mathbb{Z}^n, \ a \geq b$$

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$$\mathfrak{h}_{\mathbb{C}} = \begin{cases} \mathfrak{gl}(k, \mathbb{C}) + \mathfrak{gl}(n-k, \mathbb{C}) & (1 \leq k \leq n) \\ \mathfrak{o}(n, \mathbb{C}) & \\ \mathfrak{sp}(\frac{n}{2}, \mathbb{C}) & (n : \text{even}) \end{cases}$$

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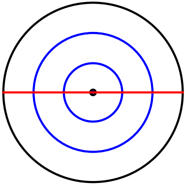

K-orbits

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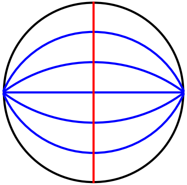

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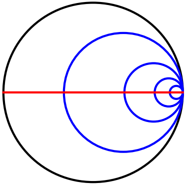

N-orbits

Non-reductive example

<u>Theorem</u> $N \subset G \supset K$

Assume $\begin{cases} G/K & \text{Hermitian symm. of non-cpt. type} \\ N & \text{max. unipotent subgp.} \end{cases}$

$\Longrightarrow N \curvearrowright G/K$ (strongly) visible

$\Downarrow$ Theorem 2

<u>Ex.24</u> π_λ: highest wt. module of scalar type

$\Longrightarrow \pi_\lambda|_N$ is MF

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Theorem 2 If $H \curvearrowright D$ strongly visibly, then any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is MF.

From line bundles to vector bundles

Observation

Properties of one-dimensional representations

- they are always irreducible
- they are always multiplicity-free

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Properties of one-dimensional representations

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We interpret these as the properties for the fiber in the line bundle case.

Propagation Theorem

$$\boxed{\begin{array}{c}\mathcal{V}\\ \downarrow \\ D\end{array}}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\boxed{\begin{matrix}\mathcal{V}\\ \downarrow \\ D\end{matrix}} \rightsquigarrow \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\begin{array}{ccc} \mathcal{V}_x & \subset & \mathcal{V} \\ \downarrow & & \downarrow \\ \{x\} & \subset & D \end{array} \quad \rightsquigarrow \quad \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$$

H-equivariant holomorphic vector bundle

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$$\begin{array}{ccccc} \boxed{H_x \curvearrowright \mathcal{V}_x} & \subset & \mathcal{V} & & \\ \downarrow & & \downarrow & \rightsquigarrow & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})} \\ \{x\} & \subset & D & & \end{array}$$

H-equivariant holomorphic vector bundle

$$H_x = \{h \in H : h \cdot x = x\}$$

Propagation Theorem

$$\begin{array}{ccc} H_x \curvearrowright \mathcal{V}_x & \subset & \mathcal{V} \\ \downarrow & & \downarrow \\ \{x\} & \subset & D \end{array} \rightsquigarrow H \curvearrowright \mathcal{O}(D, \mathcal{V})$$

H-equivariant holomorphic vector bundle

Properties of $H_x \curvearrowright \mathcal{V}_x$ "easy" $\overset{\text{Propagation ?}}{\Longrightarrow}$ Properties of $H \curvearrowright \mathcal{O}(D, \mathcal{V})$ "difficult"

Propagation Theorem

$$\begin{array}{ccccc} H_x \curvearrowright \mathcal{V}_x & \subset & \mathcal{V} & & \\ \downarrow & & \downarrow & \rightsquigarrow & H \curvearrowright \mathcal{O}(D, \mathcal{V}) \\ \{x\} & \subset & D & & \end{array}$$

H-equivariant holomorphic vector bundle

Properties of $H_x \curvearrowright \mathcal{V}_x$ (fiber) $\overset{\text{Propagation ?}}{\Longrightarrow}$ Properties of $H \curvearrowright \mathcal{O}(D, \mathcal{V})$ (sections)

$\Uparrow$

Geometry of the base space D

Propagation Theorem

$$\begin{array}{ccccc} H_x \curvearrowright \mathcal{V}_x & \subset & \mathcal{V} & & \\ \downarrow & & \downarrow & \rightsquigarrow & H \curvearrowright \mathcal{O}(D, \mathcal{V}) \\ \{x\} & \subset & D & & \end{array}$$

H-equivariant holomorphic vector bundle

Theorem 1′

Irreducibility $H_x \curvearrowright \mathcal{V}_x$ (fiber) $\xrightarrow{\text{Propagation}}$ Irreducibility $H \curvearrowright \mathcal{O}(D, \mathcal{V})$ (sections)

$\Uparrow$

H acts transitively on the base space D

Propagation Theorem

$$\begin{array}{ccccc} \boxed{H_x \curvearrowright \mathcal{V}_x} & \subset & \mathcal{V} & & \\ \downarrow & & \downarrow & \rightsquigarrow & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})} \\ \{x\} & \subset & D & & \end{array}$$

H-equivariant holomorphic vector bundle

Theorem 2′ (Progress in Mathematics, 2013)

