Bertrand systems on spaces of constant sectional curvature. The action-angle analysis. Classical, quasi-classical and quantum problems.

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$\boxed{S^3\left(0,R\right)\subset\mathbb{R}^4}$: $\left(x^1\right)^2+\left(x^2\right)^2+\left(x^3\right)^2+\left(x^4\right)^2=R^2$

$$\begin{cases} x^1= & R\sin\left(\frac{r}{R}\right)\sin\left(\vartheta\right)\cos\left(\varphi\right)\\ x^2= & R\sin\left(\frac{r}{R}\right)\sin\left(\vartheta\right)\sin\left(\varphi\right)\\ x^3= & R\sin\left(\frac{r}{R}\right)\cos\left(\vartheta\right)\\ x^4= & R\cos\left(\frac{r}{R}\right)\end{cases}\qquad \begin{array}{rcc} r & \in & [0,\pi R]\\ \vartheta & \in & [0,\pi]\\ \varphi & \in & [0,2\pi[\end{array}$$

$$\boxed{ds^2=dr^2+R^2\sin^2\left(\frac{r}{R}\right)\left(d\vartheta^2+\sin^2\left(\vartheta\right)d\varphi^2\right)}$$

restriction of $\left(dx^1\right)^2+\left(dx^2\right)^2+\left(dx^3\right)^2+\left(dx^4\right)^2$to $S^3\left(0,R\right)$

$S^2\left(0,R\right):\vartheta=\frac{\pi}{2},\ ds^2=dr^2+R^2\sin^2\left(\frac{r}{R}\right)d\varphi^2$

$\boxed{H^{3,2,+}\left(0,R\right)\subset\mathbb{R}^4}$: $\left(x^1\right)^2+\left(x^2\right)^2+\left(x^3\right)^2-\left(x^4\right)^2=-R^2\ ,\ x^4>0$

$$\begin{cases} x^1 = & R\sinh\left(\frac{r}{R}\right)\sin\left(\vartheta\right)\cos\left(\varphi\right) \\ x^2 = & R\sinh\left(\frac{r}{R}\right)\sin\left(\vartheta\right)\sin\left(\varphi\right) \\ x^3 = & R\sinh\left(\frac{r}{R}\right)\cos\left(\vartheta\right) \\ x^4 = & R\cosh\left(\frac{r}{R}\right) \end{cases} \qquad \begin{array}{ccc} r & \in & [0,\infty] \\ \vartheta & \in & [0,\pi] \\ \varphi & \in & [0,2\pi[\end{array}$$

$$\boxed{ds^2 = dr^2 + R^2\sinh^2\left(\frac{r}{R}\right)\left(d\vartheta^2 + \sin^2\left(\vartheta\right)d\varphi^2\right)}$$

restriction of $\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 - \left(dx^4\right)^2$to $H^{3,2,+}\left(0,R\right)$

$H^{2,2,+}\left(0,R\right) \subset \mathbb{R}^3 : \vartheta = \frac{\pi}{2}$, $ds^2 = dr^2 + R^2\sinh^2\left(\frac{r}{R}\right)d\varphi^2$

Representing them as subset of $\mathbb{R}^3$:

$\underline{S^3(0,R)\text{:}}$ $\overline{r} = r[\sin\left(\vartheta\right)\cos\left(\varphi\right), \sin\left(\vartheta\right)\sin\left(\varphi\right), \cos\left(\vartheta\right)]$,

$r \in [0,\pi R]$, all points on the sphere $S^2(0,\pi R) \subset \mathbb{R}^3$ being identified.

$ds^2 = \Gamma_{ij}dr^i dr^j$,

$$\boxed{\Gamma_{ij} = \frac{R^2}{r^2}\sin^2\frac{r}{R}\delta_{ij} + \frac{1}{r^2}\left(1 - \frac{R^2}{r^2}\sin^2\frac{r}{R}\right)r_i r_j}$$

$\underline{H^{3,2,+}(0,R):}$ $\overline{r} = r[\sin(\vartheta)\cos(\varphi), \sin(\vartheta)\sin(\varphi), \cos(\vartheta)]$,

$r \in [0, \infty[$

$ds^2 = \Gamma_{ij} dr^i dr^j$,

$$\boxed{\Gamma_{ij} = \frac{R^2}{r^2}\sinh^2\frac{r}{R}\delta_{ij} + \frac{1}{r^2}\left(1 - \frac{R^2}{r^2}\sinh^2\frac{r}{R}\right) r_i r_j}$$

Kinetic energy:

$$\boxed{T = \frac{m}{2}\Gamma_{ij}\frac{dr^i}{dt}\frac{dr^j}{dt}}$$

Conjugate momenta

$p_i = \frac{\partial T}{\partial \dot{r}_i} = m\Gamma_{ij}\frac{dr^j}{dt}$

$S^3:\ \overline{p} = m\frac{R^2}{r^2}\sin^2\frac{r}{R}\frac{d\overline{r}}{dt} + \frac{m}{r^2}\left(1 - \frac{R^2}{r^2}\sin^2\frac{r}{R}\right)\left(\overline{r}\frac{d\overline{r}}{dt}\right)\overline{r}$

$H^{3,2,+}:\ \overline{p} = m\frac{R^2}{r^2}\sinh^2\frac{r}{R}\frac{d\overline{r}}{dt} + \frac{m}{r^2}\left(1 - \frac{R^2}{r^2}\sinh^2\frac{r}{R}\right)\left(\overline{r}\frac{d\overline{r}}{dt}\right)\overline{r}$

<u>Isometry groups:</u>

$$\begin{array}{rcl} S^3 & - & SO(4,\mathbb{R}) \\ H^{3,2,+} & - & SO(1,3) \end{array}$$

Isotropy subgroup of the pole $r = 0$: $SO(3, \mathbb{R})$-usual rotations of the vector $\overline{r}$.

