Applications of Lusternik-Schnirelmann Category and its Generalizations

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Lecture 3: LS Category and Non-Negative Curvature

Recall that we defined a refinement of our definition of LS category.

The *1-category* of a space X, denoted $\mathrm{cat}_1(X)$, is the least integer n so that X may be covered by open sets $U_0, \ldots, U_n$ having the property that, for each U_i, there is a partial section $s_i\colon U_i \to \widetilde{X}$, where $p\colon \widetilde{X} \to X$ is the universal cover (so $p \circ s_i$ is homotopic to the inclusion $U_i \hookrightarrow X$).

Recall properties of $\mathrm{cat}_1(X)$.

(1.) $\mathrm{cat}_1(X) = \mathrm{cat}(j_1 : X \to K(\pi_1 X, 1))$.

The category on the right is ***the category of a map***.

Theorem. If $\pi_1(X) = \pi$, $B\pi = K(\pi, 1)$ and k is the maximum degree for which $j_1^* : H^k(B\pi; \mathcal{A}) \to H^k(X; \mathcal{A})$ is non-trivial (for any local coefficients $\mathcal{A}$), then

$$k \leq \mathrm{cat}_1(X) \leq \mathrm{cat}(B\pi) = \dim(B\pi).$$

Moreover, if $X = K(\pi, 1)$, then $\mathrm{cat}_1(X) = \dim(B\pi)$ (for $\dim(B\pi) > 3$).

Examples: If X is simply connected, then $\mathrm{cat}_1(X) = 0$. Also, $\mathrm{cat}_1(T^n) = n$.

The next two properties are more or less general properties of category-type invariants.

(2.) $\mathsf{cat}_1(X \times Y) \leq \mathsf{cat}_1(X) + \mathsf{cat}_1(Y)$.

(3.) If X is a CW complex and $p\colon \overline{X} \to X$ is a covering space, then $\mathsf{cat}_1(\overline{X}) \leq \mathsf{cat}_1(X)$.

Now let's turn to geometry.

Cheeger-Gromoll Theorem. If M has non-negative Ricci curvature, then there is a finite cover $\overline{M} \cong T^k \times N$ such that N is 1-connected.

Theorem. (Bochner) If M has non-negative Ricci curvature, then $b_1(M) \leq \dim(M)$. Moreover, equality holds if and only if $M \cong T^n$ (where the torus is flat).

Theorem. (Oprea) If M has non-negative Ricci curvature, then $b_1(M) \leq \mathrm{cat}(M)$. Moreover, equality holds if and only if $M \cong T^n$ (where the torus is flat).

Theorem. (Oprea-Strom) If M has non-negative Ricci curvature, then $b_1(M) \leq \mathrm{cat}_1(M)$.

Example. $M = T^2 \times S^2$ has a metric with non-negative Ricci curvature.

$b_1(M) = 2$.

$\mathrm{cat}_1(M) = 2$: This follows by $j_1 : T^2 \times S^2 \to T^2$ and

$2 \leq \mathrm{cat}_1(T^2 \times S^2) \leq \mathrm{cat}_1(T^2) + \mathrm{cat}_1(S^2) \leq 2 + 0 = 2$.

So equality holds, but M is not a torus!

Proof of Theorem.

By Cheeger-Gromoll, there is a splitting $\overline{M} \cong T^k \times N$ of a finite cover $\overline{M} \to M$.

$b_1(M) \leq b_1(\overline{M}) = b_1(T^k) = k$ by transfer for finite covers.

$k = \text{cat}_1(T^k \times N) = \text{cat}_1(\overline{M}) \leq \text{cat}_1(M)$.

Now let's look at "1-category ideas" in the context of other types of curvature.

ANSC Manifolds

A closed smooth manifold M^m is said to be *almost non-negatively (sectionally) curved* (or *ANSC*) if it admits a sequence of Riemannian metrics $\{g_n\}_{n\in\mathrm{N}}$ whose sectional curvatures and diameters satisfy

$$\sec(M, g_n) \geq -\frac{1}{n} \quad \text{and} \quad \operatorname{diam}(M, g_n) \leq \frac{1}{n}.$$

Theorem. (Yamaguchi) If M^m is an ANSC manifold, then
(1.) a finite cover of M is the total space of a fibration over a torus of dimension $b_1(M)$;
(2.) if $b_1(M) = m$, then M^m is diffeomorphic to $T^{b_1(M)}$.

Theorem. (Kapovitch-Petrunin-Tuschmann) If M is an ANSC manifold, then there is a finite cover $\overline{M}$ that is the total space of a fiber bundle

$$F \to \overline{M} \xrightarrow{p} N,$$

where $N = K(\pi, 1)$ is a nilmanifold and F is a simply connected closed manifold which is almost non-negatively curved in a generalized sense.

What is the connection between Yamaguchi and KPT ???