MF $H_x \curvearrowright \mathcal{V}_x$ (fiber) $\xrightarrow{\text{Propagation}}$ MF $H \curvearrowright \mathcal{O}(D, \mathcal{V})$ (sections)

$\Uparrow$

H acts strong visibly on the base space D

Automorphism of group action

$$H \curvearrowright D$$

Automorphism of group action

$$H \curvearrowright D$$

Lie group — manifold

Def $\sigma \in \mathrm{Aut}(H;D)$

$$\iff \begin{cases} \sigma \curvearrowright H & \text{automorphism of Lie group} \\ \sigma \curvearrowright D & \text{diffeomorphism} \\ \sigma(g \cdot x) = \sigma(g) \cdot \sigma(x) & ({}^\forall g \in H,\ {}^\forall x \in D) \end{cases}$$

Automorphism of group action

$$\underset{\text{Lie group}}{H} \curvearrowright \underset{\text{manifold}}{D}$$

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$\Longrightarrow \sigma$ preserves every H-orbit

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Write simply $\sigma \curvearrowright D$ instead of $\sigma \in \mathrm{Aut}(H;D)$

Assumptions of MF theorem

$\mathcal{V} \to D$: H-equivariant

<u>Assumption 1</u> $\quad {}^{\exists}\sigma \curvearrowright D$ anti-holomorphic s.t.
σ preserves every H-orbit.

Assumptions of MF theorem

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$$\begin{array}{ccccc} \mathcal{V} & \supset & \mathcal{V}_x & \simeq & \mathcal{V}_x^{(1)} \oplus \cdots \oplus \mathcal{V}_x^{(m)} \\ \downarrow & & \downarrow & & \text{MF as an } H_x \text{ module} \\ D & \ni & x & & \end{array}$$

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<u>Assumption 2</u> $\quad \sigma$ lifts to an anti-holomorphic bundle map of $\mathcal{V} \to D$ s.t.
$\sigma(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ $({}^{\forall} i)$ if $\sigma(x) = x$

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Note: σ permutes $\mathcal{V}_x^{(1)}, \ldots, \mathcal{V}_x^{(m)}$ if $\sigma(x) = x$.

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Note: Assumption 2 is automatic for line bundles

Propagation of MF property

Progr. Math (2013)

H: Lie group

H-equiv. holo. vector b'dle:
$\mathcal{V} \to D$

$\Downarrow$

$H \curvearrowright \mathcal{O}(D, \mathcal{V})$ = {holo. sections}

Theorem 2′ (Propagation theorem)

$H_x \curvearrowright \mathcal{V}_x$ MF $(\forall x \in D)$

$\Longrightarrow H \curvearrowright \mathcal{O}(D, \mathcal{V})$ MF

if assumptions 1 & 2 hold.

Observations of MF theorem

Points: H has infinitely many orbits on D

- propagation of MF property

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Points: H has infinitely many orbits on D

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$$\begin{array}{ccc} & \curvearrowright & \mathcal{V} \\ H & & \downarrow \\ & \curvearrowright & D \end{array}$$

fiber MF

Observations of MF theorem

Points: H has infinitely many orbits on D

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$$H \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \begin{array}{c} \mathcal{V} \\ \downarrow \\ D \end{array} \quad \Longrightarrow$$

fiber MF

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fiber MF $\qquad\qquad$ sections MF

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fiber MF $\qquad$ sections MF

$\Uparrow$

geometry of base space
. . . '(strongly) visible action'

Examples of MF theorem

Ex.20 $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C}) \simeq \mathbb{C}^{mn}$

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$\sigma(z) := \bar{z} \quad \Longrightarrow \quad$ Assumption 1 O.K.

Examples of MF theorem

Ex.20 $H = U(m) \times U(n)$

$D = M(m,n;\mathbb{C}) \simeq \mathbb{C}^{mn}$

$\sigma(z) := \overline{z}$ $\Longrightarrow$ Assumption 1 O.K.

$\mathcal{V} = D \times \mathbb{C}$ $\Longrightarrow$ Assumption 2 O.K.

Examples of MF theorem

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$\Downarrow$ Propagation theorem

$H \curvearrowright$ **Pol**(D) MF

$$GL_m \times GL_n \curvearrowright \textbf{Pol}(M(m, n; \mathbb{C}))$$

Totally real submanifold

(D, J) complex manifold

$$J_x : T_xD \to T_xD, \quad J_x^2 = -\text{id.}\ (x \in D)$$

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$T_x S \cap J_x(T_x S) = \{0\}$ for any $x \in D$,

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$\mathbb{C}^n \supset \mathbb{R}^k$ totally real

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Rem We do not request S to be of maximal dimension (e.g. $S = \mathbb{R}^n$ in $\mathbb{C}^n$.)

§3 Visible actions

(D, J) complex mfd, connected

Def. A real submanifold S of D is totally real if $T_x S$ does not contain any complex subspace.