Its Hamiltonian generators:

$\overline{L} = \overline{r} \times \overline{p}$, $\{L_i, L_j\} = \varepsilon_{ikj} L_k$

In velocity terms:

$$\boxed{\overline{L} = m\frac{R^2}{r^2}\sin^2\frac{r}{R}\overline{r} \times \frac{d\overline{r}}{dt} \text{ in } S^3}$$

$$\boxed{\overline{L} = m\frac{R^2}{r^2}\sinh^2\frac{r}{R}\overline{r} \times \frac{d\overline{r}}{dt} \text{ in } H^{3,2,+}}$$

Spherically-symmetric models: invariant under the isotropy group $SO(3, R)$

Lagrangian: $L = T - V(r)$

$\overline{L} = \overline{r} \times \overline{p}$ - constants of motion

Direction of $\overline{L}$- constant of motion

Direction of $\overline{r} \times \frac{d\overline{r}}{dt}$- constant of motion

This implies: plane motion

Involutive system of constants of motion:

- $$\underline{p_\varphi = L_3} = \begin{cases} mR^2 \sin^2 \frac{r}{R} \frac{d\varphi}{dt} & S^3(O,R) \\ mR^2 \sinh^2 \frac{r}{R} \frac{d\varphi}{dt} & H^{3,2,+}(O,R) \end{cases}$$

- $$\underline{\overline{L}^2 = \overline{L} \cdot \overline{L}} = p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2 \vartheta} = \begin{cases} m^2 R^4 \sin^4 \frac{r}{R} \left(\left(\frac{d\vartheta}{dt}\right)^2 + \sin^2 \vartheta \left(\frac{d\varphi}{dt}\right)^2 \right) & \text{on } S^3(O,R) \\ m^2 R^4 \sinh^4 \frac{r}{R} \left(\left(\frac{d\vartheta}{dt}\right)^2 + \sin^2 \vartheta \left(\frac{d\varphi}{dt}\right)^2 \right) & \text{on } H^{3,2,+}(O,R) \end{cases}$$

- $$\begin{aligned} H &= \begin{cases} \frac{1}{2m} \left(p_r^2 + \frac{L^2}{R^2 \sin^2 \frac{r}{R}} \right) + V(r) & \text{on } S^3(O,R) \\ \frac{1}{2m} \left(p_r^2 + \frac{L^2}{R^2 \sinh^2 \frac{r}{R}} \right) + V(r) & \text{on } H^{3,2,+}(O,R) \end{cases} \\ &= \begin{cases} \frac{m}{2} \left(\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2 R^2 \sin^2 \frac{r}{R}} \right) + V(r) & \text{on } S^3(O,R) \\ \frac{m}{2} \left(\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2 R^2 \sinh^2 \frac{r}{R}} \right) + V(r) & \text{on } H^{3,2,+}(O,R) \end{cases} \end{aligned}$$

$\underline{\{L_3, L\} = 0,\ \{L_3, H\} = 0,\ \{L, H\} = 0,}$

Plane motion, <u>two-dimensional reduction</u>: $\vartheta = \pi/2$ (motion in the x,y-plane)

$$\begin{cases} \left(\frac{dr}{dt}\right)^2 + R^2\left(\frac{d\varphi}{dt}\right)^2 \begin{matrix}\sin^2\\ \sinh^2\end{matrix}\left(\frac{r}{R}\right) = \frac{2}{m}\left(E - V(r)\right) \\ mR^2\left(\frac{d\varphi}{dt}\right)^2 \begin{matrix}\sin^2\\ \sinh^2\end{matrix}\left(\frac{r}{R}\right) = M \end{cases}$$

From here:

$$\begin{cases} \frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left(E - V(r)\right) - \frac{M^2}{m^2R^2}\begin{matrix}\sin^{-2}\\ \sinh^{-2}\end{matrix}\left(\frac{r}{R}\right)} \\ \frac{d\varphi}{dt} = \frac{M}{mR^2}\begin{matrix}\sin^{-2}\\ \sinh^{-2}\end{matrix}\left(\frac{r}{R}\right) \end{cases}$$

The resulting quadratures:

$$\begin{cases} \frac{dt}{dr} = \pm\left(\frac{2}{m}\left(E - V_{eff}\right)\right)^{-\frac{1}{2}} \\ \frac{d\varphi}{dt} = \frac{M}{mR^2}\begin{matrix}\sin^{-2}\\ \sinh^{-2}\end{matrix}\left(\frac{r}{R}\right) \end{cases}$$

where $V_{eff} = V + \frac{M^2}{2mR^2} \begin{matrix} \sin^{-2} \\ \sinh^{-2} \end{matrix} \left(\frac{r}{R}\right)$

The orbit itself:

$$\frac{d\varphi}{dr} = \pm \frac{M}{mR^2} \begin{matrix} \sin^{-2} \\ \sinh^{-2} \end{matrix} \left(\frac{r}{R}\right) \left(\frac{2}{m}\left(E - V_{eff}\right)\right)^{-\frac{1}{2}}$$

New variables:

$S^3(0,R)$	$H^{3,2,+}(0,R)$
$y = \frac{1}{R}\cot\frac{r}{R}$	$y = \frac{1}{R}\coth\frac{r}{R}$
y-runs over $[+\infty, -\infty]$	y-runs over $[+\infty, \frac{1}{R}]$
when r runs $[0, \pi R]$	when r runs $[0, \infty]$

Then:

$$\boxed{\begin{aligned} S^3(O,R): \;\; \varphi(r) = \varphi[y] = \;& \pm\frac{L}{m}\int \frac{dy}{\sqrt{-\frac{L^2}{m^2}y^2 + \frac{2}{m}\left(\mathcal{E}_s - V\right)}} \\ H^{3,2,+}(O,R): \;\; \varphi(r) = \varphi[y] = \;& \pm\frac{L}{m}\int \frac{dy}{\sqrt{-\frac{L^2}{m^2}y^2 + \frac{2}{m}\left(\mathcal{E}_h - V\right)}} \end{aligned}}$$

where:

$$\underline{\mathcal{E}_s = E - \frac{L^2}{2mR^2}},\; \underline{\mathcal{E}_h = E + \frac{L^2}{2mR^2}}$$

Euclidean case: $\Gamma_{ij} = \delta_{ij}$, $R = \infty$, $y = \frac{1}{r}$:

$$\boxed{\mathbb{R}^3: \quad \varphi(r) = \varphi[y] = \pm\frac{L}{m}\int \frac{dy}{\sqrt{-\frac{L^2}{m^2}y^2 + \frac{2}{m}(E - V)}}}$$

Formally and locally: <u>the same formula</u>, but $\underline{\mathcal{E}_s, \mathcal{E}_h}$ instead $\underline{E}$ in the $\mathbb{R}^3$-case.

