Integrable sigma models

Noureddine MOHAMMEDI

Laboratoire de Mathématiques et Physique Théorique (Tours, France)

09 June 2014 @ Varna

Reference:

N. Mohammedi, Nucl.Phys. **B839** (2010) 420-445 [arXiv:0806.0550 [hep-th]]

Outline

What does integrability mean ?

- **Notation:**
  The two dimensional coordinates are z and $\bar{z}$. The derivatives are $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$

What does integrability mean ?

- **Notation:**
  The two dimensional coordinates are z and $\bar{z}$. The derivatives are $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$
- **The linear system:**

$$\begin{cases} (\partial + A)\,\Psi = 0 \\ \left(\bar{\partial} + \bar{A}\right)\,\Psi = 0 \end{cases}$$

What does integrability mean ?

- **Notation:**
  The two dimensional coordinates are z and $\bar{z}$. The derivatives are $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$
- **The linear system:**

$$\begin{cases} (\partial + A)\,\Psi = 0 \\ (\bar{\partial} + \bar{A})\,\Psi = 0 \end{cases}$$

- **The zero curvature condition:**

$$\left[\partial + A\,,\,\bar{\partial} + \bar{A}\right] = 0$$

What does integrability mean ?

- **Notation:**
  The two dimensional coordinates are z and $\bar{z}$. The derivatives are $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$
- **The linear system:**

$$\begin{cases} (\partial + A)\,\Psi = 0 \\ (\bar{\partial} + \bar{A})\,\Psi = 0 \end{cases}$$

- **The zero curvature condition:**

$$\left[\partial + A\,,\,\bar{\partial} + \bar{A}\right] = 0$$

$\Longrightarrow$ The equations of motion of the theory

What does integrability mean ?

- **Notation:**
  The two dimensional coordinates are z and $\bar{z}$. The derivatives are $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$
- **The linear system:**

$$\begin{cases} (\partial + A)\,\Psi = 0 \\ \left(\bar{\partial} + \bar{A}\right)\,\Psi = 0 \end{cases}$$

- **The zero curvature condition:**

$$\left[\partial + A\,,\,\bar{\partial} + \bar{A}\right] = 0$$

$\Longrightarrow$ The equations of motion of the theory

- **Conserved quantities:**

$$\partial^{\mu} J^{(n)}_{\mu} \;=\; \partial\,\mathrm{Tr}\left(\bar{A}\Psi^{n}\right) - \bar{\partial}\,\mathrm{Tr}\left(A\Psi^{n}\right) = 0$$

The Liouville equation

- **The equation:**

$$\bar{\partial}\partial\varphi = 2\, e^{\varphi}$$

The Liouville equation

- **The equation:**

$$\bar{\partial}\partial\varphi = 2\,e^{\varphi}$$

- **The solution**

$$\varphi\,(z\,,\,\bar{z}) = \ln\left[\frac{\partial f\,\bar{\partial}\bar{f}}{\left(f+\bar{f}\right)^2}\right] \quad \text{with} \quad f = f(z) \quad \text{and} \quad \bar{f} = \bar{f}(\bar{z})$$

Liouville equation from a linear system

- **A linear system:**

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad ,$$

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Liouville equation from a linear system

- **A linear system:**

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad ,$$

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{cases} \left(\partial + e^{\varphi}\, E_+\right) \Psi = 0 \\ \left(\bar\partial - \frac{1}{2}\bar\partial\varphi\, H + E_-\right) \Psi = 0 \end{cases}$$

Liouville equation from a linear system

- **A linear system:**

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad ,$$
$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{cases} \left(\partial + e^{\varphi} E_+\right) \Psi = 0 \\ \left(\bar{\partial} - \frac{1}{2}\bar{\partial}\varphi\, H + E_-\right) \Psi = 0 \end{cases}$$

- **A little algebra:**

$$\begin{cases} \partial P = e^{Q} \\ \bar{\partial} Q = e^{P} \end{cases}$$

Solving the linear system

- **Implications:**

$$\begin{cases} \bar{\partial}\partial\,(P+Q) = 2\,e^{P+Q} \\ \bar{\partial}\partial\,(P-Q) = 0 \end{cases}$$

Solving the linear system

- **Implications:**

$$\begin{cases} \bar{\partial}\partial\,(P+Q) = 2\,e^{P+Q} \\ \bar{\partial}\partial\,(P-Q) = 0 \end{cases}$$

The second equation $\Longrightarrow$

$$P\,(z\,,\bar{z}) = Q\,(z\,,\bar{z}) + g\,(z) + \bar{g}\,(\bar{z})$$

Solving the linear system

- **Implications:**

$$\begin{cases} \bar{\partial}\partial\,(P+Q) = 2\,e^{P+Q} \\ \bar{\partial}\partial\,(P-Q) = 0 \end{cases}$$