§3 Visible actions

$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ complex mfd, connected

Def.(2003) A holomorphic action of H is visible w.r.t. a slice S if
$^{\exists}D' \underset{\text{open}}{\subset} D$,

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holomorphic

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Def.(2003) A holomorphic action of H is visible w.r.t. a slice S if
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$$\begin{cases} S \text{ meets every } H\text{-orbit} \\ J_x(T_xS) \subset T_x(H \cdot x)\ (x \in S) \end{cases}$$

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$\quad\quad \begin{array}{ccc} \mathbb{T} & & \\ \cup\!\!\!\!\!\!\;\;\; & & \\ a & & z \mapsto az \end{array}$

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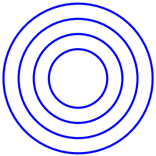
$\mathbb{T} \ni a$, $z \mapsto az$



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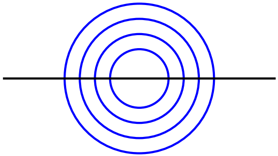

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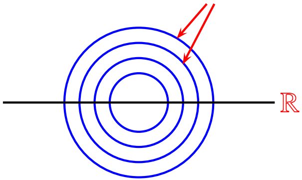
$\mathbb{R}$ meets every $\mathbb{T}$-orbit



$$J_x(T_x S) \subset T_x(H \cdot x), \quad {}^{\forall} x \in \mathbb{R} \setminus \{0\} =: S$$

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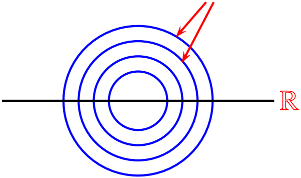

$\Longrightarrow$ $\mathbb{T}$-action on $\mathbb{C}$ is visible

Strongly visible actions

holomorphic

$H \curvearrowright D$ complex mfd, connected

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Def. ♣ ref. A holomorphic action is strongly visible if

$^\exists\sigma$ $\curvearrowleft$ H as Lie group auto

$\curvearrowright$ D as anti-holomorphic diffeo

in a compatible way

s.t. $H \cdot D^\sigma$ contains a non-empty open set of D.

(generic H-orbits meets the fixed point set D^σ)

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$H \curvearrowright D$ complex mfd, connected

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$\curvearrowright D$ as <u>anti-holomorphic</u> diffeo
in a compatible way
i.e. $\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)$

s.t. $H \cdot D^\sigma$ contains a non-empty open set of D.
(generic H-orbits meets the fixed point set D^σ)

$$D^\sigma := \{x \in D : \sigma(x) = x\}$$

Remark. Not necessarily $\sigma^2 = \mathrm{id}$

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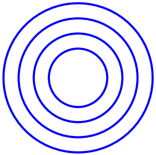

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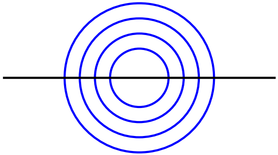
$\quad\;\; \Psi$

$\quad\;\; a \qquad z \mapsto az$



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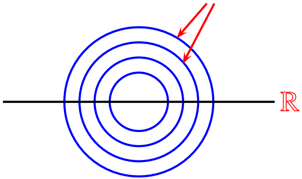
$\mathbb{R}$ meets every $\mathbb{T}$-orbit



$\sigma(a) := \overline{a}$, $\sigma(z) := \overline{z}$ anti-holomorphic.

Then $\sigma(a \cdot z) = \sigma(a) \cdot \sigma(z)$ (compatibility), and $\sigma|_{\mathbb{R}} = \mathrm{id}$.

Example of strongly visible actions

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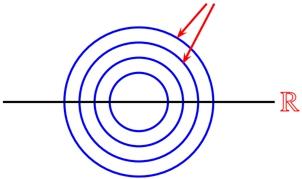

$\Longrightarrow$ $\mathbb{T}$-action on $\mathbb{C}$ is strongly visible

Strongly visible actions with slice S

$H \overset{\text{holomorphic}}{\curvearrowright} D$ complex mfd, connected

Strongly visible actions with slice S

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<u>Def.</u> ♣ ref. A holomorphic action is strongly visible w.r.t. a slice S if

$^{\exists}\sigma$ $\curvearrowleft H$

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in a compatible way

s.t.

$\sigma|_S = \mathrm{id}$

$H \cdot S$ contains a non-empty open set of D.

Remark. S is automatically totally real.

<u>Point</u> Try to find a smallest possible $S \subset D^{\sigma}$.

Strongly visible actions

<u>Proposition</u> Strongly visible $\Longrightarrow$ Visible

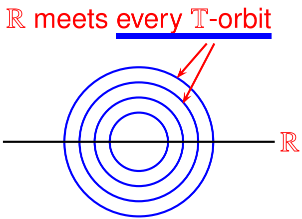
Strongly visible actions

Proposition Strongly visible $\Longrightarrow$ Visible

To be more precise,

strongly visible w.r.t. a slice S

$\Longrightarrow$ visible w.r.t. S' for some $S' \underset{\text{open dense}}{\subset} S$.