<u>When $V = 0$</u>: The <u>y-variable establishes</u> the projective mapping acting between manifolds $\mathbb{R}^3, S^3, H^{3,2,+}$. It maps locally <u>geodetic arcs</u> onto <u>geodetic arcs</u>, but, without preserving the affine parameter.

Automatically, one obtains the following Bertrand potentials:

- Euclidean space $\mathbb{R}^3$ ($\mathbb{R}^n$, as a matter of fact $\mathbb{R}^2$).

$$V_{osc} = \frac{k}{2} r^2 = \frac{k}{2} \frac{1}{y^2} \quad , \quad V_{Co} = -\frac{\alpha}{r} = -\alpha y$$

- Sphere $S^3(O,R)$ ($S^n(O,R)$, as a matter of fact $S^2(O,R)$).

$$V_{osc} = \frac{kR^2}{2} \tan^2 \frac{r}{R} = \frac{k}{2} \frac{1}{y^2} \quad , \quad V_{Co} = -\frac{\alpha}{R} \cot \frac{r}{R} = -\alpha y$$

- Pseudo-sphere $H^{3,2,+}(O,R)$ ($H^{n,2,+}(O,R)$, as a matter of fact $H^{2,2,+}(O,R)$).

$$V_{osc} = \frac{kR^2}{2} \tanh^2 \frac{r}{R} = \frac{k}{2} \frac{1}{y^2} \quad , \quad V_{Co} = -\frac{\alpha}{R} \coth \frac{r}{R} = -\alpha y$$

Remark: description of the conformal flatness

- $S^3(O,R) : \xi = a \tan \frac{r}{2R}$, $\overline{\xi} = a \tan \frac{r}{2R} \frac{\overline{r}}{r}$, $\xi \in [0, \infty]$

$$ds^2 = \frac{4R^2 a^2}{\left(a^2 + \xi^2\right)^2} \left(d\xi^2 + \xi^2 \left(d\vartheta^2 + \sin^2(\vartheta)\, d\varphi^2\right)\right)$$

If $a = R$, $\overline{\xi}$ become stereographic projection variables

$$ds^2 = \frac{4}{\left(1 + {\frac{\xi}{R^2}}^2\right)^2} \left(d\xi^2 + \xi^2 \left(d\vartheta^2 + \sin^2(\vartheta)\, d\varphi^2\right)\right)$$

- $H^{3,2,+}(O, R) : \xi = a \tanh \frac{r}{2R}$, $\overline{\xi} = a \tanh \frac{r}{2R} \frac{\overline{r}}{r}$, $\xi \in [0, a]$. If $a = R$, $\overline{\xi}$ become stereographic projection variables

$$ds^2 = \frac{4}{\left(1 - {\frac{\xi}{R^2}}^2\right)^2} \left(d\xi^2 + \xi^2 \left(d\vartheta^2 + \sin^2(\vartheta)\, d\varphi^2\right)\right)$$

Let us observe that in the spherical case the proviso "all bounded orbits" would be superfluous because due to the compactness of the configuration space all orbits are bounded. It is no longer true for the pseudosphere, where not only for the Coulomb problem but, surprisingly enough also for the degenerate oscillator, the potential energy has a finite upper bound. Therefore, there exists an ionization threshold and the continuum of nonbounded orbits above it. On the quantum level this means that there exists a continuous spectrum of energy placed above the potential supremum. Let us stress some pecularities of the Coulomb and oscillator problems on

$S^3(0,R)$. As mentioned, the Coulomb potential has the form

$$V(r) = -\frac{\alpha}{R}\cot\frac{r}{R}.$$

Due to the compactness of the configuration space there is no need to assume $\alpha > 0$. The above potential is a fundamental solution, the Green function of the Laplace equation for the g-metric tensor-Laplace-Beltrami operator on $S^3(0,R)$, i.e, it provides a spherically symmetric solution of the equation

$$\Delta V = 0, \quad r \neq 0,$$

where Δ denotes the Laplace-Beltrami operator based on the metric tensor of $S^3(0,R)$. The point is, however, that when $\alpha > 0$, the "northern" pole $r = 0$ is an attractive singularity, whereas the' "southern" one, $r = \pi R$ is the repulsive pole, and conversely if we put $\alpha < 0$. If $\alpha > 0$ there exist circular orbits with $r < \frac{\pi R}{2}$, but no circular orbits with $r > \frac{\pi R}{2}$. And quite conversely, if $\alpha < 0$ then the "southern" pole becomes attractive, and the "nothern" one-repulsive. In this way, an elementary electrostatic entity in $S^3(0,R)$ is a gigantic dipole consisting of two antipodally located point charges of the opposite signs. This agrees beautifully with the theorem proved, e.g. in the Landau and Lifshitz book that in a closed Universe the total electric charge must vanish. The diagram of V as a function of r has the vertical asymptote at $r = 0$ and the vertical asymptote with the reversed sign at $r = \pi R$. It intersects the r-axis at $r = \frac{\pi R}{2}$,. V is defined in $]0, \pi R[$.

