The two extremely opposite but hidden-related achievements of Euler: rigid body and ideal fluid. And our unifying “go between”: Affinely rigid body and affine invariance in physics

prof. Jan Jerzy Sławianowski

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Euler equations for the rigid body without translational motion of the centre of mass.

Configuration space — orthogonal, orientation-preserving matrices:

$$Q = \mathrm{SO}(3,\mathbb{R}) = \left\{\varphi \in \mathrm{GL}(3,\mathbb{R}): \ \varphi^T\varphi = I, \ \det\varphi = +1\right\}.$$

Motion — curves in Q:

$$\mathbb{R} \ni t \mapsto \varphi(t) \in Q.$$

Angular velocities — non-holonomic ones:

- spatial:

$$\Omega = \frac{d\varphi}{dt}\varphi^{-1} = \varphi\widehat{\Omega}\varphi^{-1} = -\Omega^T,$$

- co-moving:

$$\widehat{\Omega} = \varphi^{-1}\frac{d\varphi}{dt} = \varphi^{-1}\Omega\varphi = -\widehat{\Omega}^T.$$

In three dimensions there is an isomorphism between skew-symmetric second-order tensors and axial vectors:

$$\Omega = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}, \qquad \widehat{\Omega} = \begin{bmatrix} 0 & -\widehat{\Omega}_3 & \widehat{\Omega}_2 \\ \widehat{\Omega}_3 & 0 & -\widehat{\Omega}_1 \\ -\widehat{\Omega}_2 & \widehat{\Omega}_1 & 0 \end{bmatrix}, \qquad \Omega^i = \varphi^i{}_A\widehat{\Omega}^A.$$

Kinetic energy of rotations:

$$T = \sum_{A=1}^{3}\frac{I_A}{2}\left(\widehat{\Omega}^A\right)^2 = \frac{1}{2}I_{AB}\widehat{\Omega}^A\widehat{\Omega}^B, \qquad I_A, I_{AB} - \text{constant}.$$

For any $U \in \mathrm{SO}(3,\mathbb{R})$ the transformation of left translations: $\varphi \mapsto U\varphi$ preserves $\widehat{\Omega}$ and T — left-invariant kinetic energy. Right-invariant — only in the case of spherical top: $I_1 = I_2 = I_3 = I$.
Non-geodetic equations of motion:

$$\begin{aligned} I_1 \frac{d\widehat{\Omega}_1}{dt} &= (I_2 - I_3)\,\widehat{\Omega}_2\widehat{\Omega}_3 + \widehat{N}_1, \\ I_2 \frac{d\widehat{\Omega}_2}{dt} &= (I_3 - I_1)\,\widehat{\Omega}_3\widehat{\Omega}_1 + \widehat{N}_2, \\ I_3 \frac{d\widehat{\Omega}_3}{dt} &= (I_1 - I_2)\,\widehat{\Omega}_1\widehat{\Omega}_2 + \widehat{N}_3, \end{aligned}$$

where $\widehat{N}_a$ — co-moving component of torque (moment of forces), $\widehat{N} = \varphi^{-1}N$.
$\widehat{\Omega}$ — autonomous in geodetic case: $\widehat{N} = 0$, $N = 0$.
Geodetic equations — left-invariant:

$$\varphi \mapsto U\varphi.$$

Co-moving and laboratory spin:

$$\widehat{\Sigma}_A = \frac{\partial T}{\partial \widehat{\Omega}^A} = I_{\underline{A}}\Omega^{\underline{A}} = I_{AB}\Omega^B, \qquad \Sigma_a = \widehat{\Sigma}_B \left(\varphi^{-1}\right)^B{}_a.$$

Expression of the kinetic energy:

$$\mathcal{T} = \sum_{A=1}^{3} \frac{1}{2I_A}\widehat{\Sigma}_A^2 = \frac{1}{2}\widetilde{I}^{AB}\widehat{\Sigma}_A\widehat{\Sigma}_B.$$

Poisson brackets:

$$\{\Sigma_a, \Sigma_b\} = \varepsilon_{ab}{}^c\Sigma_c, \qquad \{\widehat{\Sigma}_A, \widehat{\Sigma}_B\} = -\varepsilon_{AB}{}^C\widehat{\Sigma}_C, \qquad \{\Sigma_a, \widehat{\Sigma}_B\} = 0.$$

Euler equations in spin terms:

$$\begin{aligned}
\frac{d\widehat{\Sigma}_1}{dt} &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right)\widehat{\Sigma}_2\widehat{\Sigma}_3 + \widehat{N}_1,\\
\frac{d\widehat{\Sigma}_2}{dt} &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\widehat{\Sigma}_3\widehat{\Sigma}_1 + \widehat{N}_2,\\
\frac{d\widehat{\Sigma}_3}{dt} &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right)\widehat{\Sigma}_1\widehat{\Sigma}_2 + \widehat{N}_3.
\end{aligned}$$

In doubly-invariant (spherical) rigid body:

$$\frac{d\widehat{\Sigma}_a}{dt} = \widehat{N}_a,$$

and in general:

$$\frac{d\Sigma_a}{dt} = N_a,$$

but look at the difference between the general Σ_a-spin balance and the balance for $\widehat{\Sigma}_a$, dependent on I_a-s. Geodetic motions of the isotropic rigid body ($I_1 = I_2 = I_3$) are given by:

$$\varphi(t) = \exp(t\omega)\varphi(0),$$

where ω — arbitrary skew-symmetric. In the general anisotropic case such solutions do exist only as rotations about the spatially non-moving main axes of inertia. They are Lapunov-stable for rotations about the main axes with extremal values of I_a.

What is the relationship with the hydrodynamics of the ideal incompressible fluid?

$$\varrho \frac{d\overline{v}}{dt} = \varrho \left(\frac{\partial \overline{v}}{\partial t} + (\overline{v} \cdot \nabla)\, \overline{v} \right) = -\nabla p + \varrho \overline{g}.$$

Iso-entropic motion:

$$\begin{aligned}
&\frac{ds}{dt} = \frac{\partial s}{\partial t} + \overline{v} \,\mathrm{grad}\, s = 0, \\
&\frac{\partial\left(\varrho s\right)}{\partial t} + \mathrm{div}\left(\varrho s \overline{v}\right) = 0, \\
&\frac{\partial}{\partial t}\varrho v^i = -\frac{\partial \Pi^{ik}}{\partial x^k}, \qquad \Pi^{ik} = p g^{ik} + \varrho v^i v^j, \\
&< \overline{v}_1, \overline{v}_2 >= \int\limits_D \overline{v}_1 \cdot \overline{v}_2 dx \qquad - \qquad \text{scalar product,}
\end{aligned}$$

div $\overline{v} = 0$ in D (incompressibility), $\overline{v}$ — tangent to the boundary ∂D (D — region occupied by fluid).
Kinetic energy:

$$T = \frac{\varrho}{2} < \overline{v}, \overline{v} >= \frac{\varrho}{2} \int\limits_D g_{ij} v^i v^j dx$$

(dx — Riemannian volume).

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- Time instant t, configuration: $g_t \in \mathrm{SDiff}\ D$.
- Time instant $t+\tau$, configuration: $\exp\left(\overline{v}\tau\right) g_t$ (τ — small).