G/K Hermitian symm. space

Ex.22

$$G = SL(2,\mathbb{R})$$
$$K = SO(2)$$
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$
$$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$$

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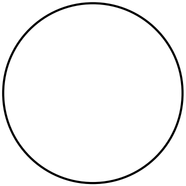

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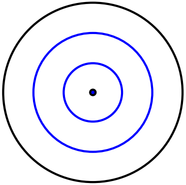

K-orbits

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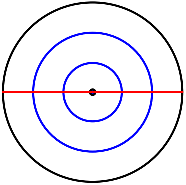
$K \curvearrowright G/K$ visible



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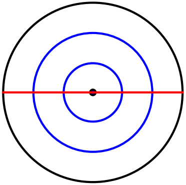

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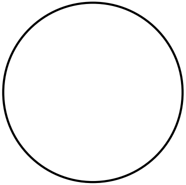

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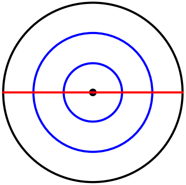

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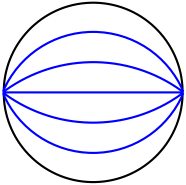

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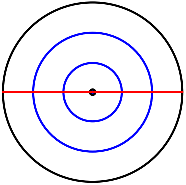

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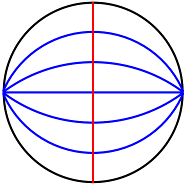

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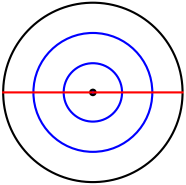

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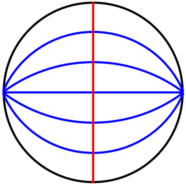
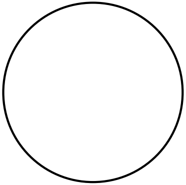


H-orbits

$N \curvearrowright G/K$



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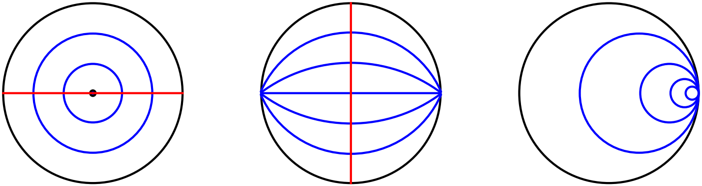
$K \curvearrowright G/K$ visible | $H \curvearrowright G/K$ visible | $N \curvearrowright G/K$



K-orbits | H-orbits | N-orbits

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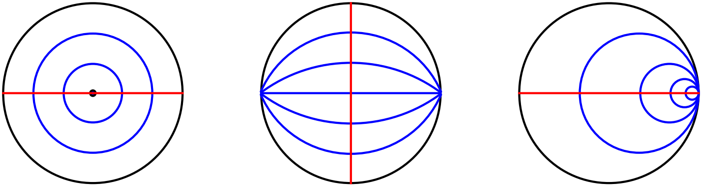

K-orbits $\quad H$-orbits $\quad N$-orbits

Visible actions on symmetric spaces

Theorem (Transf. Groups (2007))

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

§4 Compex / Riemannian / symplectic geometry

holomorphic

$H \curvearrowright (D, J)$ complex mfd, connected

Def. Action is visible if

$^{\exists}S \subset {}^{\exists}D' \subset D$ s.t.

totally real (for $S \subset D'$), open (for $D' \subset D$)

$$\begin{cases} S \text{ meets every } H\text{-orbit in } D' \\ J_x(T_xS) \subset T_x(H \cdot x) \ (x \in S) \end{cases}$$

§4 Compex / Riemannian / symplectic geometry

$H \curvearrowright (D, J)$ (holomorphic) complex mfd, connected

Def. Action is visible if
$^{\exists}S \subset {}^{\exists}D' \subset D$ s.t. (S totally real, D' open)

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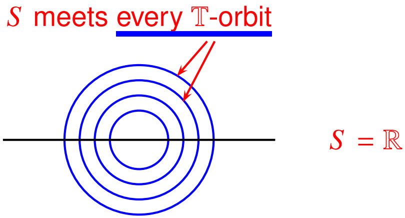

Complex / Riemannian / symplectic

$H \overset{\text{isometric}}{\curvearrowright} (D, g)$ Riemannian mfd

Def. Action is polar if $^\exists S \underset{\text{closed submfd}}{\subset} D$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit} \\ T_x S \perp T_x(H \cdot x) \quad (x \in S) \end{cases}$$

Complex / Riemannian / symplectic

$H \overset{\text{isometric}}{\curvearrowright} (D, g)$ Riemannian mfd

Def. Action is polar if $^{\exists}S \underset{\text{closed submfd}}{\subset} D$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit} \\ T_x S \perp T_x(H \cdot x) \quad (x \in S) \end{cases}$$

$H \overset{\text{symplectic}}{\curvearrowright} (D, \omega)$ symplectic mfd

Def. (Guillemin–Sternberg, Huckleberry–Wurzbacher)
Action is coisotropic (or multiplicity-free)
if $T_x(H \cdot x)^{\perp\omega} \subset T_x(H \cdot x)$ for principal orbits $H \cdot x$ in D