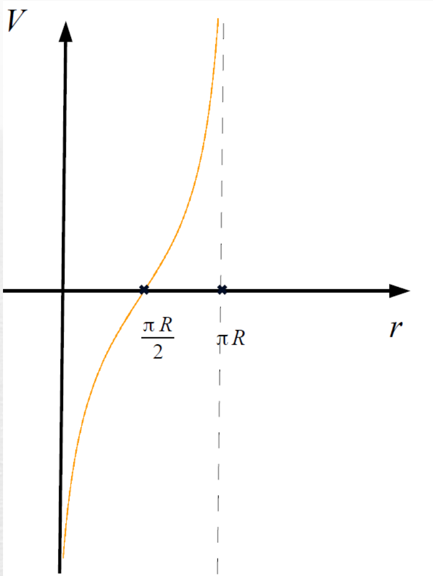

Figure 1

The degenerate oscillator potential in $S^3(0, R)$ splits the configuration space into two disjoint regions separated by the inpenetrable potential barrier placed at $r = \frac{\pi R}{2}$. This potential is invariant under the antipodal identification, thus the problem is reduced to the so-called elliptic space or Riemannian space, i.e, the quotient of sphere under the antipodal equivalence of points. Let us stress in this connection that the Coulomb problem does not project correctly from $S^3(0, R)$ to the elliptic space. The diagram of degenerate oscillator is qualitatively pictured below:

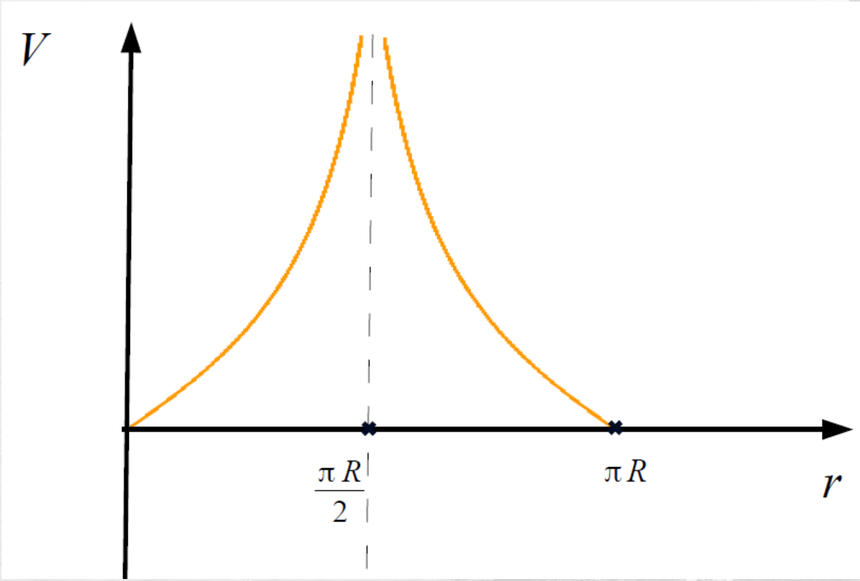

Figure 2

The diagram of the pseudospherical Coulomb potential has the (negative) vertical asymptote at $r = 0$ and the horizontal asymptote given by the value $-\frac{\alpha}{R}$.

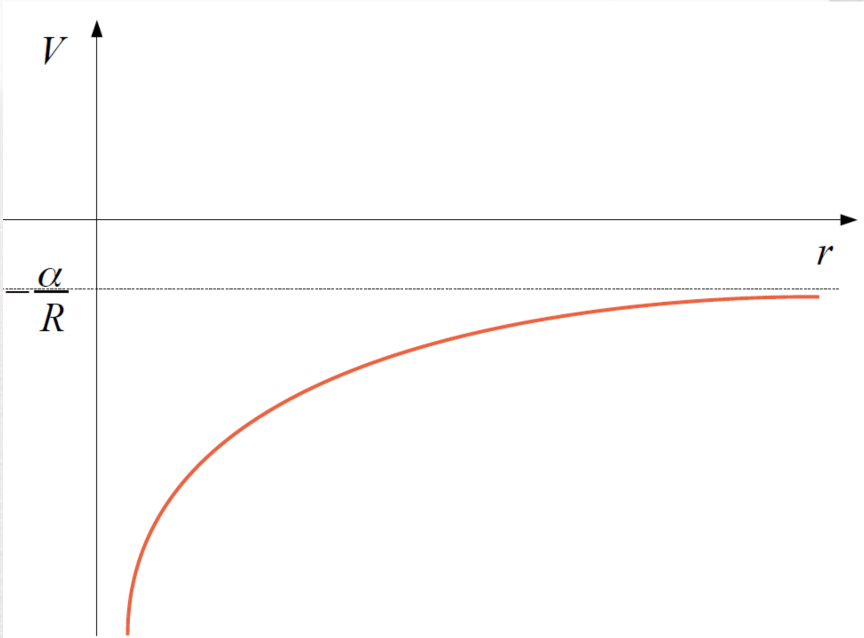

Figure 3

The diagram for the degenerate oscillator behaves parabolically at $r = 0$ and has the horizontal asymptote given by the value $\frac{kR^2}{2}$.

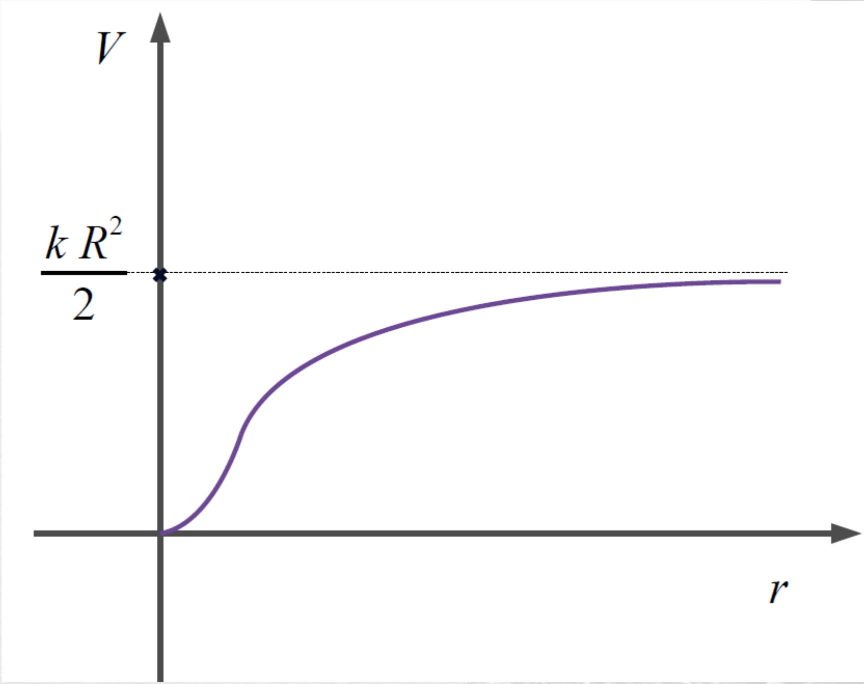

Figure 4

Obviously. in .both cases it would be rather natural to modify the potentials by additive constants so as to make them vanishing at infinity. Therefore, the "Coulomb" potential would have to be given by

$$V(r) = -\frac{\alpha}{R}\cot\frac{r}{R} + \frac{\alpha}{R}.$$

whereas the oscillatory one by

$$\frac{kR^2}{2}\tanh^2\frac{r}{R} - \frac{kR^2}{2}.$$

However, we will not do this gauging and retain the analytical form given by the above theorem.