Velocity field $\overline{v}$ obtained from $\dot{g}$ tangent at g to the group SDiff D under the right action.
Attention! SDiff D is not a Lie group!

Our idea: To admit deformations, but finite-dimensional. Affine philosophy of Thales of Miletus. GAff-invariance.

Introduction: systems on Lie groups in general.
G — a Lie group, usually linear, e.g.,
$G \subset \mathrm{GL}(N, \mathbb{R})$, $G \subset \mathrm{GL}(N, \mathbb{C})$ (but real, e.g., $U(n)$)
$G' \subset \mathrm{L}(n, \mathbb{R}) = T_e\mathrm{GL}(n, \mathbb{R})$ or $G' \subset \mathrm{L}(n, \mathbb{C}) = T_e\mathrm{GL}(n, \mathbb{C})$ — Lie algebra,

$$q\left(t^1, \ldots, t^k\right) = \exp\left(t^k E_k\right), \qquad [E_k, E_j] = C^m{}_{kj} E_m$$

G'^* — Lie co-algebra.
Typically:

$$G'^* \simeq G', \qquad < f, x >= \mathrm{Tr}(fx).$$

Motion:

$$\mathbb{R} \ni t \mapsto q(t) \in G.$$

Lie-algebraic velocities:

$$\Omega(t) = \dot{q}(t)q(t)^{-1}, \qquad \widehat{\Omega} = q(t)^{-1}\dot{q}(t),$$
$$\Omega(t) = \mathrm{Ad}_{q(t)}\widehat{\Omega}(t), \qquad \mathrm{Ad}_q(x) = qxq^{-1}.$$

In terms of dual bases $\{\ldots, E_a, \ldots\}$, $\{\ldots, E^a, \ldots\}$ in G' and G'^*:

$$\Omega = \Omega^a(t)E_a, \qquad \widehat{\Omega} = \widehat{\Omega}^a(t)E_a, \qquad \Omega^a(t) = \left(\mathrm{Ad}_{q(t)}\right)^a{}_b\widehat{\Omega}^b(t).$$

Quasi-velocities Ω^a, $\widehat{\Omega}^a$ — non-holonomic if G — non-Abelian.
$\gamma \in G'^* \otimes G'^*$ — (pseudo-)Euclidean metric in G':
Left-invariant kinetic energies on G:

$$T = \frac{1}{2}\gamma_{ab}\widehat{\Omega}^a\widehat{\Omega}^b = \frac{1}{2}\gamma\left(\widehat{\Omega}, \widehat{\Omega}\right).$$

Tangent and cotangent bundles are trivial,

$$TG = G \times G', \qquad T^*G = G \times G'^*.$$

Cotangent language — phase spaces,

$$\Sigma = \Sigma_a E^a, \qquad \widehat{\Sigma} = \widehat{\Sigma}_a E^a.$$

Trivialization:

$$\Sigma_a = \Sigma_a{}^i(q)p_i, \qquad \widehat{\Sigma}_a = \widehat{\Sigma}_a{}^i(q)p_i,$$
$$\Sigma_a\Omega^a = \widehat{\Sigma}_a\widehat{\Omega}^a = p_i\dot{q}^i.$$

Transformation properties under group translations:

- left:

$$L_g : x \mapsto gx,$$
$$\Omega \mapsto g\Omega g^{-1} = Ad_g\Omega, \qquad \widehat{\Omega} \mapsto \widehat{\Omega},$$
$$\Sigma \mapsto g\Sigma g^{-1} = Ad_g^{*-1}\Sigma, \qquad \widehat{\Sigma} \mapsto \widehat{\Sigma},$$

- right:

$$R_g : x \mapsto xg,$$
$$\Omega \mapsto \Omega, \qquad \widehat{\Omega} \mapsto g^{-1}\widehat{\Omega}g = Ad_g^{-1}\widehat{\Omega},$$
$$\Sigma \mapsto \Sigma, \qquad \widehat{\Sigma} \mapsto g^{-1}\widehat{\Sigma}g = Ad_g^{*}\widehat{\Sigma}.$$

Poisson brackets:

$$\{\Sigma_i, \Sigma_j\} = C^k{}_{ij}\Sigma_k, \qquad \{\widehat{\Sigma}_i, \widehat{\Sigma}_j\} = -C^k{}_{ij}\widehat{\Sigma}_k, \qquad \{\Sigma_i, \widehat{\Sigma}_j\} = 0,$$
$$\{\Sigma_a, f(q)\} = -\Sigma_a{}^i(q)\frac{\partial f}{\partial q^i}, \qquad \{\widehat{\Sigma}_a, f(q)\} = -\widehat{\Sigma}_a{}^i(q)\frac{\partial f}{\partial q^i}.$$

The others do vanish.
Geometrically:

- Σ_i — Hamiltonian generator of left regular translations (momentum mappings of L_G),
- $\widehat{\Sigma}_i$ — Hamiltonian generator of right regular translations (momentum mappings of R_G).

Non-holonomic representation of Legendre transformation:

$$\Sigma_a = \frac{\partial T}{\partial \Omega^a}, \qquad \widehat{\Sigma}_a = \frac{\partial T}{\partial \widehat{\Omega}^a}.$$

Left-invariant kinetic energy:

$$\mathcal{T} = \frac{1}{2}\widetilde{\gamma}^{ab}\widehat{\Sigma}_a\widehat{\Sigma}_b.$$

Right-invariant kinetic energy:

$$\mathcal{T} = \frac{1}{2}\widetilde{\gamma}^{ab}\Sigma_a\Sigma_b,$$

where

$$\widetilde{\gamma}^{ac}\gamma_{cb} = \delta^a{}_b.$$

Poisson bracket form of equations of motion:

$$\frac{df}{dt} = \{f, H\}, \qquad \text{e.g.,} \qquad H = \mathcal{T} + \mathcal{V}(q).$$

Euler equations for left-invariant models:

$$\frac{d\widehat{\Sigma}_a}{dt} = -\widetilde{\gamma}^{cd}C^b{}_{ac}\widehat{\Sigma}_d\widehat{\Sigma}_b + \widehat{N}_a,$$

e.g.,

$$\widehat{N}_a = \widehat{\Sigma}_a{}^i(q)\frac{\partial \mathcal{V}}{\partial q^i}.$$

In $\widehat{\Omega}$-terms:

$$\gamma_{ab}\frac{d\widehat{\Omega}^b}{dt} = -\gamma_{bd}C^b{}_{ac}\widehat{\Omega}^c\widehat{\Omega}^d + \widehat{N}_a$$

or in mixed terms:

$$\frac{d\widehat{\Sigma}_a}{dt} = -C^b{}_{ac}\widehat{\Omega}^c\widehat{\Sigma}_b + \widehat{N}_a.$$

In geodetic case, $\widehat{N}_a = 0$, equations are autonomously solvable with respect to $\widehat{\Sigma}$ or $\widehat{\Omega}$. Then the evolution $t \mapsto q(t)$ may be found by solving the non-autonomous system:

$$\frac{dq}{dt} = q(t)\widehat{\Omega}.$$

In geodetic models Σ_a are constants of motion:

$$\frac{d\Sigma_a}{dt} = 0,$$

but in non-geodetic case:

$$\frac{d\Sigma_a}{dt} = N_a = \{\Sigma_a, V\} = \Sigma_a{}^i\frac{\partial V}{\partial q^i}.$$