Three geometries

Complex geometry

Symplectic geometry

Riemannian geometry

Three geometries

Complex geometry

Visible action

K– (2004)

Symplectic geometry

Coisotropic action

Guillemin–Sternberg ('84)
Huckleberry–Wurzbacher ('90)

Riemannian geometry

Polar action

Bott–Samelson ('58), Conlon, Hermann, Palais, Terng, Dadok,
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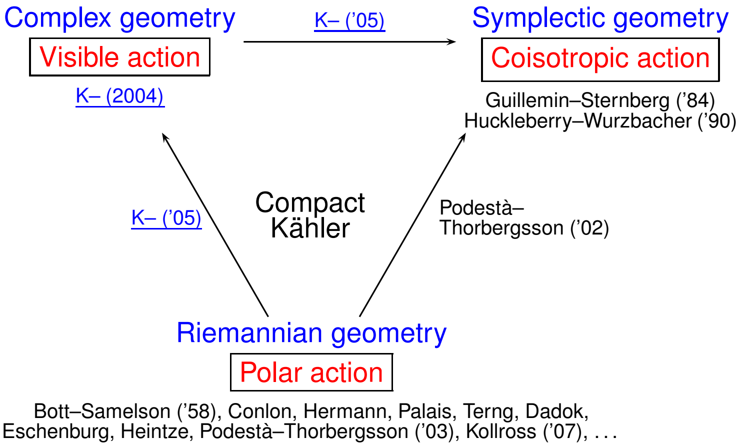

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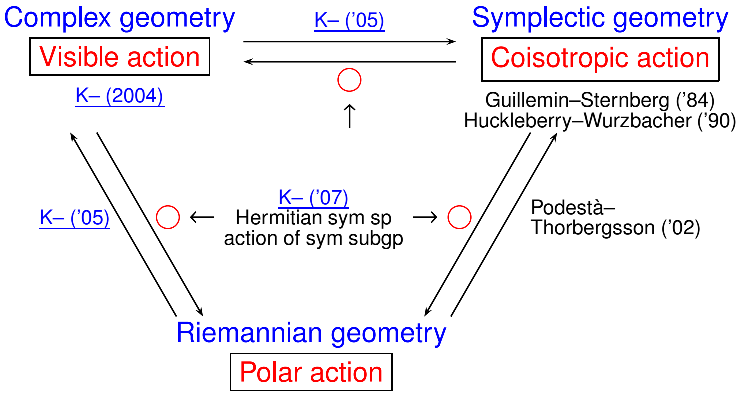

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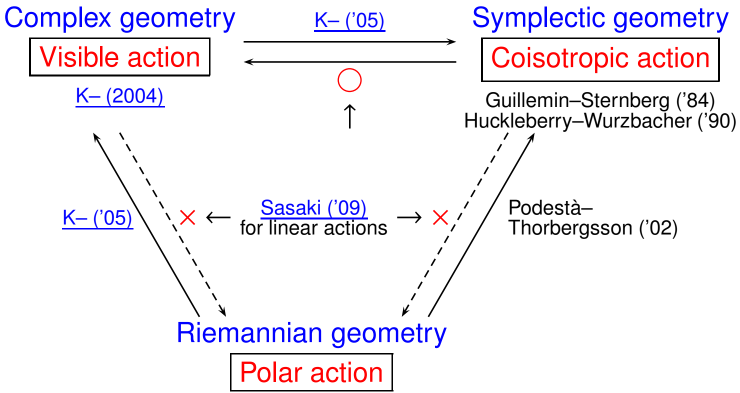

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§5 Making examples of visible actions

Ex.20	$H = U(m) \times U(n)$
	$D = M(m, n; \mathbb{C})$
$\Rightarrow$	Every H-orbit is preserved by $z \mapsto \bar{z}$

§5 Making examples of visible actions

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Proof Let $m \leq n$. Set

$$S := \left\{ \begin{pmatrix} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_m & \end{pmatrix} : a_1, \ldots, a_m \in \mathbb{R} \right\}$$

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$\Rightarrow$ Any H-orbit is of the form $H \cdot x$ ($^{\exists} x \in S$)

$$\overline{H \cdot x} \underset{\text{compatibility}}{=} \overline{H} \cdot \overline{x} \underset{x \in M(m,n;\mathbb{R})}{=} H \cdot x$$

□

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In general,

Strongly visible

(i.e. $^{\exists}\sigma$ anti-holo s.t. $(H \cdot D^{\sigma})^{\circ} \neq \emptyset$)

$\Rightarrow$ Assumption 1 of Theorem

(i.e. $^{\exists}\sigma$ anti-holo s.t. σ preserves generic H-orbits)

Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.

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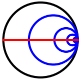
	$\Uparrow$	
	$H \curvearrowright X$ (strongly) visible	
	Base sp.	



