Applications of Lusternik-Schnirelmann Category and its Generalizations

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Lecture 4: New LS Categorical Ideas in Applied Mathematics

LS category also has a place in applied mathematics besides critical point theory. We will look at two examples, one “old” and one “new”.

In order to do this, we need to recall the “new” notion of “category” called **sectional category**.

Sectional Category.

Suppose $F \to E \xrightarrow{p} B$ is a fibration. Then the *sectional category* of p, denoted secat(p), is the least integer n such that there exists an open covering, $U_1, \ldots, U_{n+1}$, of B and, for each U_i, a map $s_i\colon U_i \to E$ having $p \circ s_i = \mathrm{incl}_{U_i}$. (That is, s_i is a local section of p).

Properties:

(0) $\mathrm{secat}(p\colon \widetilde{X} \to X) = \mathrm{cat}_1(X)$.

(1) $\mathrm{secat}(p) \leq \mathrm{cat}(B)$.

(2) If E is contractible, then $\mathrm{secat}(p) = \mathrm{cat}(B)$.

(3) If there are $x_1, \ldots, x_k \in \widetilde{H}^*(B; R)$ (any coefficient ring R) with

$$p^* x_1 = \ldots = p^* x_k = 0 \quad \text{and} \quad x_1 \cup \cdots \cup x_k \neq 0,$$

then secat$(p) \geq k$.

(4) Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration arising as a pullback of a fibration $\widehat{p} \colon \widehat{E} \to \widehat{B}$ where $\widehat{E}$ is contractible (such as a principal bundle).

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{f}} & \widehat{E} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widehat{p}} \\ B & \xrightarrow{f} & \widehat{B} \end{array}$$

Then secat$(p) = $ cat(f).

Smale's Topological Complexity of Algorithms

An *algorithm tree* is a connected directed graph G with vertices $\{R, V_1, \ldots, V_N, L_1, \ldots, L_m\}$ satisfying the following conditions.

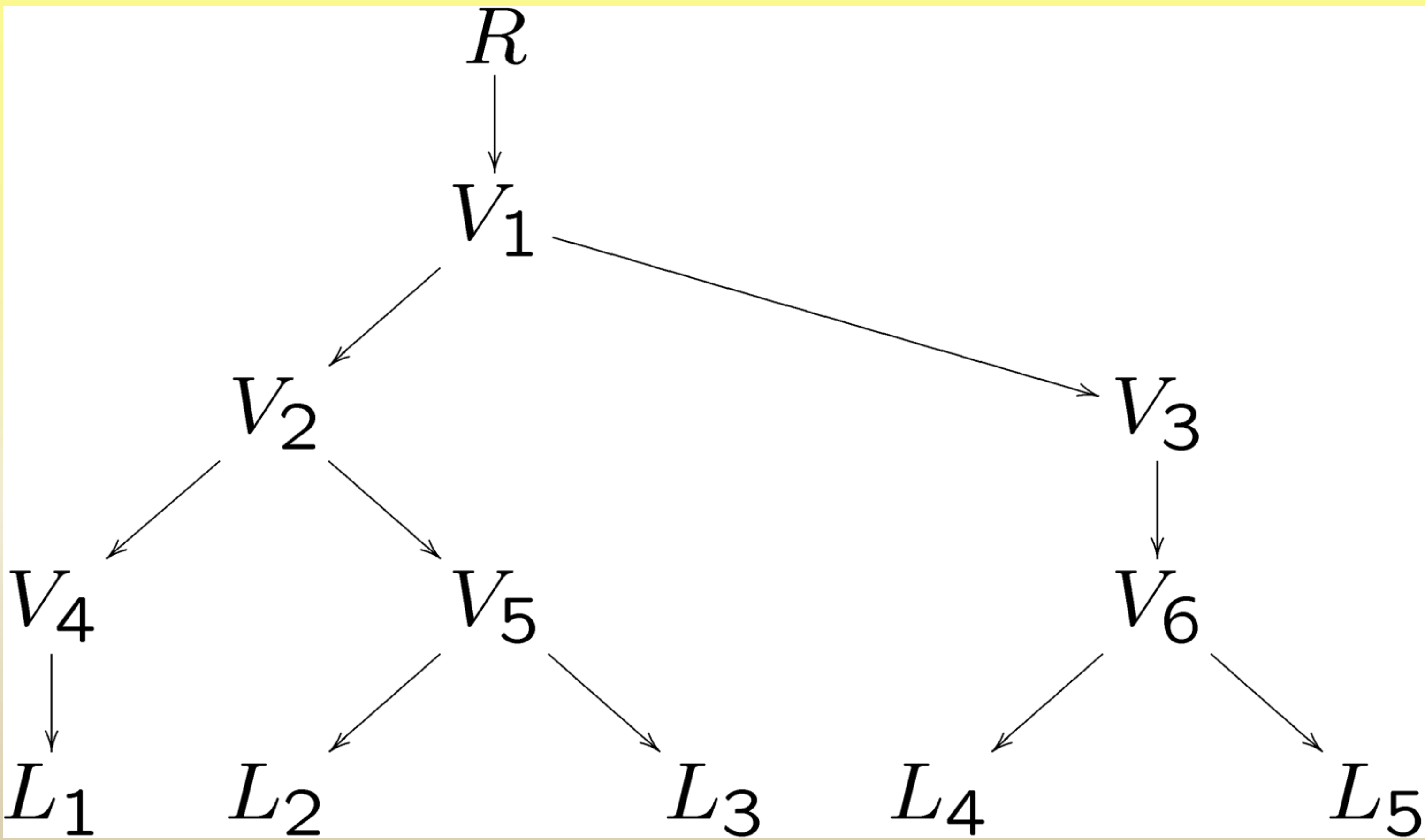
(1) There are no loops, i.e., G is a tree.