For the right-invariant models of

$$T = \frac{1}{2}\gamma_{ab}\Omega^a\Omega^b$$

equations of motion have the form:

$$\frac{d\Sigma_a}{dt} = \widetilde{\gamma}^{cd}C^b{}_{ac}\Sigma_d\Sigma_b + N_a.$$

Doubly-invariant models of T:

$$\gamma_{ab} = C^{k}{}_{la} C^{l}{}_{kb}$$

— Killing metric tensor on G'.
C is then totally γ-skew-symmetric

$$C^{ijk} = C^{i}{}_{ab} \widetilde{\gamma}^{aj} \widetilde{\gamma}^{bk} = -C^{jik} = -C^{kji} = -C^{ikj}.$$

In the geodetic case the general solution is then exponential:

$$q(t) = \exp(\Omega t) q(0) = q(0) \exp(\widehat{\Omega} t),$$

$$\widehat{\Omega} = q(0)^{-1} \Omega q(0) = Ad^{-1}_{q(0)} \Omega,$$

Ω, $\widehat{\Omega}$ — arbitrary.
In the case of one-side symmetry such solutions, so-called stationary ones do exist only for some special values of Ω, $\widehat{\Omega}$.

We are somewhere between — deformations, but finite dimensions. Affinely-rigid body, homogeneously deformable gyroscope.

$G = \mathrm{GL}(3, \mathbb{R})$, more convenient to use $\mathrm{GL}(n, \mathbb{R})$ and later on to specify $n = 3, 2$.
Better — homogeneous space. $(N, U, \rightarrow, \eta)$ — material space. $(M, V, \rightarrow, g)$ — physical space.

$$\begin{array}{c} Q = \underset{\substack{\uparrow}}{M} \times \underset{\substack{\uparrow}}{\mathrm{LI}(U, V)}, \\ \text{translational/internal motion} \end{array}$$

where $\mathrm{LI}(U, V)$ are linear isomorphisms of U onto V.
If $M = N = U = V = \mathbb{R}^n$,

$$Q = \mathrm{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n, \qquad \Phi \in Q : \quad \Phi(t, a)^i = \varphi^i{}_K(t) a^K + x^i(t)$$

Inertial objects: μ — mass distribution measure in N, it is positive and constant,

$$m = \int_N d\mu(a) \ - \ \text{total mass},$$

$$J^K = \int_N a^K d\mu(a) = 0 \ - \ a^K \text{ vanish at the material centre of mass},$$

$$J^{KL} = \int_N a^K a^L d\mu(a) \quad - \quad \text{inertial tensor, constant}$$
$$\text{(Lagrangian) mass quadrupole.}$$

Kinetic energy obtained in a usual way (summation over material points):

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} g_{ij} \frac{d\varphi^i{}_A}{dt} \frac{d\varphi^j{}_B}{dt} J^{AB}$$

Legendre transformation:

$$p_i = m g_{ij} \frac{dx^j}{dt}, \qquad p^A{}_i = g_{ij} \frac{d\varphi^j{}_B}{dt} J^{BA}$$

Kinetic part of the Hamiltonian:

$$\mathcal{T} = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} g^{ij} p^A{}_i p^B{}_j \widetilde{J}_{AB},$$

where $\widetilde{J}_{AC} J^{CB} = \delta_A{}^B$.
Cauchy deformation tensor:

$$C_{ij} = \eta_{AB} \left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j, \qquad C^{ij} = \varphi^i{}_A \varphi^j{}_B \eta^{AB}.$$

Green deformation tensor:

$$G_{AB} = g_{ij} \varphi^i{}_A \varphi^j{}_B, \qquad G^{AB} = \left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j g^{ij}.$$

Non-doubtful range of applications of models.

$$L = T - V(\varphi), \qquad H = \mathcal{T} + V(\varphi)$$

- macroscopic elasticity when the length of excited waves is comparable with the linear size of the body,
- micromorphic continua with internal degrees of freedom ruled by linear group [Eringen],
- molecular vibrations, molecular crystals,
- nuclear dynamics (collective droplet model of the atomic nuclei),
- astrophysical objects, vibrating stars, shape of Earth,
- integrable one-dimensional lattices and n-dimensional affinely-rigid body.

Drawbacks:

1. Geodetic models (without potentials — nonphysical, nonphysical - no vibrations, non-limited expansion and contraction)
2. No dynamical affine invariance — only kinematical one. Advantages of the group structure lost.

What would be affine models?
Do exist formally? Are realistic?

Canonical objects, transformations, generators

$$p_i,\ p^A{}_i \qquad \text{conjugate to} \qquad x^i,\ \varphi^i{}_A$$

Legendre:

$$p_i = \frac{\partial T}{\partial v^i} = m g_{ij} v^j, \qquad p^A{}_i = \frac{\partial T}{\partial \dot{\varphi}^i{}_A} = g_{ij} \dot{\varphi}^j{}_B J^{BA}$$

Lie-algebraic objects:

$$\Omega = \dot{\varphi}\varphi^{-1}, \qquad \Omega^i{}_j = \dot{\varphi}^i{}_A \varphi^{-1A}{}_j,$$
$$\widehat{\Omega} = \varphi^{-1}\dot{\varphi}, \qquad \widehat{\Omega}^A{}_B = \varphi^{-1A}{}_i \dot{\varphi}^i{}_B.$$

Affine velocities. Eringen's "gyration".
Their g- and η-skew-symmetric parts — angular velocity. They are always skew-symmetric in rigid motion:

$$\omega^i{}_j = \Omega^i{}_j - \Omega_j{}^i, \qquad \widehat{\omega}^A{}_B = \widehat{\Omega}^A{}_B - \widehat{\Omega}_B{}^A$$

Their conjugate affine spins — Hamiltonian generators of

$$\begin{array}{rl} \varphi \mapsto A\varphi & \quad \varphi \mapsto \varphi B \\ A \in \mathrm{GL}(V) & \quad B \in \mathrm{GL}(U) \\ \Sigma = \varphi\pi & \quad \widehat{\Sigma} = \pi\varphi \end{array}$$

Spin and vorticity:

$$S^i{}_j = \Sigma^i{}_j - \Sigma_j{}^i, \qquad V^A{}_B = \widehat{\Sigma}^A{}_B - \widehat{\Sigma}_B{}^A$$

(generators of spatial and material rotations).
Transformation rules:

$$\begin{array}{lcll} A & : & \Sigma \mapsto A\Sigma A^{-1}, & \widehat{\Sigma} \mapsto \widehat{\Sigma} \\ B & : & \Sigma \mapsto \Sigma, & \widehat{\Sigma} \mapsto B^{-1}\widehat{\Sigma}B \\ A & : & \Omega \mapsto A\Omega A^{-1}, & \widehat{\Omega} \mapsto \widehat{\Omega} \\ B & : & \Omega \mapsto \Omega, & \widehat{\Omega} \mapsto B^{-1}\widehat{\Omega}B \end{array}$$

Co-moving translational objects:

$$\widehat{v}^A = \varphi^{-1A}{}_i v^i, \qquad \widehat{p}_A = p_i \varphi^i{}_A$$

Poisson brackets:

$$\begin{aligned} \{\Sigma^i{}_j, \Sigma^k{}_l\} &= \delta^i{}_l \Sigma^k{}_j - \delta^k{}_j \Sigma^i{}_l, \\ \{\widehat{\Sigma}^A{}_B, \widehat{\Sigma}^C{}_D\} &= \delta^C{}_B \widehat{\Sigma}^A{}_D - \delta^A{}_D \widehat{\Sigma}^C{}_B, \\ \{\Sigma^i{}_j, \widehat{\Sigma}^A{}_B\} &= 0, \\ \{\widehat{\Sigma}^A{}_B, \widehat{p}_C\} &= \delta^A{}_C \widehat{p}_B, \\ \{I^i{}_j, p_k\} = \{\Lambda^i{}_j, p_k\} &= \delta^i{}_k p_j, \end{aligned}$$

where

$$I(\mathcal{O})^i{}_j := \Lambda(\mathcal{O})^i{}_j + \Sigma^i{}_j, \qquad \Lambda(\mathcal{O})^i{}_j := x^i p_j$$

and x^i are Cartesian coordinates of the $\mathcal{O}$-radius vector of the current position of the centre of mass in M.