Theorem. (Oprea-Strom) Suppose M is an ANSC manifold with associated finite cover $\overline{M}$ and fiber bundle

$$F \to \overline{M} \xrightarrow{p} N,$$

where $N = K(\pi, 1)$ is a nilmanifold and F is a simply connected closed manifold. Then

(i.) $b_1(M) \leq \dim(N) \leq \dim(\overline{M}) = \dim(M)$;

(ii.) if the universal cover $\widetilde{M}$ has non-zero Euler characteristic, then $b_1(M) \leq \dim(N) \leq \mathrm{cat}_1(M)$.

Proof. We know $b_1(M) \leq b_1(\overline{M})$. But $H_1(\overline{M};\mathbb{Q}) \cong H_1(\pi;\mathbb{Q}) \cong H_1(N;\mathbb{Q})$, so $b_1(\overline{M}) = b_1(N)$.

Now, N is a nilmanifold, so it has a (rational homotopy theoretic) minimal model $(\wedge(x_1, x_2, \ldots, x_k), d)$, where each generator has degree$(x_j) = 1$ and k is the rank of the torsionfree nilpotent group π.

By the general theory, the differential d is zero on $x_1, \ldots, x_s$ for some $2 \leq s \leq k$ and $k = \dim(N)$. (The case $s = k$ is a torus.)

Then $b_1(N) = s \leq k = \dim(N)$. Since $F \to \overline{M} \xrightarrow{p} N$ is a bundle, we see that $\dim(N) \leq \dim(\overline{M}) = \dim(M)$. This proves (i.), $b_1(M) \leq \dim(M)$.

For (ii.), because $F \simeq \widetilde{M}$ and $\chi(\widetilde{M}) \neq 0$, the bundle $F \to \overline{M} \xrightarrow{p} N$ has a transfer map $\tau \colon H^*(\overline{M}; \mathbb{Z}) \to H^*(N; \mathbb{Z})$ with $\tau \circ p^*(\alpha) = \chi(F) \cdot \alpha$, for all $\alpha \in H^*(N; \mathbb{Z})$.

This implies that

$$p^* \colon H^*(N) = H^*(K(\pi, 1)) \to H^*(\overline{M})$$

is injective on rational cohomology.

Hence, $\dim(N) \leq \mathrm{cat}_1(\overline{M})$. Thus,

$$b_1(M) \leq b_1(N) \leq \dim(N) \leq \mathrm{cat}_1(M).$$

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KPT → Yamaguchi

Theorem. (Oprea-Strom) Suppose a closed manifold M has a finite cover $\overline{M}$ that is the total space of a fiber bundle

$$F \to \overline{M} \xrightarrow{p} N,$$

where $N = K(\pi, 1)$ is a nilmanifold and F is a simply connected closed manifold. Then

(1.) a finite cover of M is the total space of a fibration over a torus of dimension $b_1(M)$;

(2.) if $b_1(M) = m = \dim(M)$, then M^m is homeomorphic to $T^{b_1(M)}$.

Proof. Consider $F \to \overline{M} \xrightarrow{p} N$:

$$b_1(M) \leq b_1(\overline{M}) = b_1(N) \leq \dim(N) \\ \leq \dim(\overline{M}) = \dim(M).$$

The general construction of the nilmanifold N via iterated principal S^1-bundles shows that we may start the iteration by a bundle over $T^{b_1(\overline{M})}$ or any torus of lower dimension. Thus, (1.) follows since a composition of fibrations is a fibration.

Now assume $b_1(M) = m = \dim(M)$.
Then $\dim(N) = m = \dim(\overline{M})$.

Hence, $\dim(F) = 0$ and (since F is connected) we have $\overline{M} = N$.

$\Rightarrow$ $b_1(N) = m = \mathrm{dim}(N)$

For a nilmanifold, this can only happen if N is a torus T^m and $\pi \cong \mathbb{Z}^m$.

Now, $\overline{M} = T^m$ covers M, so M is a $K(G, 1)$ where $G = \pi_1(M)$.

Since M is a closed m-manifold, we have that G is torsion-free.

Now, $\pi \cong \mathbb{Z}^m$ has finite index in G and $b_1(\pi) = m = b_1(M) = b_1(G)$, so $G \cong \mathbb{Z}^m$ by

Lemma. If $\pi \cong \mathbb{Z}^m$ is a finite index subgroup of a torsion-free group G and $b_1(G) = m$, then $G \cong \mathbb{Z}^m$.

Hence $M = K(\mathbb{Z}^m, 1)$ is a homotopy torus.

Hence, M is then homeomorphic to T^m.

And this is then a topological version of Yamaguchi's Bochner-type result.

There are other Bochner-type results. Here is an example.

Theorem. (Oprea-Strom) Suppose M is an ANSC manifold with associated finite cover $\overline{M}$ and fiber bundle

$$F \to \overline{M} \xrightarrow{p} N,$$

where $N = K(\pi, 1)$ is a symplectic nilmanifold and F is a simply connected closed manifold. If $\widetilde{M}$ has non-zero Euler characteristic (or more generally, p^* is injective), then

$$\mathrm{cat}_1(M) \geq \mathrm{cat}_1(\overline{M}) \geq b_1(\overline{M}) \geq \mathrm{rank}(\mathcal{Z}\pi).$$