(2) The *root* R has only one edge and that edge comes out of R.

(3) Each V_i has one edge coming into it and either one or two edges coming out of it. Those V_i with one edge coming out are called *computation vertices* and those with two edges coming out are called *branch vertices*.

(4) Each *leaf* L_j has only one edge coming into it.

Example: An Algorithm Tree G



The *Smale topological complexity* of an algorithm tree G, or of the algorithm that it describes, is defined to be the number of branch vertices in G. This is also equal to one fewer than the number of leaves in the tree:

$$\#\{L_j\} - 1 = \#\{\text{Branch } V_i\} = \tau(G).$$

The *Smale topological complexity* of a particular problem P is the minimum of the topological complexities of all algorithms which solve the problem:

$$\tau(P) = \min\{\tau(G) \,|\, G = \text{algorithm tree for } P\}.$$

Example: Root Finding Problem.

Find the roots of $P(x)$, where

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{d-1} x^{d-1} + x^d.$$

is a monic polynomial of degree d with complex coefficients.

Here, the word "find" is taken to mean "find to within a given accuracy ϵ".

Theorem (Vassiliev). There exists an algorithm tree of topological complexity $d - 1$ for the problem of determining roots of degree d monic polynomials to within given $\epsilon > 0$. Thus, the topological complexity of the problem is at most $d - 1$.

$\mathcal{P}_d$ = the set of degree d monic polynomials with complex coefficients.

Take the mapping $\pi\colon \mathbb{C}^d \to \mathcal{P}_d$ given by

$$\pi(\xi_1, \ldots, \xi_d) = \prod_{i=1}^{d} (x - \xi_i).$$

Let

$$\Delta = \{(\xi_1, \ldots, \xi_d)\colon \xi_i = \xi_j \text{ for some } i \neq j\} \subseteq \mathbb{C}^d;$$

$\pi(\Delta) = \Sigma = \{\text{Polynomials with repeated roots}\} \subseteq \mathcal{P}_d$.

Denote the restriction

$$\pi|_{\mathbb{C}^d - \Delta}\colon \mathbb{C}^d - \Delta \to \mathcal{P}_d - \Sigma$$

by π as well.

The symmetric group on d letters, $S(d)$, acts without fixed points on $\mathbb{C}^d - \Delta$, so π is a $(d\,!)$-fold covering map with $S(d)$ permuting the d coordinates ξ_i.

Theorem. (Smale) For any d there exists $\epsilon_d > 0$ such that, for $\epsilon < \epsilon_d$, the topological complexity $\tau(P(d,\epsilon))$ for the problem of finding roots of degree d monic polynomials to within ϵ is at least the sectional category of the covering $\pi\colon \mathbb{C}^d - \Delta \to \mathcal{P}_d - \Sigma$. That is,

$$\tau(P(d,\epsilon)) \geq \text{secat}(\pi).$$

Proof Sketch. (Can reduce to case of no repeated roots.)

(i) Let the solution algorithm tree be G with vertices $\{R, V_1, \ldots, V_N, L_1, \ldots, L_m\}$.

(ii) Define Z_i to be the set

$\{P(x) \in \mathcal{P}_d - \Sigma$: the output of the algorithm tree G applied to $P(x)$ exits the tree through leaf $L_i\}$.