If F is any function depending only on the configurations variables, then, obviously,

$$\begin{aligned}
\{F, \Sigma^i{}_j\} &= \varphi^i{}_A \frac{\partial F}{\partial \varphi^j{}_A}, \\
\{F, \Lambda^i{}_j\} &= x^i \frac{\partial F}{\partial x^j}, \\
\{F, \widehat{\Sigma}^A{}_B\} &= \varphi^i{}_B \frac{\partial F}{\partial \varphi^i{}_A}.
\end{aligned}$$

Canonical affine spin:

$$\begin{aligned}
K^{ij} &= \int \left(y^i - x^i\right)\left(\dot{y}^j - \dot{x}^j\right) d\mu_\varphi(y) = \int \left(y^i - x^i\right) \dot{\varphi}^j{}_K a^K d\mu(a) \\
&= \varphi^i{}_A \frac{d\varphi^j{}_B}{dt} J^{AB}
\end{aligned}$$

Dipole of distribution of linear momentum.
Affine moment of forces:

$$N^{ij} = \int \left(y^i - x^i\right) \mathcal{F}^j\,(y)\, d\mu(y),$$

where $\mathcal{F}^j$ is the force distribution.

Equations of motion:

$$m\frac{d^2x^i}{dt^2} = F^i = -g^{ij}\frac{dV}{dx^j},$$

(total force) (potential case)

$$\varphi^i{}_A\frac{d^2\varphi^j{}_B}{dt^2}J^{AB} = N^{ij} = -\varphi^i{}_A\frac{\partial V}{\partial\varphi^k{}_A}g^{kj}.$$

Balance form:

$$\begin{aligned}\frac{dk^i}{dt} &= F^i,\\ \frac{dK^{ij}}{dt} &= \frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt}J^{AB} + N^{ij},\end{aligned}$$

where $k^i = g^{ij}p_j$ and $k^i = \varphi^i{}_A\widehat{k}^A$.
By the way: Why it is so essential this form of equations of motion?
The point is that the expression for the power of forces has the form:

$$\mathcal{P} = \mathcal{P}_{\text{tr}} + \mathcal{P}_{\text{int}} = F_i\, v^i + N^{ij}\Omega_{ji}.$$

The same concerns, of course, reaction forces.

Therefore, in the case of rigid motion, when $\Omega^{i}{}_{j}$ is g-antisymmetric, the effective system of rigid-body equations of motion is a g-antisymmetric part of equations of motion of affinely-rigid body,

$$\frac{dS^{ij}}{dt} = \frac{d}{dt}\left(K^{ij} - K^{ji}\right) = N^{ij} - N^{ji} = \mathcal{N}^{ij},$$

i.e.,

$$\varphi^{i}{}_{A}\frac{d^{2}\varphi^{j}{}_{B}}{dt^{2}} - \varphi^{j}{}_{A}\frac{d^{2}\varphi^{i}{}_{B}}{dt^{2}} = N^{ij} - N^{ji} = \mathcal{N}^{ij}.$$

Similarly, in the case of incompressible affine motion, equations of motion have the form of the trace-less part of original equations of motion:

$$\varphi^{i}{}_{A}\frac{d^{2}\varphi^{j}{}_{B}}{dt^{2}}\,J^{AB} - \frac{1}{n}\,g_{ab}\,\varphi^{a}{}_{A}\frac{d^{2}\varphi^{b}{}_{B}}{dt^{2}}J^{AB}g^{ij} = N^{ij} - \frac{1}{n}\,g_{ab}\,N^{ab}\,g^{ij}.$$

And finally, in the case of spatially rotation-less motion we must take:

$$\varphi^{i}{}_{A}\frac{d^{2}\varphi^{j}{}_{B}}{dt^{2}}\,J^{AB} + \varphi^{j}{}_{A}\frac{d^{2}\varphi^{i}{}_{B}}{dt^{2}}\,J^{AB} = N^{ij} + N^{ji}.$$

Those are NON-HOLONOMIC CONSTRAINTS:

$$\Omega^{i}{}_{j} - \Omega_{j}{}^{i} = \Omega^{i}{}_{j} - g_{jk}\,g^{il}\,\Omega^{k}{}_{l} = 0.$$

No really Euler form — the non-dynamical term does not vanish ever — affine symmetry of degrees of freedom broken to the orthogonal one:

$$\frac{d\varphi^{i}{}_{A}}{dt}\frac{d\varphi^{j}{}_{B}}{dt}J^{AB} = 2\frac{\partial T_{\mathrm{int}}}{\partial g_{ij}}, \qquad \frac{dK^{ij}}{dt} = 2\frac{\partial T_{\mathrm{int}}}{\partial g_{ij}} + N^{ij}.$$

Similarly:

$$
\begin{aligned}
\frac{d\widehat{k}^A}{dt} &= -\widehat{k}^B \widetilde{J}_{BC} \widehat{K}^{CA} + \widehat{F}^A, \\
\frac{d\widehat{K}^{AB}}{dt} &= -\widehat{K}^{AC} \widetilde{J}_{CD} \widehat{K}^{DB} + \widehat{N}^{AB},
\end{aligned}
$$

or, using non-holonomic velocities,

$$
\begin{aligned}
m\frac{d\widehat{v}^A}{dt} &= -m\widehat{\Omega}^A{}_B \widehat{v}^B + \widehat{F}^A, \\
J^{AC}\frac{d\widehat{\Omega}^B{}_C}{dt} &= -\widehat{\Omega}^B{}_D \widehat{\Omega}^D{}_C J^{CA} + \widehat{N}^{AB}.
\end{aligned}
$$

What would be affine models?

