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Affinely-rigid body and oscillatory two-dimensional models

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Geometric description of the affinely-rigid body

We are given two Euclidean spaces (N, U, η) and (M, V, g), respectively the material and physical spaces. Here N and M are the basic point spaces, U and V are their linear translation spaces, and $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$ are their metric tensors. The space N is used for labelling the material points, and elements of M are geometric spatial points.

The configuration space of the affinely-rigid body

$$Q := \mathrm{AfI}(N, M)$$

consists of affine isomorphisms of N onto M. The material labels $a \in N$ are parametrized by Cartesian coordinates a^K (Lagrange variables). Cartesian coordinates in M will be denoted by y^i and the corresponding geometric points by y. The configuration $\Phi \in Q$ is to be understood in such a way that the material point $a \in N$ occupies the spatial position $y = \Phi(a)$.

Let $\overline{\mu}$ denote the co-moving (Lagrangian) mass distribution in N; obviously, it is constant in time. Lagrange coordinates a^K in N will be always chosen in such a way that their origin $a^K = 0$ coincides with the centre of mass $\mathcal{C}$:

$$\int a^K d\overline{\mu}(a) = 0.$$

The configuration space may be identified then with $M \times \mathrm{LI}(U, V)$,

$$Q = \mathrm{AfI}(N, M) \simeq M \times \mathrm{LI}(U, V) = M \times Q_{\mathrm{int}},$$

where $\mathrm{LI}(U, V)$ denotes the manifold of all linear isomorphisms of U onto V. The Cartesian product factors refer respectively to the translational motion (M) and the internal or relative motion $(\mathrm{LI}(U, V))$.

The motion is described as a continuum of instantaneous configurations:

$$\Phi(t,a)^i = \phi^i{}_K(t)a^K + x^i(t), \tag{1}$$

where $x(t)$ is the centre of mass position and $\phi(t)$ tells us how constituents of the body are placed with respect to the centre of mass. The quantities $\left(x^i, \phi^i{}_K\right)$ are our generalized coordinates.

Obviously, if we put $U = V = \mathbb{R}^n$, then Q_{int} reduces to $\mathrm{GL}(n,\mathbb{R})$ and Q becomes the semi-direct product $\mathbb{R}^n \times_s \mathrm{GL}(n,\mathbb{R})$; $\mathbb{R}^n$ is then interpreted as an Abelian group with addition of vectors as a group operation.

Inertia of affinely-constrained systems of material points is described by two constant quantities:

$$m = \int d\overline{\mu}(a), \qquad J^{KL} = \int a^K a^L d\overline{\mu}(a),$$

i.e. the total mass m and the co-moving second-order moment $J \in U \otimes U$. More precisely, it is so in the usual theory based on the d'Alembert principle, when the kinetic energy is obtained by summation (integration) of usual (based on the metric g) kinetic energies of constituents,

$$T = \frac{1}{2} g_{ij} \int \frac{\partial \Phi^i}{\partial t}\frac{\partial \Phi^j}{\partial t} d\overline{\mu}(a).$$

Substituting to this general formula the above affine constraints (1) we obtain:

$$T = T_{\text{tr}} + T_{\text{int}} = \frac{m}{2} g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2} g_{ij}\frac{d\phi^i_A}{dt}\frac{d\phi^i_B}{dt} J^{AB}.$$

Obviously, if we analytically identify U and V with $\mathbb{R}^n$ and $\mathrm{LI}(U,V)$ with $\mathrm{GL}(n,\mathbb{R})$, then

$$T_{\text{int}} = \frac{1}{2}\mathrm{Tr}\left(\dot{\phi}^T \dot{\phi} J\right).$$

Some two-dimensional problems

Now, let us discuss the two-dimensional affinely-rigid body. Considered is a discrete or continuous system of material points subject to constraints according to which during any admissible motion all affine relations between constituents of the body are invariant (the material straight lines remain straight lines, their parallelism is conserved, and all mutual ratios of segments placed on the same straight lines are constant). The conception of the affinely-rigid body is a generalization of the usual metrically-rigid body, in which during any admissible motion all distances (metric relations) between its constituents are constant. We do not take into account the motion of the centre of mass. When translational motion is neglected, the configuration space Q may be analytically identified with the linear group $\mathrm{GL}(2,\mathbb{R})$, i.e., the group of non-singular real 2×2 matrices. The most adequate description of degrees of freedom is that based on the two-polar decomposition of matrices:

$$\phi = ODR^T, \tag{2}$$

where

$$O = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \ D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \ R = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}.$$