The detailed analysis shows that for the Coulomb problem on the sphere $S^3(0,R)$ the values of constants of motion E, L are constrained by the following weak inequality .

$$E \geq -\frac{m\alpha^2}{2L^2} + \frac{L^2}{2mR^2}.$$

The equality case corresponds. to circular orbits. The corresponding function $L \to E(L)$ is increasing, has the negative vertical asymptote at $L = 0$, tends to infinity when $L \to \infty$ intersecting the L-axis at some point L_0. It is easy to see that $L_0 = \sqrt{m\alpha R}$; this expression, as expected, tends to infinity together with the "radius of the Universe".

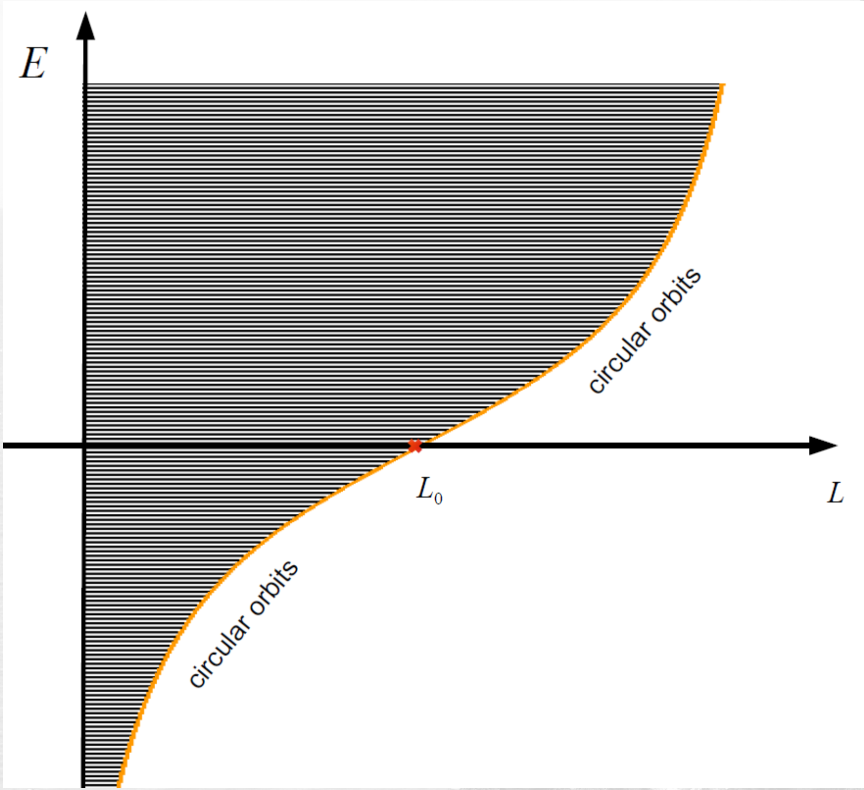

Figure 5

Similarly, for the pseudo sphere with pseudoradius R, we obtain

$$E \geq -\frac{m\alpha^2}{2L^2} - \frac{L^2}{2mR^2},$$

the equality case corresponding again to circular orbits. The extremal point of the dependence $L \to E(L)$ corresponds to the value $L_0 = \sqrt{m\alpha R}$ and again tends to infinity together with the "cosmological" pseudoradius R. The resulting extremal value of E equals, obviously, $-\frac{\alpha}{R}$ and just coincides with the threshold of classical unbounded orbits or quantum continuous spectrum. The diagram of the function $L \to E(L)$ for circular orbits has the negative vertical asymptote at $L = 0$, increases to $-\frac{\alpha}{R}$ at L_0 and then decreases to minus infinity when $L \to \infty$.

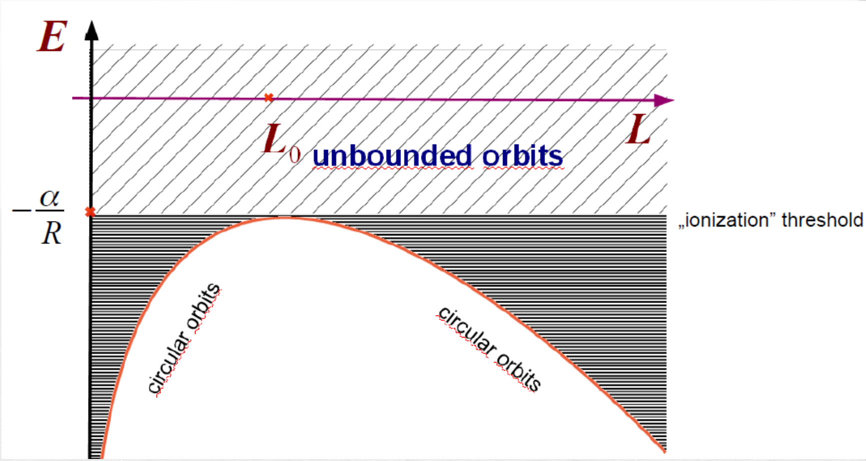

Figure 6

Let us quote also the corresponding relationships for the degenerate oscillator. In the spherical space $S^3(0,R)$ we have

$$E \geq L\omega_0 + \frac{L^2}{2mR^2}, \quad \omega_0 = \sqrt{\frac{k}{m}},$$

the equality case again corresponding to circular orbits.