∞ many orbits

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

Classification theory of visible actions

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  - geometry of three involutions
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Two involutions	$\longleftrightarrow$	semisimple symmetric space
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Three involutions (special case)	$\longleftrightarrow$	visible action (2004–) (special case)

Visible actions on symmetric spaces

Theorem (geometry of three involutions '07)

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{any symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

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$\Downarrow$ Propagation theorem

Thm $V_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$ (λ generic) is an algebraic MF direct sum of irreducible $\mathfrak{g}'$-modules if

- nilradical of $\mathfrak{p}_{\mathbb{R}}$ is abelian
- $G'_{\mathbb{R}} \cdot o$ is closed in $G_{\mathbb{R}}/P_{\mathbb{R}}$
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$G'_\mathbb{R}$	$\subset$	$G_\mathbb{R}$	$\supset$	$P_\mathbb{R}$
subgp		real reductive		real parabolic

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(Generalized) Cartan involutions

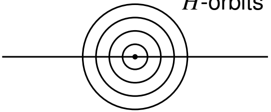
<u>Observation</u>

$$D = G/K$$

Suppose we have a decomposition

$$G = H\ A\ K$$

Set $S := A \cdot o \subset D$



$\Longrightarrow S$ is a candidate of ‘slice’ for (strongly) visible action

Classification theory of visible actions

Grassmannian $U(n)/(U(p) \times U(q)) \simeq Gr_p(\mathbb{C}^n) \quad (n = p + q)$

Ex.(symmetric case) $\quad n_1 + n_2 = p + q = n$

$\Longrightarrow$ $U(n_1) \times U(n_2)$ acts on $Gr_p(\mathbb{C}^n)$
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$\Longleftrightarrow \min(n_1 + 1, n_2 + 1, n_3 + 1, p, q) \leq 2$

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For type B, C, D and exceptional groups (Y. Tanaka, Tohoku J. (2013), J. Math. Soc. Japan (2013), B. Austrian Math Soc. (2013), J. Algebra (2014))

$\Downarrow$ Propagation theorem

MF property of the following

- $GL_m \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ Ex.16
- $GL_{m-1} \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ Ex.18 (Kac)
- the Stembridge list of $\pi_\lambda \otimes \pi_\nu$ Ex.11
- $GL_n \downarrow (GL_p \times GL_q)$ Ex.12
- $GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3})$ Ex.13
- ∞-dimensional versions

......

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Idea: induced action preserving visibility

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$$H \subset G$$

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$H \curvearrowright Y$ visible w.r.t. S

$\rightsquigarrow$ ⇐ certain assumption

$G \curvearrowright X := G \times_H Y$ visible w.r.t. $S \simeq [\{e\}, S]$

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Ex. $H = U(p) \times U(q)$, $Y = M(p, q; \mathbb{C})$ $(p \geq q)$

$G = U(p+q)$, $X = T^*(G/H) = T^*(Gr_p(\mathbb{C}^{p+q}))$

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$\rightsquigarrow$ momentum map: nilpotent orbit for $GL(p+q, \mathbb{C})$ for partition $(2^q, 1^{p-q})$ is spherical (Panyushev)

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Examples of (strongly) visible actions

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Triunity of visible actions

$$\begin{pmatrix} H & & L \\ & \rotatebox{45}{$\subset$} \quad \rotatebox{-45}{$\supset$} & \\ & G & \\ & \cup & \\ & G^{\sigma} & \end{pmatrix} := \begin{pmatrix} \mathbb{T}^n & & U(1) \times U(n-1) \\ & \rotatebox{45}{$\subset$} \quad \rotatebox{-45}{$\supset$} & \\ & U(n) & \\ & \cup & \\ & O(n) & \end{pmatrix}$$

Triunity of visible actions

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Geometry

(visible actions)

$\mathbb{P}^{n-1}\mathbb{R}$ meets every $\mathbb{T}^n$-orbit on $\mathbb{P}^{n-1}\mathbb{C}$

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$$\begin{array}{ccc} \mathbb{P}^{n-1}\mathbb{R} & \mathbb{T}^n & \mathbb{P}^{n-1}\mathbb{C} \\ \| & \| & \| \\ G^{\sigma}/G^{\sigma} \cap L & H & G/L \end{array}$$

$\updownarrow$

Group

$G = HG^{\sigma}L \Rightarrow H \curvearrowright G/L$

Triunity of visible actions

$$\left(\begin{array}{ccc} H & & L \\ & \rotatebox{-45}{$\supset$}\ \ \rotatebox{45}{$\subset$} & \\ & G & \\ & \cup & \\ & G^{\sigma} & \end{array}\right) := \left(\begin{array}{ccc} \mathbb{T}^n & & U(1)\times U(n-1) \\ & U(n) & \\ & \cup & \\ & O(n) & \sigma(g) := \overline{g} \end{array}\right)$$

with $\mathbb{T}^n \subset U(n) \supset U(1)\times U(n-1)$.