This decomposition is connected with the algebraic Gram-Schmid orthogonalization. It is also know in literature as the "singular value decomposition". The matrices $O, R \in \mathrm{SO}(2,\mathbb{R})$ are orthogonal ($O^T O = R^T R = \mathrm{Id}$, $\det O = \det R = 1$), D is diagonal and positive. The orthogonal group $\mathrm{SO}(2,\mathbb{R})$ is a commutative group of plane rotations. Spatial rotations are described by the action of $\mathrm{SO}(2,\mathbb{R})$ on $\mathrm{GL}(2,\mathbb{R})$ through the left regular translations, material rotations are represented by the action of the rotation subgroup through the right multiplication. In the non-degenerate case ($D_1 \neq D_2$), the decomposition is unique up to the permutation of the diagonal elements of D accompanied by the simultaneous multiplying of O and R on the right-side by the appropriate special orthogonal matrices (ones having in each row and column zeros but once ± 1 as elements). This implies that the potential energy of doubly isotropic models depends only on D and is invariant with respect to the permutations of its nonvanishing matrix elements. The deformation invariants D_1, D_2 are important mechanical quantities. They are scalar measures of deformation, i.e. tell us how strongly the body is deformed, but do not contain any information concerning the orientation of deformation in the physical or material space. The orthogonal matrices O and R describe the space and body orientations of the strain. Incidentally, let us mention that the complexification of $\mathrm{GL}(2,\mathbb{R})$ to $\mathrm{GL}(2,\mathbb{C})$ and then the restriction to the other, completely opposite (because compact), real form $\mathrm{U}(2)$ sheds some light on our model and establishes also certain kinship with the three-dimensional rigid body.

We shall consider only highly symmetric model, where J is isotropic, i.e., its matrix has the form μI, μ denoting a positive constant, and I is the 2×2 identity matrix. The isotropic kinetic energy is as follows:

$$\begin{aligned} T &= \frac{\mu}{2}\left[\left({D_1}^2 + {D_2}^2\right)\left(\left(\frac{d\varphi}{dt}\right)^2 + \left(\frac{d\psi}{dt}\right)^2\right) - 4D_1D_2\frac{d\varphi}{dt}\frac{d\psi}{dt}\right. \\ &+ \left.\left(\frac{dD_1}{dt}\right)^2 + \left(\frac{dD_2}{dt}\right)^2\right]. \end{aligned} \tag{3}$$

The matrices O and R do not enter into this equation, hence the angles φ, ψ are cyclic variables. In these coordinates the Hamilton-Jacobi equation is non-separable even in the interaction-free case. However, the separability becomes possible in new variables, obtained by the $\pi/4$-rotation in the plane of the deformation invariants D_1, D_2 and by an appropriate modification of the angular variables. Thus, we introduce the following new coordinates:

$$\alpha = \frac{1}{\sqrt{2}}\left(D_1 + D_2\right), \quad \beta = \frac{1}{\sqrt{2}}\left(D_1 - D_2\right), \quad \eta = \varphi - \psi, \quad \gamma = \varphi + \psi.$$

The kinetic energy becomes then

$$T = \frac{\mu}{2}\left[\alpha^2\left(\frac{d\eta}{dt}\right)^2 + \beta^2\left(\frac{d\gamma}{dt}\right)^2 + \left(\frac{d\alpha}{dt}\right)^2 + \left(\frac{d\beta}{dt}\right)^2\right]. \tag{4}$$

This form is both diagonal and separable.

The classical Stäckel theorem leads to the following general form of separable potentials:

$$V(\varphi,\psi,\alpha,\beta)=\frac{V_\eta\,(\varphi-\psi)}{\alpha^2}+\frac{V_\gamma\,(\varphi+\psi)}{\beta^2}+V_\alpha(\alpha)+V_\beta(\beta). \tag{5}$$

In this formula V_η, V_γ, V_α, V_β are arbitrary (but regular enough) functions of a single variable (indicated as an argument). We consider doubly-isotropic models in which the potential energy does not depend on variables φ, ψ (equivalently η, γ), i.e. $V_\eta=0$ and $V_\gamma=0$. Performing the Legendre transformation we obtain the corresponding Hamiltonian $H=H_\alpha+H_\beta$ in the form:

$$H=\frac{1}{2\mu}\left(\frac{(p_\varphi-p_\psi)^2}{4\alpha^2}+{p_\alpha}^2\right)+\frac{1}{2\mu}\left(\frac{(p_\varphi+p_\psi)^2}{4\beta^2}+{p_\beta}^2\right)+V_\alpha(\alpha)+V_\beta(\beta), \tag{6}$$

where p_φ, p_ψ, p_α, p_β are the canonical momenta conjugate to φ, ψ, α, β, respectively, and

$$\begin{aligned}H_\alpha &= \frac{1}{2\mu}\left(\frac{(p_\varphi-p_\psi)^2}{4\alpha^2}+{p_\alpha}^2\right)+V_\alpha(\alpha),\\ H_\beta &= \frac{1}{2\mu}\left(\frac{(p_\varphi+p_\psi)^2}{4\beta^2}+{p_\beta}^2\right)+V_\beta(\beta).\end{aligned} \tag{7}$$

The quantities $H_\alpha, H_\beta, p_\varphi, p_\psi$ form a Poisson-involutive system of constants of motion.