- **Left affinely invariant:**

$$
T_{\text{int}} = \frac{1}{2}\mathcal{L}^B{}_A{}^D{}_C \widehat{\Omega}^A{}_B \widehat{\Omega}^C{}_D, \qquad \frac{d\Sigma^i{}_j}{dt} = N^i{}_j
$$

- **Right affinely invariant:**

$$
T_{\text{int}} = \frac{1}{2}\mathcal{R}^j{}_i{}^l{}_k \Omega^i{}_j \Omega^k{}_l, \qquad \frac{d\widehat{\Sigma}^A{}_B}{dt} = \widehat{N}^A{}_B
$$

- **Doubly affinely invariant:**

$$
T_{\text{int}} = \frac{A}{2}\mathrm{Tr}\left(\Omega^2\right) + \frac{B}{2}\left(\mathrm{Tr}\ \Omega\right)^2 = \frac{A}{2}\mathrm{Tr}\left(\widehat{\Omega}^2\right) + \frac{B}{2}\left(\mathrm{Tr}\ \widehat{\Omega}\right)^2
$$

Comment to d'Alembert:

$$T_{\mathrm{int}} = \frac{1}{2}\mathcal{A}^{K}{}_{i}{}^{L}{}_{j}\frac{d\varphi^{i}{}_{K}}{dt}\frac{d\varphi^{j}{}_{L}}{dt},$$

where

$$\mathcal{A}^{K}{}_{i}{}^{L}{}_{j} = g_{ij}J^{KL}.$$

Translational motion is described in both cases by the following kinetic energies:

$$T_{\mathrm{tr}} = \frac{m}{2}C_{ij}\frac{dx^{i}}{dt}\frac{dx^{j}}{dt} = \frac{m}{2}\eta_{AB}\widehat{v}^{A}\widehat{v}^{B}$$

or

$$T_{\mathrm{tr}} = \frac{m}{2}g_{ij}\frac{dx^{i}}{dt}\frac{dx^{j}}{dt} = \frac{m}{2}G_{AB}\widehat{v}^{A}\widehat{v}^{B}$$

Equations of motion:

- $\mathcal{L}$-affine invariant:

  $$\frac{dp_i}{dt} = Q_i, \qquad \frac{d\Sigma^{i}{}_{j}}{dt} = -\frac{1}{m}\widetilde{C}^{ik}p_k p_j + Q^{i}{}_{j},$$

  where

  $$Q_i = -\frac{\partial V}{\partial x^i}, \qquad Q^{i}{}_{j} = -\varphi^{i}{}_{A}\frac{\partial V}{\partial \varphi^{j}{}_{A}},$$

  or in the other form:

  $$\frac{dp_i}{dt} = Q_i, \qquad \frac{dI(\mathcal{O})^{i}{}_{j}}{dt} = Q_{\mathrm{tot}}(\mathcal{O})^{i}{}_{j},$$

  where

  $$\begin{aligned} I(\mathcal{O})^{i}{}_{j} &= \Lambda(\mathcal{O})^{i}{}_{j} + \Sigma^{i}{}_{j} = x^{i}p_{j} + \Sigma^{i}{}_{j}, \\ Q_{\mathrm{tot}}(\mathcal{O})^{i}{}_{j} &= Q_{\mathrm{tr}}(\mathcal{O})^{i}{}_{j} + Q^{i}{}_{j} = x^{i}Q_{j} + Q^{i}{}_{j}. \end{aligned}$$

- $\mathcal{R}$-affine invariant:

$$\frac{dp_i}{dt} = Q_i, \qquad \frac{d\widehat{\Sigma}^A{}_B}{dt} = \widehat{Q}^A{}_B,$$

where

$$\widehat{Q}^A{}_B = -\frac{\partial V}{\partial \varphi^i{}_A}\varphi^i{}_B = \left(\varphi^{-1}\right)^A{}_i Q^i{}_j \varphi^j{}_B,$$

- Left affine, right metrical:

$$T_{int} = \frac{I}{2}\eta_{KL}\widehat{\Omega}^K{}_M \widehat{\Omega}^L{}_N \eta^{MN} + \frac{A}{2}\widehat{\Omega}^K{}_L \widehat{\Omega}^L{}_K + \frac{B}{2}\widehat{\Omega}^K{}_K \widehat{\Omega}^L{}_L$$

(drunk missile, effective mass), because

$$p_i = C_{ij}(\varphi)\frac{dx^j}{dt} \neq g_{ij}\frac{dx^j}{dt}.$$

- Right affine, left metrical:

$$T_{int} = \frac{I}{2} g_{ik}\Omega^i{}_j \Omega^k{}_l g^{jl} + \frac{A}{2}\Omega^i{}_j \Omega^j{}_i + \frac{B}{2}\Omega^i{}_i \Omega^j{}_j$$

(Arnold discretized?).

For the doubly (left- and right-) affinely invariant models of internal dynamics the general solution is given by the matrix exponent:

$$\varphi(t) = \exp(Et)\,\varphi_0 = \varphi_0 \exp\left(\widehat{E}t\right).$$

Here φ_0 is an arbitrary initial configuration and E, $\widehat{E}$ are arbitrary values of affine velocity respectively in the spatial and co-moving representation.

Situation becomes a little more complicated when there is only one-side affine invariance and one-side metrical invariance. Namely, just like in the case of metrically-rigid body there appear then some stationarity conditions.

So, let us assume geodetic left-affinely invariant and right-metrically invariant kinetic energy and look for solutions

$$\varphi(t) = \varphi_0 \exp(Ft).$$

It turns out that it is a solution for arbitrary φ_0 but only for the η-normal F,

$$\left[F, F^{\eta\,T}\right] = FF^{\eta\,T} - F^{\eta\,T}F = 0,$$

where

$$\left(F^{\eta\,T}\right)^A{}_B = \eta_{BD}\, F^D{}_C\, \eta^{CA}.$$

This holds in particular when, e.g.,

$$F^{\eta\,T} = -F, \qquad F^{\eta\,T} = F.$$

And conversely, let us assume geodetic left-metrically and right-affinely invariant model of the kinetic energy. Then there are stationary solutions of the form:

$$\varphi(t) = \exp(Et)\varphi_0.$$

Here again φ_0 is arbitrary and E is g-normal,

$$\left[E, E^{gT}\right] = EE^{gT} - E^{gT}E = 0,$$

where

$$\left(E^{gT}\right)^i{}_j = g_{jl}\, E^l{}_k\, g^{ki}.$$

This holds, e.g. when

$$F^{gT} = -F, \qquad F^{gT} = F.$$

Obviously, the exponential solutions do exist only in geodetic case, however, this case is essentially important and geometrically distinguished.

Coordinates, analytical description.

$$G[\varphi] \in U^* \otimes U^*, \qquad C[\varphi] \in V^* \otimes V^*.$$

Two "metric-like" tensors in analogy to

$$\eta \in U^* \otimes U^*, \qquad g \in V^* \otimes V^*.$$

Raising their first indices, one obtains the mixed tensors:

$$\widehat{G}[\varphi] \in U \otimes U^*, \qquad \widehat{C}[\varphi] \in V \otimes V^*,$$

analytically:

$$\widehat{G}[\varphi]^A{}_B = \eta^{AC}\, G[\varphi]_{CB}, \qquad \widehat{C}[\varphi]^i{}_j = g^{ik}\, C[\varphi]_{kj}.$$

Any $\varphi \in \mathrm{LI}(U, V)$ may be represented by:

$$\lambda_a \in \mathbb{R}, \qquad R_a \in U, \qquad L_a \in V, \qquad a = 1, \ldots, n,$$

where

$$\begin{aligned} \widehat{G} R_a &= \lambda_a R_a = \exp\left(2q^a\right) R_a, \\ \widehat{C} L_a &= \lambda_a^{-1} L_a = \exp\left(-2q^a\right) L_a. \end{aligned}$$

The bases

$$L = \left(\ldots, L_a, \ldots\right), \qquad R = \left(\ldots, R_a, \ldots\right)$$

may be identified with

$$L : \mathbb{R}^n \to V, \qquad R : \mathbb{R}^n \to U,$$

and their duals $(\ldots, L^a, \ldots)$, $(\ldots, R^a, \ldots)$ may be identified with linear mappings:

$$L : V \to \mathbb{R}^n, \qquad R : U \to \mathbb{R}^n.$$

Identifying the diagonal matrix Diag $\left(\ldots, e^{q^a}, \ldots\right)$ with the linear mapping

$$D : \mathbb{R}^n \to \mathbb{R}^n,$$

we can write the two-polar decomposition:

$$\varphi = LDR^{-1}.$$

In matrix terms: L, R are orthogonal and D is diagonal. Therefore, φ is represented as a pair of rigid (materially) bodies in $\mathbb{R}^n$ and with the n-tuple of one-dimensional coordinates $(\ldots, q^a, \ldots)$ — logarithmic deformation invariants.