(iii) Define (since an algorithm exists) an *input-output map* $\phi: \mathcal{P}_d - \Sigma \rightarrow \mathbb{C}^d$ by $\phi(P(x)) = (z_1, z_2, \ldots, z_d)$, where each z_i satisfies $|z_i - \xi_i| < \epsilon$, for ξ_i the true roots of $P(x)$.

(iv) The branch inequalities say that Z_i is a semi-algebraic set, so the Tietze extension theorem gives an open set U_i containing Z_i and an extension $\phi: U_i \rightarrow \mathbb{C}^d$.

These are "sections to within ϵ". They can be deformed to actual sections, so

$$\text{secat}(\pi) \leq m - 1 = \#(\text{branches}) = \tau(P(d,\epsilon)).$$

Combined with Vassiliev's result, we have

$$d - 1 \geq \tau(P(d,\epsilon)) \geq \text{secat}(\pi).$$

The covering map $\pi\colon \mathbb{C}^d - \Delta \to \mathcal{P}_d - \Sigma$ is a principal bundle induced by some classifying map $f\colon \mathcal{P}_d - \Sigma \to K(S(d), 1)$. Then, we have $\text{secat}(\pi) = \text{cat}(f)$

So how do we calculate $\text{cat}(f)$?

In fact, $\mathbb{C}^d - \Delta = \mathbb{R}^2[d]$, the *ordered configuration space* of d points in $\mathbb{R}^2$.

The symmetric group $S(d)$ acts freely on $\mathbb{R}^2[d]$ by permuting the ordering and the resulting quotient is the d*th unordered configuration space*, denoted $\mathbb{R}^2(d)$, which is $P_d - \Sigma$.

Now, $\mathbb{R}^2(d) = K(Br(d), 1)$, where $Br(d)$ is the *braid group* and there is a homomorphism $Br(d) \to S(d)$ arising from the classifying map $f\colon \mathbb{R}^2(d) \to K(S(d), 1)$ of the covering projection $\mathbb{R}^2[d] \to \mathbb{R}^2(d)$.

Theorem (Arnold-Fuchs-Vassiliev).

$$H^{d-1}(K(Br(d),1);\pm\mathbb{Z}) = \begin{cases} 0 & \text{if } d \neq p^q, \text{ for } p \text{ prime} \\ \mathbb{Z}_p & \text{if } d = p^q, \text{ for } p \text{ prime} \end{cases}$$

Moreover, the homomorphism $f^*\colon H^*(K(S(d),1);\pm\mathbb{Z}) \to H^*(K(Br(d),1);\pm\mathbb{Z})$ is surjective.

Corollary. For d equal to a power of some prime, there exists $\epsilon_d > 0$ such that, for $\epsilon < \epsilon_d$,

$$\tau(P(d,\epsilon)) = d - 1.$$

Proof. Since f^* is surjective and $H^{d-1}(Br(d);\pm\mathbb{Z}) \neq 0$, there exists $u \in H^{d-1}(S(d);\pm\mathbb{Z})$ with $f^*(u) \neq 0$. Hence,

$$\text{secat}(\pi) = \text{cat}(f) \geq \text{wgt}(u) \geq d - 1.$$

Farber's Topological Complexity of Motion Planning

A mechanical system $\mathcal{S}$ is described by its totality of states $X = X(\mathcal{S})$; this is the *configuration space* of $\mathcal{S}$.

Example. (1) A planar pendulum has configuration space S^1.

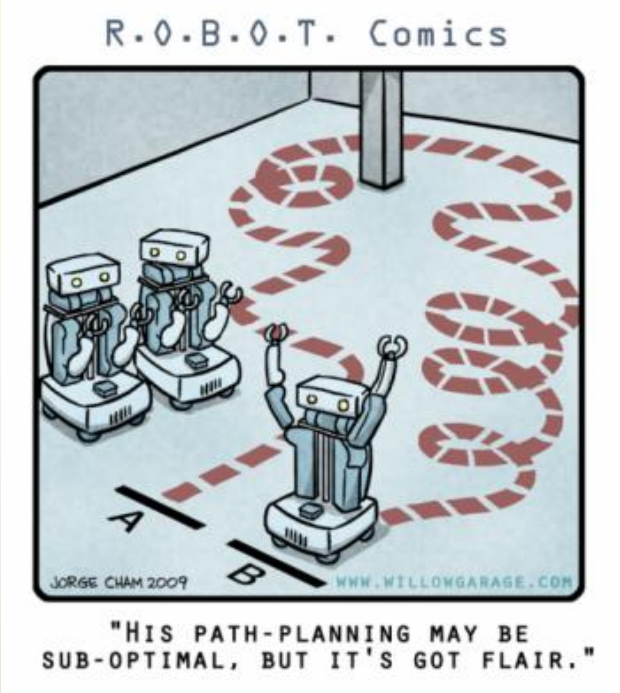
(2) A planar double pendulum has configuration space $T^2 = S^1 \times S^1$.