The stationary Hamilton-Jacobi equation has the following form:

$$\left(\frac{1}{4\alpha^2}+\frac{1}{4\beta^2}\right)\left(\left(\frac{\partial S}{\partial\varphi}\right)^2+\left(\frac{\partial S}{\partial\psi}\right)^2\right)+\left(\frac{1}{2\beta^2}-\frac{1}{2\alpha^2}\right)\frac{\partial^2 S}{\partial\varphi\partial\psi} \tag{8}$$

$$+\left(\frac{\partial S}{\partial\alpha}\right)^2+\left(\frac{\partial S}{\partial\beta}\right)^2=2\mu\left(E-\left(V_\alpha(\alpha)+V_\beta(\beta)\right)\right),$$

where E is a fixed value of the energy. Due to the fact that the variables φ, ψ have the cyclic character, we may write:

$$S=S_\varphi(\varphi)+S_\psi(\psi)+S_\alpha(\alpha)+S_\beta(\beta)=a\varphi+b\psi+S_\alpha(\alpha)+S_\beta(\beta)$$

and the action variables are as follows:

$$J_\varphi=\oint p_\varphi d\varphi=2\pi a,\quad J_\alpha=\pm\oint\sqrt{2\mu\left(E_\alpha-V_\alpha(\alpha)\right)-\frac{\left(J_\varphi-J_\psi\right)^2}{16\pi^2\alpha^2}}d\alpha, \tag{9}$$

$$J_\psi=\oint p_\psi d\psi=2\pi b,\quad J_\beta=\pm\oint\sqrt{2\mu\left(E_\beta-V_\beta(\beta)\right)-\frac{\left(J_\varphi+J_\psi\right)^2}{16\pi^2\beta^2}}\,d\beta, \tag{10}$$

where E_α, E_β, a, b are separation constants.

Remark. Let us observe that the isotropic kinetic energy

$$T = \frac{\mu}{2}\mathrm{Tr}\left(\dot{\phi}^T\dot{\phi}\right) \tag{11}$$

may be simply written as

$$T = \frac{\mu}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \dot{u}^2\right), \tag{12}$$

where x, y, z, u are simply the matrix elements of ϕ,

$$\phi = \left[\begin{array}{cc} x & y \\ z & u \end{array}\right]. \tag{13}$$

This is formally the expression for the material point with the mass μ in $\mathbb{R}^4$ or the quadruple of such material points in $\mathbb{R}$. However, in the mechanics of deformable bodies these generalized coordinates are not very useful for dynamical models.

It is both convenient and instructive to use also other generalized coordinates in the affine kinematics. We mean coordinates in which the problem is separable; as mentioned, the separability in various coordinates corresponds geometrically to some degeneracy of the problem. And besides, those coordinates suggest some modifications of the potential V leading to new models of deformative dynamics, more realistic than the harmonic oscillator and at the same time admitting also some analytical treatment. As expected, in doubly isotropic models the most natural candidates are to be sought among orthogonal coordinates on the plane of the deformation invariants (D_1, D_2). The most natural of them are just the variables α, β introduced above: they are obtained from D_1, D_2 by the rotation by $\pi/4$ in $\mathbb{R}^2$. Together with the modified angular variables η, γ they provide a system of T-orthogonal coordinates in $\mathbb{R}^4$, i.e., in the space of variables x, y, z, u. To be more precise, they are orthogonal coordinates for the metric element $dx^2 + dy^2 + dz^2 + du^2$ on which the kinetic energy T is based. And moreover, as said above, they are the nice separation variables for T in the Stäckel sense. Other natural T-separating variables are obtained as some byproducts of α, β. The most natural of them are polar variables in the $\mathbb{R}^2$-plane of the pairs (α, β). In certain problems it is analytically convenient to use the modified "polar" variables r, ϑ given by

$$\alpha = \sqrt{r}\cos\frac{\vartheta}{2}, \quad \beta = \sqrt{r}\sin\frac{\vartheta}{2}.$$

Obviously, the "literal" polar variables ρ, ϵ are defined by

$$\alpha = \rho\cos\epsilon, \quad \beta = \rho\sin\epsilon; \quad \rho = \sqrt{r}, \quad \epsilon = \frac{\vartheta}{2}.$$