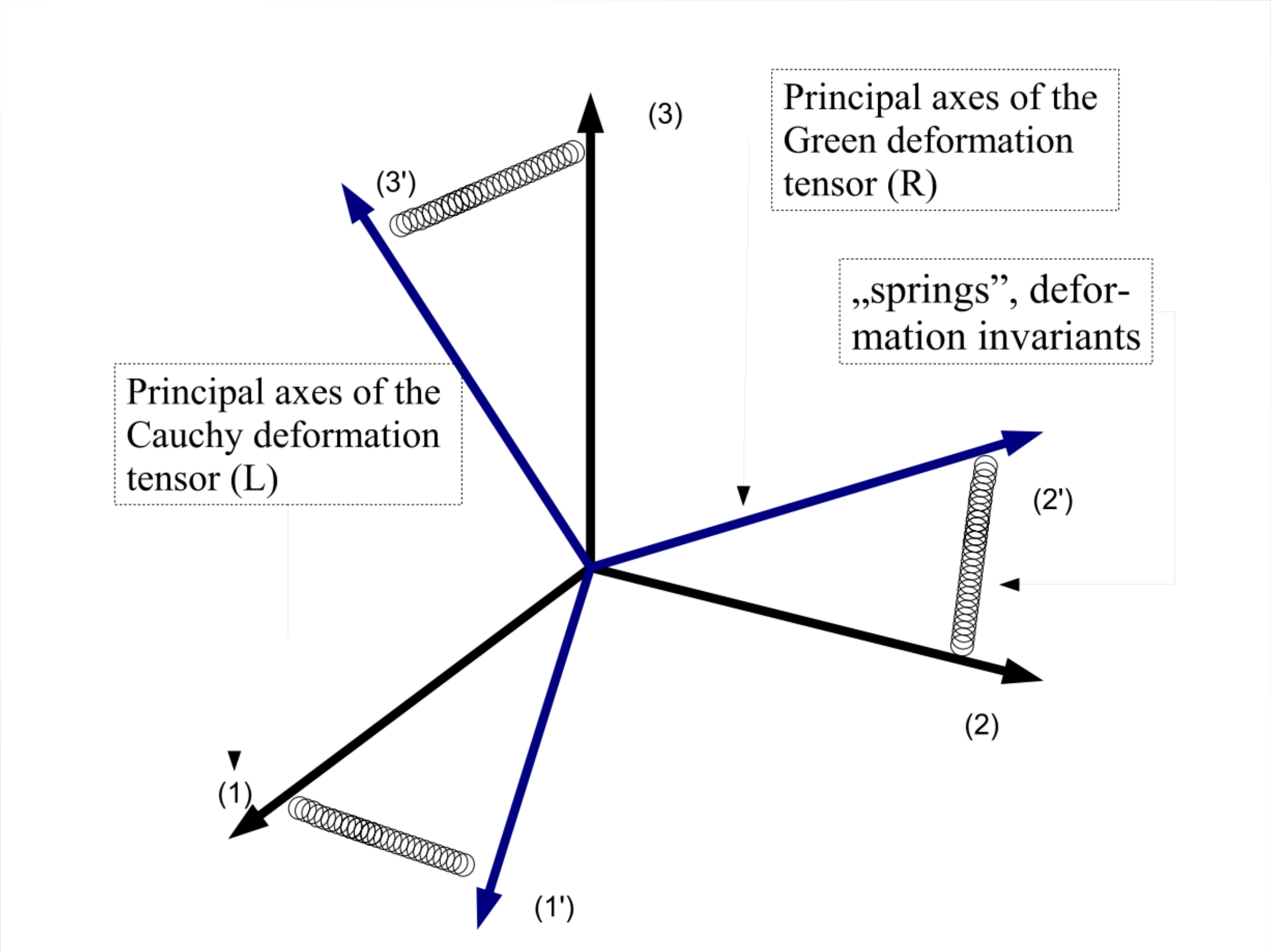
Two-polar decomposition in the matrix form.

$$\varphi = LDR^{-1},$$

where $L, R \in \mathrm{SO}(n, \mathbb{R})$ are orthogonal (isometric) and D is diagonal.

$$q = \frac{1}{n}\left(q^1 + \ldots + q^n\right)$$

are centre of deformation invariants, p — its conjugate momentum.



Principal axes of the Green deformation tensor (R)
„springs”, deformation invariants
Principal axes of the Cauchy deformation tensor (L)
(3)
(3')
(2')
(2)
(1)
(1')

Angular velocities and canonical momenta:

$$\widehat{\chi}^a{}_b = L^a{}_i \frac{dL^i{}_b}{dt}, \quad \text{its conjugate is} \quad \widehat{\rho}$$

$$\widehat{\vartheta}^a{}_b = R^a{}_K \frac{dR^K{}_b}{dt} \quad \text{its conjugate is} \quad \widehat{\tau}$$

$$M := -\widehat{\rho} - \widehat{\tau}, \qquad N := \widehat{\rho} - \widehat{\tau}$$

and then the second-order Casimir invariant has the form

$$C(2) = \operatorname{Tr}\left(\Sigma^2\right) = \operatorname{Tr}\left(\widehat{\Sigma}^2\right),$$

therefore,

$$C(2) = \sum_a p_a^2 + \frac{1}{16}\sum_{a,b} \frac{(M^a{}_b)^2}{\operatorname{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{16}\sum_{a,b} \frac{(N^a{}_b)^2}{\operatorname{ch}^2 \frac{q^a - q^b}{2}},$$

where the symbols are used: $Q^a = D^{aa}$, $q^a = \ln Q^a$.