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$\updownarrow$

Group $\quad G = HG^{\sigma}L \Rightarrow H \curvearrowright G/L$

$\updownarrow$

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Triunity of visible actions

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$\updownarrow$

Group $\quad G = HG^{\sigma}L \Rightarrow H \curvearrowright G/L$

$\updownarrow$

Group $\quad G = LG^{\sigma}H \Rightarrow L \curvearrowright G/H$

$\updownarrow$

Group $\quad (G\times G) = \mathrm{diag}(G)(G^{\sigma}\times G^{\sigma})(H\times L) \Rightarrow$ diag.

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$\Downarrow$ Propagation theorem

Three kinds of MF results:

- (Taylor series) $\mathbb{T}^n \curvearrowright \mathcal{O}(\mathbb{C}^n)$ Ex.2
- ($GL_n \downarrow GL_{n-1}$) Restriction $\pi|_{GL_{n-1}}$ Ex.14
- (Pieri) $\pi \otimes S^k(\mathbb{C}^n)$ Ex.9

$\otimes$-product rep.

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n, \ a \geq b$$

Ex.11 (Stembridge 2001, K– 2002)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module if

1) $\min(a-b, p, q) = 1$ (and ν is any),

or

2) $\min(a-b, p, q) = 2$ and

★ ν is of the form $\nu = (\underbrace{x \cdots x}_{n_1} \ \underbrace{y \cdots y}_{n_2} \ \underbrace{z \cdots z}_{n_3}) \ (x \geq y \geq z)$,

or

3) $\min(a-b, p, q) \geq 3$, ★ &

$\min(x-y, \ y-z, n_1, n_2, n_3) = 1.$

Restriction $(GL_n \downarrow G_{n_1} \times GL_{n_2} \times GL_{n_3})$

Ex.12 $(GL_n \downarrow (GL_p \times GL_q))$ $\quad n = p + q$

$$\pi^{GL_n}_{(\underbrace{x,\dots,x}_{n_1},\underbrace{y,\dots,y}_{n_2},\underbrace{z,\dots,z}_{n_3})}|_{GL_p \times GL_q} \text{ is MF}$$

if $\min(p,q) \leq 2$ or

if $\min(n_1, n_2, n_3, x-y, y-z) \leq 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

Ex.13 $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$ ([5])

$n = n_1 + n_2 + n_3$

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Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.

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Theorem (Propagation thm of MF property)

Sections

$\Longrightarrow$ $H \curvearrowright \mathcal{O}(X, \mathcal{V})$ multiplicity-free

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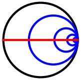
Theorem (Propagation thm of MF property)

Fiber: $H_x \curvearrowright \mathcal{V}_x$ multiplicity-free $\Longrightarrow$ Sections: $H \curvearrowright \mathcal{O}(X, \mathcal{V})$ multiplicity-free

$\Uparrow$

$H \curvearrowright X$ (strongly) visible

Base sp.



∞ many orbits

Heuristic idea of Theorem

Setting $\mathcal{V} \to D$: G-equiv. holomorphic vector bundle

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<u>Setting</u> $\mathcal{V} \to D$: G-equiv. holomorphic vector bundle

$$\rightsquigarrow \quad \mathcal{R}_x := \mathrm{End}_{G_x}(\mathcal{V}_x) \underset{\text{subring}}{\subset} \mathrm{End}_{\mathbb{C}}(\mathcal{V}_x)$$

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Ex. $S = \{pt\}$, $G_x \curvearrowright \mathcal{V}_x$ <u>irred.</u> $\Rightarrow G \curvearrowright \mathcal{H}$ <u>irred.</u>

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Reproducing kernel

Prototype (Scalar valued) holomorphic functions

$$\mathbb{C}^n \quad \supset \quad D \quad \text{complex domain}$$

Reproducing kernel

Prototype (Scalar valued) holomorphic functions

$$\begin{array}{ccc l} \mathbb{C}^n & \supset & D & \text{complex domain} \\ \mathcal{H} & \subset & \mathcal{O}(D) & \\ \text{Hilbert space} & & \{\text{holomorphic functions on } D\} & \end{array}$$

Reproducing kernel

Prototype (Scalar valued) holomorphic functions

$\mathbb{C}^n$	$\supset$	D	complex domain
$\mathcal{H}$	$\subset$	$\mathcal{O}(D)$	
Hilbert space		{holomorphic functions on D}	

Definition (reproducing kernel)

Let $\{\varphi_l\}$ be an orthonormal basis of $\mathcal{H}$.

$$K_{\mathcal{H}}(z,w) := \sum_l \varphi_l(z)\overline{\varphi_l(w)}$$

is independent of the choice of the basis.