The natural metric on the manifold of 2×2 matrices,

$$ds^2 = \mathrm{Tr}\left(d\phi^T d\phi\right) = dx^2 + dy^2 + dz^2 + du^2,$$

becomes then

$$\begin{aligned} ds^2 &= r\cos^2\frac{\vartheta}{2}d\eta^2 + r\sin^2\frac{\vartheta}{2}d\gamma^2 + \frac{1}{4r}dr^2 + \frac{r}{4}d\vartheta^2 \\ &= d\rho^2 + \rho^2 d\epsilon^2 + \rho^2\cos^2\epsilon\; d\eta^2 + \rho^2\sin^2\epsilon\; d\gamma^2 \\ &= d\rho^2 + \frac{1}{4}\rho^2 d\vartheta^2 + \rho^2\cos^2\frac{\vartheta}{2}\; d\eta^2 + \rho^2\sin^2\frac{\vartheta}{2}\; d\gamma^2. \end{aligned}$$

Obviously, kinetic energy is then expressed as follows

$$\begin{aligned} T &= \frac{\mu}{2}\left(\frac{1}{4r}\left(\frac{dr}{dt}\right)^2 + \frac{r}{4}\left(\frac{d\vartheta}{dt}\right)^2 + r\cos^2\frac{\vartheta}{2}\left(\frac{d\eta}{dt}\right)^2 + r\sin^2\frac{\vartheta}{2}\left(\frac{d\gamma}{dt}\right)^2\right) \\ &= \frac{\mu}{2}\left(\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\epsilon}{dt}\right)^2 + \rho^2\cos^2\epsilon\left(\frac{d\eta}{dt}\right)^2 + \rho^2\sin^2\epsilon\left(\frac{d\gamma}{dt}\right)^2\right) \\ &= \frac{\mu}{2}\left(\left(\frac{d\rho}{dt}\right)^2 + \frac{1}{4}\rho^2\left(\frac{d\vartheta}{dt}\right)^2 + \rho^2\cos^2\frac{\vartheta}{2}\left(\frac{d\eta}{dt}\right)^2 + \rho^2\sin^2\frac{\vartheta}{2}\left(\frac{d\gamma}{dt}\right)^2\right). \end{aligned}$$

The above crowd of expressions is due to the fact that different conventions are better suited to different analogies: the two-dimensional homogeneously deformable body and three-dimensional spherical top with dilatations. Physically we are interested here in the first problem, however, certain aspects of the second one (spherical top with dilatations) are formally useful and the mysterious link between them is interesting in itself.

Let us notice that (r, ϑ) may be interpreted as polar coordinates in the two-dimensional space of quantities $2D_1D_2$, ${D_1}^2 - {D_2}^2$,

$$2D_1D_2 = r\cos\vartheta, \quad {D_1}^2 - {D_2}^2 = r\sin\vartheta, \tag{14}$$

or, inverting these formulas,

$$r = \rho^2 = {D_1}^2 + {D_2}^2, \quad \tan\vartheta = \tan(2\epsilon) = \frac{1}{2}\left(\frac{D_1}{D_2} - \frac{D_2}{D_1}\right). \tag{15}$$

Therefore, ϑ refers to the shear degrees of freedom, whereas $r = \rho^2$ is some kind of the measure of size. More precisely, dilatation is measured by the product D_1D_2, thus,

$$r = \frac{2D_1D_2}{\cos\vartheta} \tag{16}$$

contains an "admixture" of the shear parameter ϑ. Nevertheless, just like D_1D_2 it is a homogeneous function of degree 2 of (D_1, D_2). The shear parameter ϑ is evidently a homogeneous function of degree zero.

It is also convenient to parametrize deformation invariants as follows:

$$D_1 = \exp\left(\frac{a+b}{2}\right), \quad D_2 = \exp\left(\frac{a-b}{2}\right).$$

Then

$$\alpha = \frac{1}{\sqrt{2}}(D_1 + D_2) = \sqrt{2}e^{\frac{a}{2}}\cosh\frac{b}{2}, \quad \beta = \frac{1}{\sqrt{2}}(D_1 - D_2) = \sqrt{2}e^{\frac{a}{2}}\sinh\frac{b}{2},$$

$$D_1D_2 = e^a, \quad {D_1}^2 + {D_2}^2 = 2e^a\cosh b, \quad {D_1}^2 - {D_2}^2 = 2e^a\sinh b, \quad \frac{D_1}{D_2} = e^b,$$

$$\sin\vartheta = \tanh b, \quad \cos\vartheta = \frac{1}{\cosh b}, \quad \tan\vartheta = \sinh b.$$

These simple formulas shed some light onto the link between two-dimensional homogeneously deformable body and three-dimensional top. Nevertheless, this link is still rather mysterious and obscure.

For the completeness let us also mention about other orthogonal coordinates on the plane of deformation invariants:

(i) Elliptic variables (κ, λ), where

$$\alpha = \sqrt{2} \cosh \kappa \cos \lambda, \quad \beta = \sqrt{2} \sinh \kappa \sin \lambda.$$

(ii) Parabolic variables (ξ, δ), where

$$\alpha = \frac{1}{2} \left(\xi^2 - \delta^2 \right), \quad \beta = \xi \delta.$$

(iii) Two-polar variables (e, f), where

$$\alpha = \frac{c \sinh e}{\cosh e - \cos f}, \quad \beta = \frac{c \sin f}{\cosh e - \cos f},$$

and c is a constant.