Lattice structure when $I = 0$, $B = 0$:

$$\mathcal{T}_{\text{latt}} = \frac{1}{2\alpha}\sum_a p_a^2 + \frac{1}{32\alpha}\sum_{a,b} \frac{(M^a{}_b)^2}{\operatorname{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b} \frac{(N^a{}_b)^2}{\operatorname{ch}^2 \frac{q^a - q^b}{2}}$$

- Hyperbolic Sutherland-like lattices:

$$\begin{aligned}\mathcal{T}^{\mathrm{aff}}_{\mathrm{int}} &= \frac{1}{4An}\sum_{a,b}(p_a-p_b)^2+\frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}\\ &- \frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}+\frac{1}{2n(A+nB)}p^2,\\ \mathcal{T}^{\mathrm{aff-metr}}_{\mathrm{int}} &= \mathcal{T}^{\mathrm{aff}}_{\mathrm{int}}\left[A\to I+A\right]+\frac{I}{2(I^2-A^2)}\|V\|^2,\\ \mathcal{T}^{\mathrm{metr-aff}}_{\mathrm{int}} &= \mathcal{T}^{\mathrm{aff}}_{\mathrm{int}}\left[A\to I+A\right]+\frac{I}{2(I^2-A^2)}\|S\|^2.\end{aligned}$$

- Calogero-Moser-like lattices:

$$\mathcal{T}_{\mathrm{int}}=\frac{1}{2I}\sum_a P_a^2+\frac{1}{8I}\sum_{a,b}\frac{(M^a{}_b)^2}{(Q^a-Q^b)^2}+\frac{1}{8I}\sum_{a,b}\frac{(N^a{}_b)^2}{(Q^a+Q^b)^2}.$$

- Usual Sutherland-like lattices:

$$\begin{aligned}\mathcal{T}_{\mathrm{int}} &= \frac{1}{2A}\sum_a p_a^2-\frac{B}{2A(A+nB)}p^2\\ &+ \frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}}+\frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}}.\end{aligned}$$

$Q \in \mathrm{GL}(2,\mathbb{R})$:

$$H^{\text{eff}}_{M,N} = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + U^{\text{eff}}_{M,N} + \mathcal{V}\left(q^1, q^2\right),$$

where

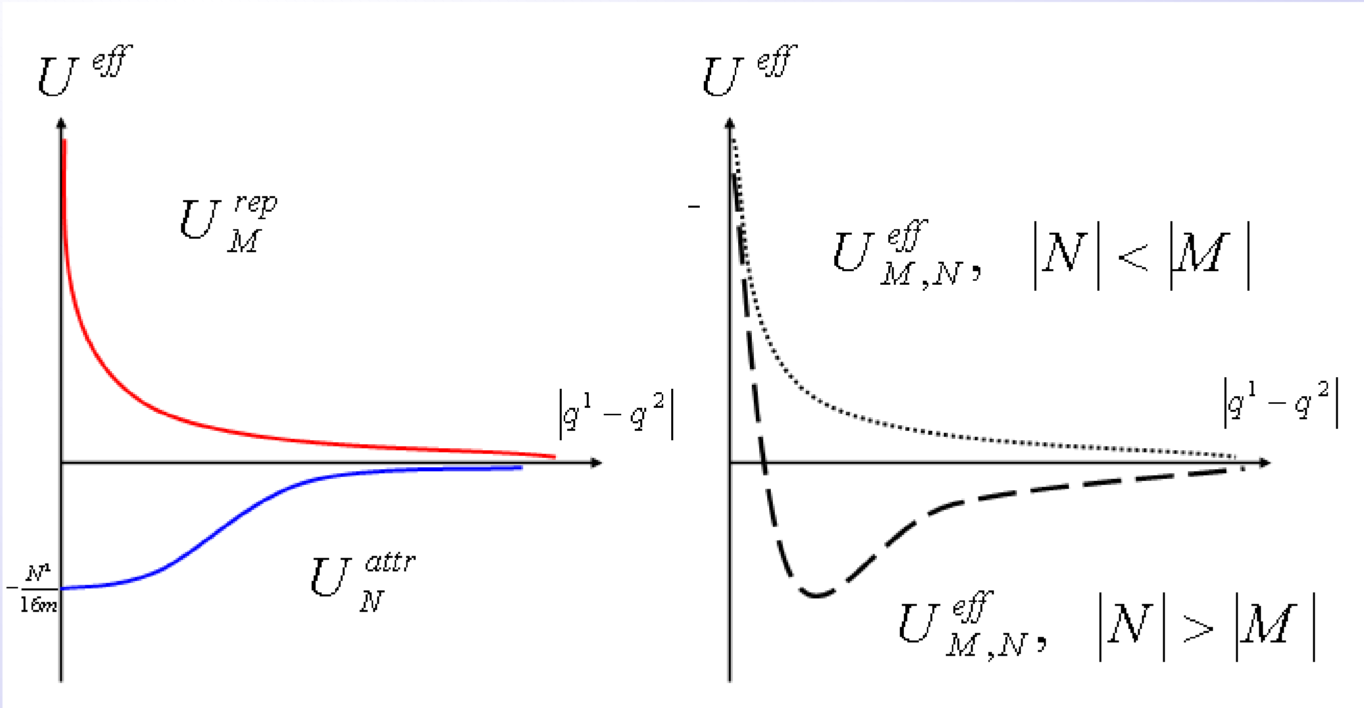
$$U^{\text{eff}}_{M,N} = \frac{M^2}{16m\,\mathrm{sh}^2\frac{q^1-q^2}{2}} - \frac{N^2}{16m\,\mathrm{ch}^2\frac{q^1-q^2}{2}}$$

Let us put again:

$$\begin{aligned} x &:= q^2 - q^1, & q &= \frac{1}{2}\left(q^1 + q^2\right), \\ M &:= M^1{}_2, & N &:= N^1{}_2, \\ p_x &:= \frac{1}{2}\left(p_2 - p_1\right), & p &= p_1 + p_2. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{T}^{\text{aff}}_{\text{int}} &= \frac{p_x^2}{A} + \frac{M^2}{16A\mathrm{sh}^2\frac{x}{2}} - \frac{N^2}{16A\mathrm{ch}^2\frac{x}{2}} + \frac{p^2}{4(A+2B)}, \\ \mathcal{T}^{\text{aff}}_{\text{int}}[x] &= \frac{p_x^2}{A} + \mathcal{V}^{\text{eff}}_{M,N}(x), \\ \mathcal{V}^{\text{eff}}_{M,N} &= \frac{M^2}{16A\mathrm{sh}^2\frac{x}{2}} - \frac{N^2}{16A\mathrm{ch}^2\frac{x}{2}}, \\ \mathcal{H} &= \frac{p_x^2}{A} + \mathcal{V}^{\text{eff}}_{M,N}(x) + U(x) + \frac{p^2}{4(A+2B)} + W(q). \end{aligned}$$



As mentioned, the most convenient way of discussing and solving equations of motion is that based on Poisson brackets,

$$\frac{dF}{dt} = \{F, H\},$$

where F runs over some maximal system of (functionally) independent functions on the phase space.