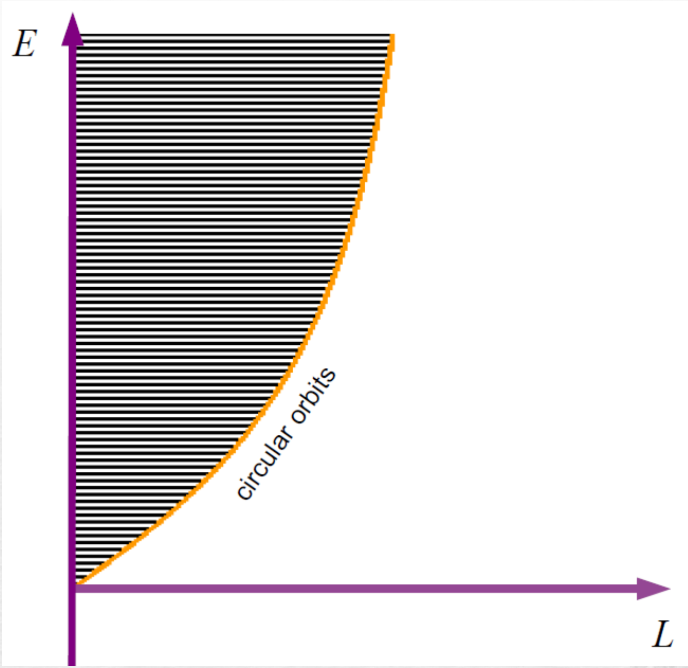

Figure 7

On the pseudosphere $H^{3,2,+}(0, R)$ the situation is more complicated because of the "saturation" property of the degenerate oscillator potential. Namely. we obtain then

$$E \geq L\omega_0 - \frac{L^2}{2mR^2}, \quad \omega_0 = \sqrt{\frac{k}{m}},$$

but above the threshold $E = \frac{kR^2}{2}$ there is a continuum of unbounded classical orbits and the quantum continuous spectrum.

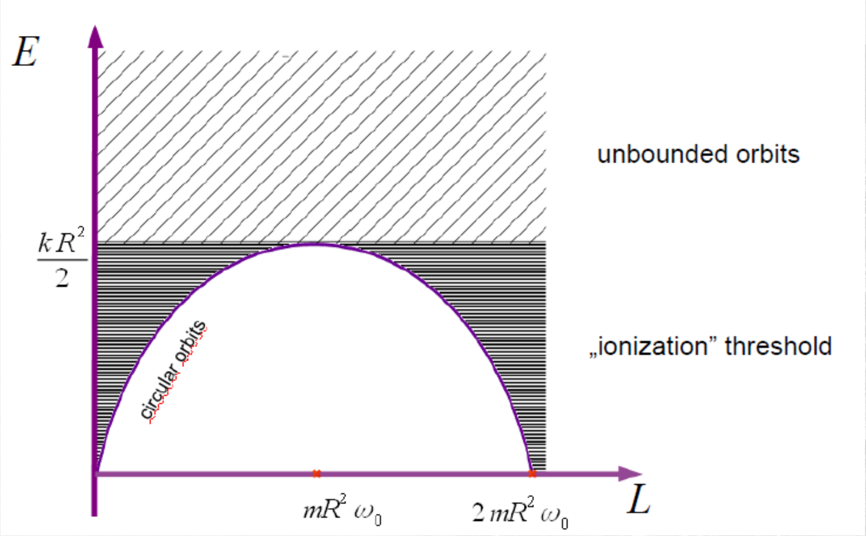

Figure 8

Hamilton-Jacobi equation and action-angle variables

$$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{R^2}\begin{pmatrix}\sin\\ \sinh\end{pmatrix}^{-2}\left(\frac{r}{R}\right)\left(\frac{\partial S}{\partial \vartheta}\right)^2 + \frac{1}{R^2}\begin{pmatrix}\sin\\ \sinh\end{pmatrix}^{-2}\left(\frac{r}{R}\right)\sin^{-2}(\vartheta)\left(\frac{\partial S}{\partial \varphi}\right)^2 = 2m\left(E - V\right)$$

$$S(r,\vartheta,\varphi) = S_r(r) + S_\vartheta(\vartheta) + S_\varphi(\varphi)$$

$$\begin{cases} \frac{dS}{d\varphi} = \alpha_\varphi = M \\ \left(\frac{dS}{d\vartheta}\right)^2 + \sin^{-2}(\vartheta)\left(\frac{\partial S}{\partial \varphi}\right)^2 = \alpha_\vartheta^2 = L^2 \\ \left(\frac{dS}{dr}\right)^2 - 2m\left(E - V\right) = -\dfrac{\alpha_\vartheta^2}{R^2\begin{pmatrix}\sin\\ \sinh\end{pmatrix}^2\left(\frac{r}{R}\right)} \end{cases}$$

$$J_\varphi = \oint p_\varphi d\varphi = \oint M d\varphi = 2\pi M = 2\pi\alpha_\varphi$$

$$J_\vartheta = \oint p_\vartheta d\vartheta = \oint \pm\sqrt{\alpha_\vartheta^2 - \frac{\alpha_\varphi^2}{\sin^2(\vartheta)}}\, d\vartheta = 2\pi\left(\alpha_\vartheta - \alpha_\varphi\right) = 2\pi\left(L - M\right)$$

$$J_\vartheta + J_\varphi = 2\pi L = 2\pi\alpha_\vartheta$$

$$\boxed{J_r = \oint p_r dr = \oint \pm \sqrt{2m\left(E - V(r)\right) - \frac{\left(J_\vartheta + J_\varphi\right)^2}{4\pi^2 R^2 \begin{pmatrix} \sin \\ \sinh \end{pmatrix}^2 \left(\frac{r}{R}\right)}}\, dr}$$

One-fold degeneracy: J_ϑ, J_φ enter through $(J_\vartheta + J_\varphi)$. Characteristic feature of all spherically-symmetric models.