For our analysis of the deformative motion the parabolic (ξ, δ) and two-polar variables (e, f) are non-useful, because the corresponding Hamilton-Jacobi equations are non-separable even in the non-physical geodetic models, i.e., ones with vanishing potentials. In the elliptic coordinates (κ, λ) the metric underlying the kinetic energy takes on the form:

$$\begin{aligned} ds^2 &= \operatorname{Tr} \left(d\phi^T d\phi \right) = \left(\cosh^2 \kappa - \cos^2 \lambda \right) d\kappa^2 \\ &+ \left(\cosh^2 \kappa - \cos^2 \lambda \right) d\lambda^2 + \cosh^2 \kappa \cos^2 \lambda d\eta^2 + \sinh^2 \kappa \sin^2 \lambda d\gamma^2. \end{aligned}$$

The general Stäckel-separable Hamiltonians $H=T+V$ in variables $(\alpha,\beta,\eta,\gamma)$, $(r,\vartheta,\eta,\gamma)$ and $(\kappa,\lambda,\eta,\gamma)$ have respectively the form:

$$\begin{aligned} H &= \frac{1}{2\mu}\left(\left({p_\alpha}^2+\frac{{p_\eta}^2}{\alpha^2}\right)+\left({p_\beta}^2+\frac{{p_\gamma}^2}{\beta^2}\right)\right) \\ &+ V_\alpha(\alpha)+V_\beta(\beta)+\frac{V_\eta(\eta)}{\alpha^2}+\frac{V_\gamma(\gamma)}{\beta^2}, \end{aligned} \tag{17}$$

$$\begin{aligned} H &= \frac{1}{2\mu}\left(4r{p_r}^2+\frac{1}{r}\left(\frac{{p_\varphi}^2+{p_\psi}^2+2p_\varphi p_\psi\cos\vartheta}{\sin^2\vartheta}+4{p_\vartheta}^2\right)\right) \\ &+ V_r(r)+\frac{V_\vartheta(\vartheta)}{r}+\frac{V_\eta(\eta)}{r\cos^2\frac{\vartheta}{2}}+\frac{V_\gamma(\gamma)}{r\sin^2\frac{\vartheta}{2}}, \end{aligned} \tag{18}$$

$$\begin{aligned} H &= \frac{1}{4\mu}\left(\frac{{p_\kappa}^2}{(\cosh^2\kappa-\cos^2\lambda)}+\frac{{p_\lambda}^2}{(\cosh^2\kappa-\cos^2\lambda)}\right. \\ &+ \left.\frac{{p_\eta}^2}{\cosh^2\kappa\cos^2\lambda}+\frac{{p_\gamma}^2}{\sinh^2\kappa\sin^2\lambda}\right) \\ &+ \frac{V_\kappa(\kappa)}{2\left(\cosh^2\kappa-\cos^2\lambda\right)}+\frac{V_\lambda(\lambda)}{2\left(\cosh^2\kappa-\cos^2\lambda\right)} \\ &+ \frac{V_\eta(\eta)}{2\cosh^2\kappa\cos^2\lambda}+\frac{V_\gamma(\gamma)}{2\sinh^2\kappa\sin^2\lambda}. \end{aligned} \tag{19}$$

Let us observe that, obviously,

$$\cosh^2\kappa - \cos^2\lambda = \sinh^2\kappa + \sin^2\lambda$$

and it does not matter what is written in the corresponding denominators above. Making use of this fact we immediately see that when the problem is doubly isotropic, i.e., V_η, V_γ are constant, then obviously (p_η, p_γ), equivalently (p_φ, p_ψ), are constants of motion but also there is a separation of the Hamilton-Jacobi equation in the variables κ, λ. Therefore, there are two additional constants of motion and the problem is integrable. Those constants of motion are given by

$$\begin{aligned} K &= \frac{h_\kappa \cos^2\lambda - h_\lambda \cosh^2\kappa}{2\left(\cosh^2\kappa - \cos^2\lambda\right)} = \frac{h_\kappa \cos^2\lambda - h_\lambda \cosh^2\kappa}{2\left(\sinh^2\kappa + \sin^2\lambda\right)}, \\ L &= \frac{h_\kappa \sin^2\lambda - h_\lambda \sinh^2\kappa}{2\left(\sinh^2\kappa + \sin^2\lambda\right)} = \frac{h_\kappa \sin^2\lambda - h_\lambda \sinh^2\kappa}{2\left(\cosh^2\kappa - \cos^2\lambda\right)}, \end{aligned}$$