The second equation $\Longrightarrow$

$$P\,(z\,,\bar{z}) = Q\,(z\,,\bar{z}) + g\,(z) + \bar{g}\,(\bar{z})$$

$$\partial P = e^{Q} \qquad \Longrightarrow$$

$$Q = -g(z) - \ln\left(-\int dz e^{-g(z)} + \bar{h}(\bar{z})\right)$$

Solving the linear system

$$\bar{\partial} Q = e^{P} \qquad \Longrightarrow$$

$$\bar{h}(\bar{z}) = -\int d\bar{z} e^{\bar{g}(\bar{z})}$$

- **Extracting the Liouville field:**

$$\varphi = P + Q = -g(z) + \bar{g}(\bar{z}) - 2\ln\left(-\int dz e^{-g(z)} - \int d\bar{z} e^{\bar{g}(\bar{z})}\right)$$

Solving the linear system

$$\bar{\partial} Q = e^{P} \qquad \Longrightarrow$$

$$\bar{h}(\bar{z}) = -\int d\bar{z} e^{\bar{g}(\bar{z})}$$

- **Extracting the Liouville field:**

$$\varphi \;=\; P + Q \;=\; -g(z) + \bar{g}(\bar{z}) - 2\ln\left(-\int dz e^{-g(z)} - \int d\bar{z} e^{\bar{g}(\bar{z})}\right)$$

$$g(z) = -\ln\left(-\partial f(z)\right) \qquad , \qquad \bar{g}(\bar{z}) = \ln\left(-\bar{\partial}\bar{f}(\bar{z})\right)$$

The theory

- **The action:**

$$S = \int \mathrm{d}z \mathrm{d}\bar{z} \left[\eta_{ij} \partial \varphi^i \bar{\partial} \varphi^j - V(\varphi) \right] \quad .$$

The theory

- **The action:**

$$S = \int \mathrm{d}z\mathrm{d}\bar{z}\left[\eta_{ij}\partial\varphi^i\bar{\partial}\varphi^j - V\left(\varphi\right)\right] \quad .$$

- **The equations of motion:**

$$\mathcal{E}^i \;\equiv\; \bar{\partial}\partial\varphi^i + \frac{1}{2}\eta^{ij}\partial_j V = 0 \quad .$$

- **The linear system:**

$$\begin{aligned}\left[\partial + \alpha_i\left(\varphi\right)\partial\varphi^i + \gamma\left(\varphi\right)\right]\Psi &= 0\\ \left[\bar{\partial} + \beta_j\left(\varphi\right)\bar{\partial}\varphi^j + \rho\left(\varphi\right)\right]\Psi &= 0 \quad .\end{aligned}$$

The theory

- **The requirement:**

$$\begin{aligned} \mathcal{F} &= \mathcal{E}^i \mu_i = 0 \\ \mathcal{F} &= \left[\partial + \alpha_i(\varphi)\,\partial\varphi^i + \gamma(\varphi)\ ,\ \bar{\partial} + \beta_j(\varphi)\,\bar{\partial}\varphi^j + \rho(\varphi)\right] \end{aligned}$$

The theory

- **The requirement:**

$$\begin{aligned} \mathcal{F} &= \mathcal{E}^i \mu_i = 0 \\ \mathcal{F} &= \left[\partial + \alpha_i(\varphi)\,\partial\varphi^i + \gamma(\varphi)\,,\,\bar{\partial} + \beta_j(\varphi)\,\bar{\partial}\varphi^j + \rho(\varphi)\right] \end{aligned}$$

- **The conditions:**

$$\begin{gathered} \beta_i - \alpha_i = \mu_i \\ \partial_i \beta_j - \partial_j \alpha_i + [\alpha_i\,,\beta_j] = 0 \\ \partial_i \rho + [\alpha_i\,,\,\rho] = 0 \\ \partial_j \gamma + [\beta_j\,,\,\gamma] = 0 \\ [\gamma\,,\,\rho] = \tfrac{1}{2}\eta^{kl}\partial_l V\,\mu_k \quad . \end{gathered}$$

The master equations

- **A little algebra:**

$$[H_i\ ,\ H_j] = 0$$

The master equations

- **A little algebra:**

$$[H_i\ ,\ H_j] = 0$$

- **The determination of the potential:**

$$\left[e^{-H_i\varphi^i}\,\widehat{\omega}_0\,e^{H_j\varphi^j}\ ,\ \rho_0\right] = \frac{1}{2}\eta^{kl}\partial_l V\,H_k$$

The master equations

- **A little algebra:**

$$[H_i \, , \, H_j] = 0$$

- **The determination of the potential:**

$$\left[e^{-H_i \varphi^i} \, \widehat{\omega}_0 \, e^{H_j \varphi^j} \, , \, \rho_0 \right] = \frac{1}{2} \eta^{kl} \partial_l V \, H_k$$

Are there other solutions beyond Toda type theories ????