(3) A planar n-pendulum (or planar robot arm with n bars) has configuration space T^n.

(4) If n particles (or robots) move in a space Y to avoid collisions, then the configuration space is

$$F(Y,n) = \{(y_1,\ldots,y_n) \in Y^n \,|\, y_i \neq y_j,\ \forall i \neq j\}.$$

(5) If n robots move along a set wire system in the floor of a factory, say, then they move on a graph Γ.
The configuration space is then $F(\Gamma,n)$.



The Motion Planning Problem.

Let X be the configuration space of a system $\mathcal{S}$. The *motion planning problem* is to find a continuous path $\gamma\colon I \to X$ with $\gamma(0) = A$ and $\gamma(1) = B$ for any $A, B \in X$.

We make the following requirements:
(1) The process of finding the path should work for all pairs of points;
(2) The process should be fully automated (i.e. algorithmic).

This is a Motion Planning Algorithm.

Let $ev\colon X^I \to X \times X$ be the evaluation fibration $ev(\gamma) = (\gamma(0), \gamma(1))$. A *motion planning algorithm* is a continuous section

$$s\colon X \times X \to X^I, \quad ev \circ s = 1_{X \times X}.$$

Theorem. A motion planning algorithm s exists if and only if X is contractible!

Proof. If $X \simeq *$, let $H\colon X \times I \to X$ have $H(x, 0) = x$ and $H(x, 1) = B_0$. Given A, B, define

$$\gamma(t) = \begin{cases} H(A, 2t) & 0 \le t \le 1/2 \\ H(B, 2-2t) & 1/2 \le t \le 1. \end{cases}$$

If s exists, define $H\colon X \times I \to X$ by $H(A,t) = s(A, B_0)(t)$. Then $H(A,0) = A$ and $H(A,1) = B_0$ since s is a section.

So, in general, there is no Motion Planning Algorithm.

What do we do? **Measure the deviation**!

Definition. The *Topological Complexity* of the motion planning algorithm problem for X is

$$\mathrm{TC}(X) = \mathrm{secat}(ev\colon X^I \to X \times X).$$

Topological Complexity is estimated by LS Category.

Theorem. The following estimates hold:

$$\mathsf{cat}(X) \le \mathsf{TC}(X) \le \mathsf{cat}(X \times X). \le 2\,\mathsf{cat}(X)$$

Proposition. If X is a Lie group, then $\mathsf{TC}(X) = \mathsf{cat}(X)$.

Example. $\mathsf{TC}(T^n) = \mathsf{cat}(T^n) = n$.

Example. $\mathsf{TC}(\mathbb{R}P^3) = \mathsf{TC}(SO(3)) = 3$.

Example.

$$\mathsf{TC}(S^n) = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even.} \end{cases}$$

$$\mathsf{TC}(F(\mathbb{R}^m, n)) = \begin{cases} 2n-2 & m \text{ odd} \\ 2n-3 & m \text{ even.} \end{cases}$$

Theorem. Let Γ be a connected graph with at least one vertex of degree ≥ 3. Then

$$\mathsf{TC}(F(\Gamma, n)) \leq 2\, m(\Gamma),$$

where $m(\Gamma)$ is the number of vertices of degree ≥ 3.

If $n \geq m(\Gamma) > 2$, then $\mathsf{TC}(F(\Gamma, n)) = 2\, m(\Gamma)$.

The Topological Complexity of real projective spaces has a fascinating connection to a classical question in topology.

Theorem. For $n \neq 1, 3, 7$, $\mathsf{TC}(\mathbb{R}P^n)$ is the smallest k such that $\mathbb{R}P^n$ admits an immersion into $\mathbb{R}P^k$.

Open Question. If G is a discrete group of finite cohomological dimension $\mathrm{cd}(G) < \infty$, can $\mathrm{TC}(K(G,1)) = \mathrm{TC}(G)$ be described by the algebraic structure of G?

New Theorem. (Grant-Lupton-Oprea) If A and B are complementary subgroups of G (i.e. $AB = G$ and $A \cap B = \emptyset$), then

$$\mathsf{cd}(A \times B) \leq \mathsf{TC}(G).$$

The mathematical facts worthy of being studied are those which, by their analogy with other facts, are capable of leading us to the knowledge of a mathematical law just as experimental facts lead us to the knowledge of a physical law. They reveal the kinship between other facts, long known, but wrongly believed to be strangers to one another.

----- Henri Poincaré