where the auxiliary quantities h_κ, h_λ are not constants of motion and are respectively given by

$$h_\kappa = \frac{1}{2\mu}\left({p_\kappa}^2 + 2\mu V_\kappa - \frac{\frac{1}{4}(p_\varphi - p_\psi)^2 + 2\mu V_\kappa}{\cosh^2\kappa} + \frac{\frac{1}{4}(p_\varphi + p_\psi)^2 + 2\mu V_\kappa}{\sinh^2\kappa}\right),$$

$$h_\lambda = \frac{1}{2\mu}\left({p_\lambda}^2 + 2\mu V_\lambda + \frac{\frac{1}{4}(p_\varphi - p_\psi)^2 + 2\mu V_\lambda}{\cos^2\lambda} + \frac{\frac{1}{4}(p_\varphi + p_\psi)^2 + 2\mu V_\lambda}{\sin^2\lambda}\right);$$

we remember that V_κ, V_λ are constants here.

Therefore, we have the involutive system of constants of motion (their Poisson brackets do vanish), and

$$H = K + L$$

has the vanishing Poisson brackets with all of them, i.e., with p_φ, p_ψ (i.e., with p_η, p_γ), K, L.

The elliptic coordinates and the corresponding separable models are not very interesting for applications. From this point of view the "polar" coordinates (r, ϑ), or equivalently (ρ, ϵ), are much more useful. The configurational metric tensor is then expressed as follows:

$$\begin{aligned} ds^2 &= \mathrm{Tr}\left(d\phi^T d\phi\right) = \frac{1}{4r} dr^2 + \frac{r}{4} d\vartheta^2 + r d\varphi^2 - 2r\cos\vartheta d\varphi d\psi + r d\psi^2 \\ &= d\rho^2 + \frac{1}{4}\rho^2\left(d\vartheta^2 + d(2\varphi)^2 - 2\cos\vartheta d(2\varphi) d(2\psi) + d(2\psi)^2\right) \\ &= \frac{1}{4r}\left(dr^2 + r^2\left(d\Theta^2 + d\Phi^2 - 2\cos\Theta d\Phi d\Psi + d\Psi^2\right)\right), \end{aligned}$$

where, obviously, the doubled angles are used, $\Theta = \vartheta$, $\Phi = 2\varphi$, $\Psi = 2\psi$. This expression is very interesting in itself. We used here three alternative systems of symbols, each of them convenient and suggestive in some areas of applications. It is seen that the expression

$$d\sigma^2 = d\Theta^2 + d\Phi^2 - 2\cos\Theta d\Phi d\Psi + d\Psi^2$$

is exactly, up to a constant multiplier, identical with the doubly-invariant (i.e., both left- and right-invariant) squared metric element on the rotation group in three dimensions, $\mathrm{SO}(3, \mathbb{R})$, or on its covering group $\mathrm{SU}(2)$. This identification is based on interpreting Φ, Θ, Ψ as Euler angles. More precisely, to be literal in this analogy, one should change the sign at Ψ, then one obtains the usual expression

$$d\sigma^2 = d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2.$$

This metric underlies the kinetic energy expression for the spherical top,

$$T = \frac{I}{2}\left(\left(\frac{d\Theta}{dt}\right)^2 + \left(\frac{d\Phi}{dt}\right)^2 + 2\cos\Theta\frac{d\Phi}{dt}\frac{d\Psi}{dt} + \left(\frac{d\Psi}{dt}\right)^2\right).$$

In mechanics of gyroscopic systems Φ, Θ, Ψ are referred to respectively as the precession, nutation and rotation angles. This, of course, has nothing to do with our object, i.e., homogeneously deformable two-dimensional body; such a body has only one rotational degree of freedom. The analogy is formal, nevertheless instructive and effective in the computational sense. The idea has to do with the "concentric" parametrization of $\mathbb{R}^4$.

As mentioned, the Cartesian variables x, y, z, u, i.e., matrix elements of the configuration matrix ϕ, are non-effective when investigating deformations. This was just the reason to use the two-polar decomposition and the corresponding coordinates $(D_1, D_2, \varphi, \psi)$ or $(\alpha, \beta, \varphi, \psi)$. The two "radii" (D_1, D_2) or (α, β) have to do with the purely scalar deformation; (φ, ψ) (equivalently (η, γ)) are angular variables of compact topology (orientation of deformations in the physical space and in the body). The "concentric" parametrization consists in encoding the possibility of unbounded motion in the radial variable in $\mathbb{R}^4$,

$$\rho = \sqrt{r} = \sqrt{x^2 + y^2 + z^2 + u^2} = \sqrt{{D_1}^2 + {D_2}^2} = \sqrt{\mathrm{Tr}(\phi^{\mathrm{T}}\phi)} = \sqrt{\mathrm{Tr}G},$$

where the symbol G is used for the Green deformation tensor expressed in the Cartesian coordinates. More geometrically, we are dealing here with the deformation invariant:

$$\rho = \sqrt{\eta^{AB} G_{AB}} = \sqrt{g_{ij} {\phi^i}_A {\phi^j}_B \eta^{AB}},$$

g, η denotes respectively the spatial and material (reference) metric tensors.