A simplified model

- **The action:**

$$S = \int \mathrm{d}z\mathrm{d}\bar{z}\, g_{ij}\left(\varphi\right) \partial\varphi^i \bar{\partial}\varphi^j \quad .$$

A simplified model

- **The action:**

$$S = \int \mathrm{d}z\mathrm{d}\bar{z}\, g_{ij}\left(\varphi\right)\partial\varphi^i\bar{\partial}\varphi^j \quad .$$

- **The equations of motion:**

$$\mathcal{E}^l \;\equiv\; \bar{\partial}\partial\varphi^l + \Gamma^l_{ij}\partial\varphi^i\bar{\partial}\varphi^j = 0 \quad .$$

A simplified model

- **The action:**

$$S = \int \mathrm{d}z\mathrm{d}\bar{z}\, g_{ij}\left(\varphi\right) \partial\varphi^i \bar{\partial}\varphi^j \quad .$$

- **The equations of motion:**

$$\mathcal{E}^l \equiv \bar{\partial}\partial\varphi^l + \Gamma^l_{ij}\partial\varphi^i\bar{\partial}\varphi^j = 0 \quad .$$

The Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\right)$$

A simplified model

- **Notation:**

$$
\begin{aligned}
J &= (K_i - L_i)\,\partial\varphi^i \\
\bar{J} &= (K_i + L_i)\,\bar{\partial}\varphi^i \quad .
\end{aligned}
$$

A simplified model

- **Notation:**

$$\begin{aligned} J &= (K_i - L_i)\,\partial\varphi^i \\ \bar{J} &= (K_i + L_i)\,\bar{\partial}\varphi^i \quad . \end{aligned}$$

- **The Lax pair:**

$$\begin{aligned} \left[\partial + \frac{1}{1+\lambda}\,J\right]\Psi &= 0 \\ \left[\bar{\partial} + \frac{1}{1-\lambda}\,\bar{J}\right]\Psi &= 0 \quad . \end{aligned}$$

The geometry

- **The consistency conditions:**

The geometry

- **The consistency conditions:**

$$\partial_i K_j + \partial_j K_i - 2\Gamma^l_{ij}\, K_l = 0$$

The geometry

- **The consistency conditions:**

$$\partial_i K_j + \partial_j K_i - 2\Gamma^l_{ij} K_l = 0$$

$$\partial_i L_j - \partial_j L_i = 0 \quad \Longrightarrow \quad L_i = \partial_i \Lambda$$

The geometry

- **The consistency conditions:**

$$\partial_i K_j + \partial_j K_i - 2\Gamma^l_{ij}\, K_l = 0$$

$$\partial_i L_j - \partial_j L_i = 0 \qquad \Longrightarrow \qquad L_i = \partial_i \Lambda$$

$$\partial_i L_j + \partial_j L_i - 2\Gamma^l_{ij}\, L_l - [L_i\,,\,K_j] - [L_j\,,\,K_i] = 0$$

The geometry

- **The consistency conditions:**

$$\partial_i K_j + \partial_j K_i - 2\Gamma^l_{ij}\, K_l = 0$$

$$\partial_i L_j - \partial_j L_i = 0 \qquad \Longrightarrow \qquad L_i = \partial_i \Lambda$$

$$\partial_i L_j + \partial_j L_i - 2\Gamma^l_{ij}\, L_l - [L_i\,,\, K_j] - [L_j\,,\, K_i] = 0$$

$$\partial_i K_j - \partial_j K_i + [K_i\,,\, K_j] = [L_i\,,\, L_j] \quad ,$$

The geometry

- **The consistency conditions:**

$$\partial_i K_j + \partial_j K_i - 2\Gamma^l_{ij} K_l = 0$$

$$\partial_i L_j - \partial_j L_i = 0 \quad \Longrightarrow \quad L_i = \partial_i \Lambda$$

$$\partial_i L_j + \partial_j L_i - 2\Gamma^l_{ij} L_l - [L_i , K_j] - [L_j , K_i] = 0$$

$$\partial_i K_j - \partial_j K_i + [K_i , K_j] = [L_i , L_j] \quad ,$$

- **Interpretation:**

$$\begin{aligned} \partial \bar{J} + \bar{\partial} J &= 2K_i \,\mathcal{E}^i \\ \partial \bar{J} - \bar{\partial} J + \left[J , \bar{J}\right] &= 2L_i \,\mathcal{E}^i \quad . \end{aligned}$$