$\boxed{\text{Bertrand potentials:}}$

- $S^3(O, R)$:
  - Isotropic degenarate oscilator: $V = \frac{kR^2}{2}\tan^2\left(\frac{r}{R}\right)$ (singular on the "equator" $r = \frac{1}{2}\pi R$)

$$E = \frac{1}{2\pi}\omega_0\left(2J_r + J_\vartheta + J_\varphi\right) + \frac{\left(2J_r + J_\vartheta + J_\varphi\right)^2}{8\pi^2 m R^2}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$, $\nu_0 = \frac{1}{2\pi}\omega_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$. Total degenacy- E expressed through the integer-coefficients combination $J_r, J_\vartheta, J_\varphi$. After the apprioate change of action-angle variables, $J_1 = J = 2J_r + J_\vartheta + J_\varphi$

$$E = \nu_0 J + \frac{J^2}{8\pi^2 m R^2}$$

Fundamental frequency:

$$\nu = \frac{dE}{dJ} = \nu_0 + \frac{J}{4\pi^2 m R^2},$$

i.e.,

$$\nu(E) = \sqrt{\nu_0^2 + \frac{E}{2\pi^2 m R^2}}, \qquad \omega(E) = \sqrt{\omega_0^2 + \frac{2E}{mR^2}}.$$

No isochronism. ω depends on initial conditions, but only through the energy parameter E. Coefficient 2 at J_r in E-two radial turning points for r per one period in the φ-variable.

– Coulomb problem, $V = -\frac{\alpha}{R}\cot\left(\frac{r}{R}\right)$
  Two conjugate antipodally placed singular poles-one attractive and one repulsive

$$E = -\frac{2\pi^2 m\alpha^2}{J^2} + \frac{J^2}{8\pi^2 m R^2}$$

$J = J_r + J_\vartheta + J_\varphi$
Equal coefficients-one radial turning point for r per one period in the φ-variable.
Very interesting: linear superposition of the usual formula for the Kepler problem in $\mathbb{R}^3$ and free geodetic motion in $S^3(O, R)$.

- $H^{3,2,+}(O, R)$

– isotropic degenerate oscilator $V = \frac{kR^2}{2}\tanh^2\left(\frac{r}{R}\right)$
Remark: it is bounded (!), $\sup V = \frac{kR^2}{2}$. There exists dissociation threshold

$$E = \nu_0 J - \frac{J^2}{8\pi^2 m R^2}, \quad J = 2J_r + J_\vartheta + J_\varphi$$

where $\omega_0 = \sqrt{\frac{k}{m}}$, $\nu_0 = \frac{1}{2\pi}\omega_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

$$\nu(E) = \sqrt{\nu_0^2 - \frac{E}{2\pi^2 m R^2}}, \qquad \omega(E) = \sqrt{\omega_0^2 - \frac{2E}{mR^2}}.$$

Valid below the dissociation threshold $\sup V = \frac{kR^2}{2}$.

– Coulomb problem, $V = -\frac{\alpha}{R}\coth\left(\frac{r}{R}\right)$

$$E = -\frac{2\pi^2 m\alpha^2}{J^2} - \frac{J^2}{8\pi^2 m R^2}$$

$J = J_r + J_\vartheta + J_\varphi$ Valid below the dissociation threshold $\sup V = -\frac{\alpha}{R}$.

- Bohr-Sommerfeld quantum conditions: $J = nh$

– $S^3\,(O,R)$

  * degenerate isotropic oscilator:

$$E_n = n\nu_0 h + \frac{n^2h^2}{8\pi^2 mR^2} = n\omega_0\hbar + \frac{n^2\hbar^2}{2mR^2}$$

  $n = 0, 1, 2, \ldots$

  * Coulomb:

$$E_n = -\frac{2\pi^2 m\alpha^2}{n^2h^2} + \frac{n^2h^2}{8\pi^2 mR^2} = -\frac{m\alpha^2}{2n^2\hbar^2} + \frac{n^2\hbar^2}{2mR^2},$$

  $n = 1, 2, 3, \ldots$ Valid for both sings of E_n! Purely discrete spectrum-compact configuration space.

– $H^{3,2,+}(O,R)$

  * degenerate isotropic oscillator:

$$E_n = n\nu_0 h - \frac{n^2h^2}{8\pi^2 mR^2} = n\omega_0\hbar - \frac{n^2\hbar^2}{2mR^2}$$

  Valid for such values of n that

  · $E_n \geq 0$

· $E_n < \sup V = \frac{kR^2}{2} = \frac{m{\omega_0}^2 R^2}{2}$ - satisfied automatically
Finite number of discrete energy levels for bounded states:

$$n \leq 2\frac{R^2}{\hbar}\sqrt{km}$$

$\ast$ attractive Coulomb problem

$$E_n = -\frac{2\pi^2 m\alpha^2}{n^2 h^2} - \frac{n^2 h^2}{8\pi^2 mR^2} = -\frac{m\alpha^2}{2n^2\hbar^2} - \frac{n^2\hbar^2}{2mR^2}.$$

Valid for such values of n that $E_n < -\frac{\alpha}{R}$ - satisfied automatically

– Free geodetic motion in $S^3(O, R)$:
Special case of Coulomb with $\alpha = 0$, but not of that oscilator with $k = 0$

$$E = \frac{J}{8\pi^2 mR^2} = \frac{(J_r + J_\vartheta + J_\varphi)^2}{8\pi^2 mR^2}$$

Bohr-Sommerfeld spectrum:

$$\begin{aligned} E_n &= \frac{n^2 h^2}{8\pi^2 mR^2} = \frac{n^2\hbar^2}{2mR^2}, \quad n = 0, 1, 2, 3 \ldots \\ &= E_k = \frac{2k^2\hbar^2}{mR^2}, \quad k = 0, \frac{1}{2}, 1, \ldots \end{aligned}$$

– Free geodetic motion in the Riemann elliptic space $S^3(O,R)$/antipodal identification

$$E = \frac{J}{2\pi^2 mR^2} = \frac{\left(J_r + J_\vartheta + J_\varphi\right)^2}{2\pi^2 mR^2}$$