Degrees of freedom orthogonally transversal to the radial variable ρ (or equivalently r) describe the geometrically bounded aspect of motion. Those modes of motion are encoded in the concentric spheres in $\mathbb{R}^4$, in particular, in the unit sphere given by equation $\rho = 1$, i.e., $r = 1$. But it is well-known that the group $\mathrm{SU}(2)$, i.e., the group of unitary unimodular matrices and the covering group of $\mathrm{SO}(3, \mathbb{R})$, may be naturally identified with the unit sphere $S^3(0, 1) \subset \mathbb{R}^4$. And in this way this sphere may be parametrized with the use of the Euler angles Φ, Θ, Ψ. The parametrization of $\mathbb{R}^4$ with the use of variables $(\rho, \Phi, \Theta, \Psi)$ or (r, Φ, Θ, Ψ) is rather nonusual, however well-suited to the description of the three-dimensional rigid body with imposed dilatations or, as we see, to the description of the two-dimensional homogeneously deformable body. In other applications one uses rather spherical systems of coordinates in $\mathbb{R}^4$, e.g., r, λ, μ, ν, where

$$\begin{aligned} x^1 &= r \sin\lambda \cos\mu \cos\nu, \\ x^2 &= r \sin\lambda \cos\mu \sin\nu, \\ x^3 &= r \sin\lambda \sin\mu, \\ x^4 &= r \cos\lambda. \end{aligned}$$

Let us mention that the isotropic harmonic oscillator may be described obviously in terms of those variables, and the expression of Hamiltonian through the action variables J_r, J_λ, J_μ, J_ν, in analogy to (32) below, is given by

$$H = \omega(2J_r + J_\lambda + J_\mu + J_\nu), \tag{20}$$

where the degeneracy, i.e., the resonance between J_r, J_λ, J_μ, J_ν is explicitly seen.

One can also use certain mixed type parametrizations in $\mathbb{R}^4$, e.g., representing it as $\mathbb{R}^3 \times \mathbb{R}$, $\mathbb{R}^2 \times \mathbb{R}^2$ and taking spherical coordinates in $\mathbb{R}^3$ or polar ones in one or two copies of $\mathbb{R}^2$. In all such coordinate systems the isotropic harmonic oscillator is separable and this is some aspect of its very high, total degeneracy.

However, it is hard to realize a wider class of realistic applications of these coordinates, e.g., in elastic and similar problems. Unlike this, the apparently exotic parametrization in terms of the "radial distance" ρ and "Euler angles" Φ, Θ, Ψ offers certain models of potentials which are both separable and qualitatively physical.

We have quoted the general Stäckel-separable Hamiltonian in variables $(r, \vartheta, \varphi, \psi)$ (18). It is doubly isotropic when the shape functions V_η, V_γ are put as constants. Obviously, the corresponding terms $V_\eta / \cos^2(\vartheta/2)$, $V_\gamma / \sin^2(\vartheta/2)$ may be simply included into $V_\vartheta(\vartheta)$. We have the following four constants of motion in involution, responsible for separability:

- p_φ, p_ψ, i.e., equivalently p_η, p_γ,
- $h_\vartheta = \frac{1}{2\mu}\frac{1}{\sin^2\vartheta}\left({p_\varphi}^2 + {p_\psi}^2 + 2p_\varphi p_\psi \cos\vartheta\right) + \frac{2}{\mu}{p_\vartheta}^2 + V_\vartheta(\vartheta)$,
- $H = T + V = H_r + \frac{h_\vartheta}{r}$, where, however, the two indicated terms in H, namely

$$H_r = \frac{2}{\mu} r {p_r}^2 + V_r(r), \qquad \frac{h_\vartheta}{r}$$

  are not constants of motion when taken separately.

The term V_r stabilizes the radial mode of motion which without this term would be unbounded, therefore physically non-applicable in elastic problems. The term V_ϑ is responsible for the shear dynamics. Let us stress that in spite of the "angular" character of ϑ the shear mode of motion is also non-compact. It is just seen from the fact that the shear is algebraically expressed by the quantity $\tan\vartheta$, which is unbounded. Therefore, in certain problems some non-constant expression for V_ϑ is also desirable. Even if we use V_r proportional to $r = \rho^2$, any model with non-vanishing V_ϑ introduces some anharmonicity.