A deformation of the $SU(2)$ principal chiral sigma model

- **The $SU(2)$ Lie algebra:**

$$[T_a\,,\,T_b] = f_{ab}^c T_c$$

A deformation of the $SU(2)$ principal chiral sigma model

- **The $SU(2)$ Lie algebra:**

$$[T_a\,,\,T_b] = f_{ab}^{c} T_c$$

- **The $SU(2)$ theory:**

$$S\left(g\right) = \int \mathrm{d}z\mathrm{d}\bar{z}\left[\mathrm{Tr}\left(g^{-1}\partial g\, g^{-1}\bar{\partial} g\right) - 2C\,\mathrm{Tr}\left(T_3 g^{-1}\partial g\right)\mathrm{Tr}\left(T_3 g^{-1}\bar{\partial} g\right)\right] \quad ,$$

A deformation of the $SU(2)$ principal chiral sigma model

- **The $SU(2)$ Lie algebra:**

$$[T_a\,,\,T_b] = f^c_{ab} T_c$$

- **The $SU(2)$ theory:**

$$S\,(g) = \int \mathrm{d}z\mathrm{d}\bar{z}\,\left[\mathrm{Tr}\,\left(g^{-1}\partial g\, g^{-1}\bar{\partial} g\right) - 2C\,\mathrm{Tr}\,\left(T_3 g^{-1}\partial g\right)\,\mathrm{Tr}\,\left(T_3 g^{-1}\bar{\partial} g\right)\right] \quad ,$$

- **The currents of the Lax pair**

$$\begin{aligned} J &= \partial g g^{-1} + \sqrt{C}\,\partial\left(g T_3 g^{-1}\right) \\ \bar{J} &= \bar{\partial} g g^{-1} - \sqrt{C}\,\bar{\partial}\left(g T_3 g^{-1}\right) \end{aligned}$$

A unique model ??

- **The action:**

$$S = \int \mathrm{d}z\mathrm{d}\bar{z} \left[Q_{ij}(\varphi)\, \partial\varphi^i \bar{\partial}\varphi^j - V(\varphi) \right] \quad .$$

A unique model ??

- **The action:**

$$S = \int \mathrm{d}z\mathrm{d}\bar{z} \left[Q_{ij}(\varphi)\,\partial\varphi^i \bar{\partial}\varphi^j - V(\varphi) \right] \quad .$$

The metric tensor and the torsion

$$g_{ij} = \frac{1}{2}\left(Q_{ij} + Q_{ji}\right) \quad , \quad b_{ij} = \frac{1}{2}\left(Q_{ij} - Q_{ji}\right) \quad .$$

A unique model ??

- **The action:**

$$S = \int \mathrm{d}z\mathrm{d}\bar{z}\left[Q_{ij}\left(\varphi\right)\partial\varphi^i\bar{\partial}\varphi^j - V\left(\varphi\right)\right] \quad .$$

The metric tensor and the torsion

$$g_{ij} = \frac{1}{2}\left(Q_{ij} + Q_{ji}\right) \quad , \quad b_{ij} = \frac{1}{2}\left(Q_{ij} - Q_{ji}\right) \quad .$$

- **Question:**

Are there integrable theories of this type ???

A unique model ??

- **The answer:**

$$
\begin{aligned}
S \;=\; & \int_{\partial\Sigma} \mathrm{d}z\mathrm{d}\bar{z} < h^{-1}\partial h \,,\, h^{-1}\bar{\partial} h > \\
& + \frac{1}{6}\int_{\Sigma} \mathrm{d}^3 y \epsilon^{\mu\nu\sigma} < h^{-1}\partial_\mu h \,,\, \left[h^{-1}\partial_\nu h \,,\, h^{-1}\partial_\sigma h \right] > \\
& + 2\int_{\partial\Sigma} \mathrm{d}z\mathrm{d}\bar{z} \, < \rho_0 \,,\, h^{-1}\omega_0 h >
\end{aligned}
$$

A unique model ??

- **Equations of motion:**

$$\partial\left(h^{-1}\bar{\partial}h\right)-\left[\rho_0\,,\,h^{-1}\omega_0 h\right]=0\quad .$$

A unique model ??

- **Equations of motion:**

$$\partial\left(h^{-1}\bar{\partial}h\right) - \left[\rho_0\,,\, h^{-1}\omega_0 h\right] = 0 \quad .$$

- **The linear system:**

$$\begin{aligned}\left(\partial + \lambda\, h^{-1}\omega_0 h\right)\Psi &= 0\\ \left(\bar{\partial} + h^{-1}\bar{\partial}h + \frac{1}{\lambda}\,\rho_0\right)\Psi &= 0 \quad .\end{aligned}$$

Things to do

- **The general solution to the master equations of integrability**

Things to do

- **The general solution to the master equations of integrability**
- **The equivalent of the G/H coset theories**

Things to do

- The general solution to the master equations of integrability
- The equivalent of the G/H coset theories
- Relation to Poisson-Lie duality

Thank you for your attention