Bohr-Sommerfeld spectrum:

$$\begin{aligned} E_n &= \frac{\left(2J_n\right)^2}{8\pi^2 mR^2} = \frac{(2n)^2 h^2}{8\pi^2 mR^2} = \frac{(2n)^2\hbar^2}{2mR^2}, \quad n = 0,1,2,3\ldots \\ &= E_k = \frac{2k^2\hbar^2}{mR^2}, \quad k = 0,1,2\ldots \end{aligned}$$

Schrödinger quantization:

A complete system of commuting operators:

- $\widehat{H} = -\frac{\hbar^2}{2m}\Delta + V(r)$-Hamiltonian
  $\Delta\Psi = \sum\limits_{i,j} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial q^i}\left(\sqrt{|g|}g^{ij}\frac{\partial\Psi}{\partial q^j}\right)$-Laplace-Beltrami, i.e., $\Delta = g^{ij}\nabla_i\nabla_j$, ∇- Levi-Civita differentiation
- $\widehat{\overline{L}}^2 = \sum\limits_{i=1}^{3}\left(\widehat{L}_i\right)^2$, eigenvalues $\hbar^2 l(l+1)$, $l = 0,1,2,\ldots$ $\widehat{L}_a = \frac{\hbar}{i}\varepsilon_{abc}r^b\frac{\partial}{\partial r^c}$

- $\widehat{L}_3$, eigenvalues $m\hbar$, $m = -l, -l+1, \ldots 0, \ldots l-1, l$,

Therefore, the standard separation of variables:

$$\Psi_{nlm}\left(r, \vartheta, \varphi\right) = f_{nl}(r) Y_{lm}(\vartheta, \varphi)$$

Scalar product:

$$\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi}_1\left(r, \vartheta, \varphi\right) \Psi_2\left(r, \vartheta, \varphi\right) d\mu\left(r, \vartheta, \varphi\right)$$

$d\mu = \sqrt{|g|} dr d\vartheta d\varphi$

$L^2(\mu)$- Hilbert space of wave functions

- $S^3(O, R) : \sqrt{|g|} dr d\vartheta d\varphi = R^2 \sin^2 \frac{r}{R} \sin\vartheta dr d\vartheta d\varphi$
- $H^{3,2,+}(O, R) : \sqrt{|g|} dr d\vartheta d\varphi = R^2 \sinh^2 \frac{r}{R} \sin\vartheta dr d\vartheta d\varphi$

Radial equations obtained from $\widehat{H}\Psi = E\Psi$

- $S^3(O, R)$:

$$\frac{d^2 f_{nl}}{dr^2} + \frac{2}{R} \cot \frac{r}{R} \frac{d f_{nl}}{dr} - \frac{l(l+1)}{R^2 \sin^2 \frac{r}{R}} f_{nl} - \frac{2mV}{\hbar^2} f_{nl} + \frac{2mE}{\hbar^2} f_{nl} = 0$$

- $H^{3,2,+}(O,R)$:

$$\frac{d^2 f_{nl}}{dr^2} + \frac{2}{R}\coth\frac{r}{R}\frac{df_{nl}}{dr} - \frac{l(l+1)}{R^2\sinh^2\frac{r}{R}} f_{nl} - \frac{2mV}{\hbar^2} f_{nl} + \frac{2mE}{\hbar^2} f_{nl} = 0$$

The obvious degeneracy with respect to m.

For Bertrand models- additional degeneracy with respect to l-just like in the Euclidean space

The quantum counterpart of classical degeneracy:

$$\begin{aligned} E &= E\left(J_r + J_\vartheta + J_\varphi\right) \qquad \text{Kepler} \\ E &= E\left(2J_r + J_\vartheta + J_\varphi\right) \qquad \text{oscillator} \end{aligned}$$

Spectra on $S^3(O,R)$

- Free motion:

$$E_j = \frac{2\hbar^2}{mR^2} j(j+1) \qquad j = 0, \frac{1}{2}, 1, \ldots$$

  Totat l-degeneracy
  In the Riemann elliptic space $S^3(O,R)$/antipodes the same formula with integer j's

- Degenerate oscilator:

$$E_j = \left((2j-1) + \frac{3}{2}\right)\hbar\widetilde{\omega} + \frac{2j(j+1)\hbar^2}{mR^2},$$

  $j = \frac{1}{2}, 1, \ldots$ $\widetilde{\omega} = \frac{\hbar}{8m}\left(\sqrt{1+64m^2\omega_0^2\hbar^{-2}} - 1\right)$
  Total l-degeneracy
  New specifically quantum features:

  - $\frac{3}{2}$ appears - expected
  - $j^2 \to j(j+1)$- expected
  - $\omega \to \widetilde{\omega}$- non-expected

- Coulomb problem:

$$E_j = -\frac{m\alpha^2}{2\left(2j+1\right)^2\hbar^2} + \frac{2j\left(j+1\right)\hbar^2}{mR^2}$$

  $j = \frac{1}{2}, 1, \frac{3}{2} \ldots$ $l = 0, 1, 2 \ldots 2j$
  Expected quantum modifications.

Modified numbering of energy levels

- Free motion

$$E_n = \frac{n\,(n+2)\,\hbar^2}{2mR^2}, \quad n = 0, 1, 2, \ldots$$

or

$$E_n = \frac{(n-1)\,(n+1)\,\hbar^2}{2mR^2}, \quad n = 1, 2, 3, \ldots$$

- Degenerate oscillator

$$E_n = \left(n + \frac{3}{2}\right)\hbar\Omega + \frac{(n+1)\,(n+3)\,\hbar^2}{2mR^2}$$

$$\Omega = \frac{\hbar}{2mR^2}\left(\sqrt{1 + 4m^2R^4\omega_0^2\hbar^{-2}} - 1\right)$$

$$n = 0, 1, 2, \ldots$$

- Coulomb problem

$$E_n = -\frac{m\alpha^2}{2n^2\hbar^2} + \frac{(n-1)\,(n+1)\,\hbar^2}{2mR^2}$$

$$l = 0, 1, 2, \ldots (n-1) \qquad n = 1, 2, 3, \ldots$$

THANK YOU FOR ATTENTION