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The most convenient and geometrically distinguished choice is q^a, p_a, $M^a{}_b$, $N^a{}_b$, L, R or, more precisely, some coordinates on $\mathrm{SO}(n, \mathbb{R})$ parameterizing L and R. In d'Alembert models Q^a, P_a are more convenient than q^a, p_a.
An important point is that q^a, p_a, $M^a{}_b$, $N^a{}_b$ generate some Poisson subalgebra, because their Poisson brackets may be expressed by them alone without any use of L, R-variables. And Hamiltonians also depend only on q^a, p_a, $M^a{}_b$, $N^a{}_b$, whereas L, R are non-holonomically cyclic variables. This enables one to perform a partial reduction of the problem. In fact, the following subsystem of equations is closed:

$$\frac{dq^a}{dt} = \{q^a, H\} = \frac{\partial H}{\partial p_a},$$

$$\frac{dM^a{}_b}{dt} = \{M^a{}_b, H\} = \{M^a{}_b, M^c{}_d\}\frac{\partial H}{\partial M^c{}_d} + \{M^a{}_b, N^c{}_d\}\frac{\partial H}{\partial N^c{}_d},$$

$$\frac{dp_a}{dt} = \{p_a, H\} = -\frac{\partial H}{\partial q^a},$$

$$\frac{dN^a{}_b}{dt} = \{N^a{}_b, H\} = \{N^a{}_b, M^c{}_d\}\frac{\partial H}{\partial M^c{}_d} + \{N^a{}_b, N^c{}_d\}\frac{\partial H}{\partial N^c{}_d}.$$

Obviously,

$$\{q^a, p_b\} = \delta^a{}_b, \qquad \{q^a, M^c{}_d\} = \{p_a, M^c{}_d\} = \{q^a, N^c{}_d\} = \{p_a, N^c{}_d\} = 0.$$

Poisson brackets of M, N-quantities follow directly from those for $\hat{\rho}$, $\hat{\tau}$, and the latter ones correspond exactly to the structure constants of $\mathrm{SO}(n,\mathbb{R})$, thus,

$$\{\hat{\rho}_{ab},\hat{\rho}_{cd}\} = \hat{\rho}_{ad}\delta_{cb} - \hat{\rho}_{cb}\delta_{ad} + \hat{\rho}_{db}\delta_{ac} - \hat{\rho}_{ac}\delta_{db},$$

$$\{\hat{\tau}_{ab},\hat{\tau}_{cd}\} = \hat{\tau}_{ad}\delta_{cb} - \hat{\tau}_{cb}\delta_{ad} + \hat{\tau}_{db}\delta_{ac} - \hat{\tau}_{ac}\delta_{db}, \qquad \{\hat{\rho}_{ab},\hat{\tau}_{cd}\} = 0,$$

where the raising and lowering of indices are meant in the Kronecker-delta sense. From these Poisson brackets we obtain the following ones:

$$\{M_{ab},M_{cd}\} = \{N_{ab},N_{cd}\} = M_{cb}\delta_{ad} - M_{ad}\delta_{cb} + M_{db}\delta_{ac} - M_{ac}\delta_{db},$$

$$\{M_{ab},N_{cd}\} = N_{cb}\delta_{ad} - N_{ad}\delta_{cb} - N_{ac}\delta_{db} + N_{db}\delta_{ac}.$$

The subsystem for $(q^a, p_a, M^a{}_b, N^a{}_b)$ may be in principle autonomously solvable. When the time dependence of $\hat{\rho} = (N-M)/2$ and $\hat{\tau} = -(N+M)/2$ is known, then performing the inverse Legendre transformation we can obtain the time dependence of angular velocities $\hat{\chi}$, $\hat{\vartheta}$:

$$\hat{\chi}^a{}_b = \frac{\partial H}{\partial \hat{\rho}^b{}_a}, \qquad \hat{\vartheta}^a{}_b = \frac{\partial H}{\partial \hat{\tau}^b{}_a}$$

(some care must be taken when differentiating with respect to skew-symmetric matrices). And finally the evolution of L, R is given by the following time-dependent systems:

$$\frac{dL}{dt} = L\hat{\chi}, \qquad \frac{dR}{dt} = R\hat{\vartheta}.$$

Let us now consider the geodetic models on $\mathrm{SL}(n, \mathbb{R})$. The number of degrees of freedom equals $(n^2 - 1) = \dim \mathrm{SL}(n, \mathbb{R})$. We are interested in models describing elastic, bounded vibrations. The fundamental question is the following:

- Does a $2(n^2 - 1)$-dimensional family of bounded solutions exist? (below- dissociation-threshold situations)
- Does a $2(n^2 - 1)$-dimensional family of non-bounded, escaping solutions exist? (above-dissociation-threshold situations)

The answer is affirmative. Let us present an outline of the reasoning supporting the statement that there is an open family of bounded and an open family of escaping solutions within the general solution of the doubly-invariant geodetic problem on $\mathrm{SL}(n, \mathbb{R})$.
Let $\alpha \in \mathfrak{sl}(n)$ $(\mathrm{Tr}\alpha = 0)$ be similar to an antisymmetric matrix $\lambda = -\lambda^T \in \mathfrak{so}(n)$, $\alpha = \chi\lambda\chi^{-1}$ for some $\chi \in \mathrm{SL}(n, \mathbb{R})$. Then every motion

$$\Psi(t) = e^{\alpha t}\Psi_0 = \chi e^{\lambda t}\chi^{-1}\Psi_0$$

is bounded. The structure constants (simplicity of $\mathrm{SL}(n, \mathbb{R})$) imply that the set of such α-s is $(n^2 - 1)$-dimensional, although $\dim \mathrm{SO}(n, \mathbb{R}) = n(n-1)/2$. Nevertheless, it is not so that these $n^2 - 1$ velocity parameters combine additively with $n^2 - 1$ parameters of Ψ_0 so as to result in $2(n^2 - 1)$ parameters (initial conditions) in the phase space. The reason is that appropriate correlations between Ψ_0 and λ may repeat the same orbits. In dimensions $n = 2, 3$ the above solutions are always periodic. In higher dimensions they may be so but need not. Take for example $n = 4$, represent $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$ and assume that λ is a block matrix consisting of two 2×2 skew-symmetric blocks. Any of these blocks has essentially one parameter. If the ratio of these angular velocity parameters is an irrational number, then the resulting motion is non-periodic, its orbit is not closed and because of this it is not a Lie subgroup in the usual sense, although it is an algebraic subgroup.

The closures of such orbits are two-dimensional submanifolds. But one can also show that there are bounded non-periodic solutions in two and three dimensions as well. The point is that the mentioned matrices λ may be slightly perturbed by small symmetric matrices κ and we can take the solutions

$$\Psi(t) = \chi e^{(\lambda+\kappa)t}\chi^{-1}\Psi_0.$$

The afore-mentioned periodic orbits (corresponding to $\kappa = 0$) are stable in the sense that for some open range of $\kappa = \kappa^T \in \mathfrak{sl}(n)$, i.e., for some open range of $\alpha = \lambda + \kappa \in \mathfrak{sl}(n)$ the resulting motion is still bounded although no longer periodic. And there is sufficiently much of the above matrices α so as not to interfere with the arbitrariness of Ψ_0. Thus, the corresponding family of solutions contains an open subset (in the sense of initial conditions) of the general solution.
Quite similarly, if we took symmetric $\lambda = \lambda^T \in \mathfrak{sl}(n)$, then the corresponding solutions $\Psi(t) = \chi e^{\lambda t}\chi^{-1}\Psi_0 = e^{\chi\lambda\chi^{-1}t}\Psi_0$ would be non-bounded (escaping). And it will be so if we slightly perturb λ by “small”antisymmetric matrices $\epsilon = -\epsilon \in \mathfrak{so}(n)$ from some open neighbourhood of the null element. And again we conclude that the general solution contains an open subset of unbounded (escaping) trajectories.
The quantum counterpart is obvious: In quantized geodetic models there exists a discrete energy spectrum of physically bounded situations, and above it —- the continuous spectrum corresponding to the dissociated body. There is an obvious analogy with the $E < 0$ and $E > 0$ situations for the Coulomb problem.

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Thank you for your attention!

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