Examples of reproducing kernels

$$\{\varphi_j\} \quad \subset \quad \mathcal{H} \quad \subset \quad \mathcal{O}(D)$$

orthonormal basis   Hilbert space

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<u>Example 1</u> (weighted Bergman space)

$$D := \{z \in \mathbb{C} : |z| < 1\}$$

Fix $\lambda > 1$.

$$\mathcal{H} := \{f \in \mathcal{O}(D) : \|f\|_\lambda < \infty\}$$

$$\|f\|_\lambda := \left(\int_D |f(x+iy)|^2(1-x^2-y^2)^{\lambda-2}dxdy\right)^{\frac{1}{2}}$$

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$$K_{\mathcal{H}}(z,w) = \sum_{l=0}^{\infty} \frac{\Gamma(\lambda+l)}{\pi\Gamma(\lambda-1)} z^l \overline{w}^l$$

$$= \frac{\lambda-1}{\pi}(1 - z\overline{w})^{-\lambda}$$

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$$K_{\mathcal{F}}(z,w) = \sum_{l=0}^{\infty} \frac{z^l\overline{w}^l}{\pi l!} = \frac{1}{\pi} e^{z\overline{w}}$$

Properties of reproducing kernel

$$\begin{array}{c}\mathcal{O}(D)\\ \cup\\ \mathcal{H}\end{array} \rightsquigarrow K_{\mathcal{H}}(z,w) = \sum_j \varphi(x)\overline{\varphi(w)}$$

Hilbert space

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<u>Corollary</u> Suppose a group G acts on D
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Then G acts on $\mathcal{H}$ as a unitary representation
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$$K_{\mathcal{H}}(gz, gw) = K_{\mathcal{H}}(z,w) \quad {}^{\forall}g \in G, {}^{\forall}z, {}^{\forall}w \in D. \qquad (\star)$$

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$$(\star) \iff K_{\mathcal{H}}(gz,gz) = K_{\mathcal{K}}(z,z) \quad {}^{\forall}g \in G, {}^{\forall}z \in D$$

Scalar-valued reproducing kernel

$$\underbrace{\mathcal{H}}_{\text{Hilbert space}} \subset \mathcal{O}(D)$$

Assume that for each $x \in D$,

$$\begin{array}{rccc} \mathrm{ev}_x : & \mathcal{H} & \to & \mathbb{C} \\ & \Psi & & \Psi \\ & f & \mapsto & f(x) \end{array}$$ is continuous.

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$$\begin{aligned} K_{\mathcal{H}}(z, w) &= \mathrm{ev}_w \circ \mathrm{ev}_z^* \\ &= \sum_j \varphi_j(z) \varphi_j(w) \end{aligned}$$

Operator-valued reproducing kernel

$\mathcal{V} \to D$: holomorphic vector bundle

$\mathcal{H} \underset{\text{Hilbert space}}{\hookrightarrow} \mathcal{O}(D, \mathcal{V})$

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$$K_{\mathcal{H}}(x, y) := \mathrm{ev}_y \circ \mathrm{ev}_x^* \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_x^*, \mathcal{V}_y)$$

operator-valued reproducing kernel

Operator-valued reproducing kernel

$$\mathcal{V} = \coprod_{x \in D} \mathcal{V}_x \longrightarrow D$$

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$\underset{\text{Hilbert space}}{\mathcal{H}} \hookrightarrow \mathcal{O}(D, \mathcal{V})$

$\Updownarrow$ one to one

$K_{\mathcal{H}} \in \mathcal{O}(D \times D, \mathcal{H}om(\mathcal{V}^*, \mathcal{V}))$
positive definite operator-valued reproducing kernel

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$\mathcal{H}$: unitarity, irreducibility, MF, $\ldots \Longleftrightarrow$ Properties on $K_{\mathcal{H}}$

‘Visible’ approach to multiplicity-free theorems

Theorem

fiber $\xrightarrow{\text{visible action}}$ sections

'Visible' approach to multiplicity-free theorems

Thm (K– '08) $\pi|_H$ is multiplicity-free if
π: highest wt. rep. of scalar type
(G, H): semisimple symmetric pair
(Hua, Kostant, Schmid, K– : explicit formula)

Fact (É. Cartan '29, I. M. Gelfand '50)
$L^2(G/K)$ is multiplicity-free

Theorem

Multiplicity-free space
Kac '80, Benson–Ratcliff '91
Leahy '98

Stembridge's list (2001) of multiplicity-free $\otimes$ product of finite dim'l reps (GL_n)

fiber $\xrightarrow{\text{visible action}}$ sections

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Hermitian symm sp. (K– ’07) ↖ Theorem ↗ Crown domain

Vector sp. (Sasaki ’09) ↙ ↘ Grassmann mfd. (K– ’07)

Multiplicity-free space
Kac ’80, Benson–Ratcliff ’91
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Stembridge’s list (2001) of
multiplicity-free ⊗ product of
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fiber $\xrightarrow{\text{visible action}}$ sections

'Visible' approach

To give a simple principle that explains the property MF for both finite and infinite dimensional reps

MF (multiplicity-free) theorem

Propagation of MF property from fiber to sections

$\Uparrow$

Visible actions on complex mfds

Analysis of group action with infinitely many orbits

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Thank you !!

(Short story by Soseki, 1908)

“He uses the hammer and chisel without any forethought, and he can make the eyebrows and nose as live.”

“Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake.”

$\overset{\exists 1}{\hookleftarrow}$