Particularly interesting is the following simple model:

$$V = V_r(r) + \frac{V_\vartheta(\vartheta)}{r} = \frac{C}{2}r + \frac{2C}{r\cos\vartheta} = C\left(\frac{1}{D_1 D_2} + \frac{{D_1}^2 + {D_2}^2}{2}\right). \tag{21}$$

The model is perhaps phenomenological and academic, however, from the "elastic" point of view it has very physical properties: it prevents the collapse to the point or straight-line, because the term $1/D_1 D_2$ is singularly repulsive there, and at the same time it prevents the unlimited expansion, because the harmonic oscillatory term $C({D_1}^2+{D_2}^2)/2 = C(\alpha^2+\beta^2)/2$ grows infinitely then. There is a stable continuum of relative equilibria at the non-deformed configurations when $D_1 = D_2 = 1$. Expansion along some axis results in contraction along the perpendicular axis, because

$$\frac{\partial^2 V}{\partial D_1 \partial D_2} > 0$$

at $D_1 = D_2 = 1$. This qualitatively physical potential of nonlinear hyperelastic vibrations is separable, therefore, at the same time it is also in principle analytically treatable. Its structure seems to suggest some three-dimensional models with the attractive harmonic term proportional to $({D_1}^2+{D_2}^2+{D_3}^2)$ and some collapse-preventing term, e.g., one proportional to $(D_1 D_2 D_3)^{-p}$ or $(D_1 D_2)^{-p}+(D_3 D_1)^{-p}+(D_2 D_3)^{-p}$, $p > 0$, however, there is no chance then for separability and integrability.

In the chapter below we begin with some problems concerning the harmonic oscillator,

$$\begin{aligned} V(\alpha,\beta) &= \frac{C}{2}\left(\alpha^2+\beta^2\right) = \frac{C}{2}\left({D_1}^2+{D_2}^2\right) \\ &= \frac{C}{2}(x^2+y^2+z^2+u^2) = \frac{C}{2}\mathrm{Tr}\left(\phi^T\phi\right), \ C > 0. \end{aligned} \tag{22}$$

and then discuss some natural anharmonic modifications.

Harmonic oscillator and certain anharmonic alternatives

The expressions J_α, J_β depend on potentials $V_\alpha(\alpha)$, $V_\beta(\beta)$, respectively. After specifying the form of these potentials we can obtain the Hamilton function H as some function of our action variables, i.e., $H = E(J_\alpha, J_\beta, J_\varphi, J_\psi)$. We can find the explicit dependence of the energy E on the action variables and the possible further degeneracy. We will also perform the usual Bohr-Sommerfeld quantization procedure for our model.

Hence, we consider the model of the harmonic oscillator potential (22). Some physical comments are necessary here. Namely, the potential (22) describes only the attractive forces which prevent the unlimited expansion of the body. Its non-physical feature is that it does not prevent the collapse, i.e., the contraction to the null-dimensional singularity. It attracts to the configuration $D_1 = D_2 = 0$ instead than to the non-deformed state $D_1 = D_2 = 1$. Nevertheless, the model may be useful in some range of initial conditions. Except the subset of measure zero in the manifold of those conditions, the collapse to $D_1 D_2 = 0$ is prevented by the centrifugal repulsion. And the collapse missbehaviour of (22) is not very malicious when the system is discrete. Obviously, (11) and (22) describe the isotropic harmonic oscillator in $\mathbb{R}^4$ or the quadruple of identical one-dimensional oscillators in $\mathbb{R}$. In this sense the solution is obvious and a priori known. Nevertheless, the model is a useful step towards investigating more realistic ones. And another point is very important. Namely, the very strong degeneracy of this model has to do, as usually, with the separability of the Hamilton-Jacobi equation in several coordinate systems.

After some calculations we obtain the dependence of the energy $E = E_\alpha + E_\beta$ on the action variables as follows:

$$E = \frac{\omega}{4\pi}\left[4J + |J_\varphi - J_\psi| + |J_\varphi + J_\psi|\right], \; J = J_\alpha + J_\beta, \tag{23}$$

where $\omega = \sqrt{C/\mu}$ and

$$\begin{aligned} E_\alpha &= \frac{\omega}{4\pi}\left(4J_\alpha + |J_\varphi - J_\psi|\right), \\ E_\beta &= \frac{\omega}{4\pi}\left(4J_\beta + |J_\varphi + J_\psi|\right). \end{aligned}$$

Then performing the Bohr-Sommerfeld quantization procedure, i.e. supposing that $J = nh$, $J_\varphi = mh$, $J_\psi = lh$, where h is the Planck constant and $n = 0, 1, \dots;\ m, l = 0, \pm 1, \dots$, we obtain the energy spectrum in the following form:

$$E = \frac{1}{2}\hbar\omega\left[4n + |m-l| + |m+l|\right]. \tag{24}$$

We may rewrite this formula as follows:

(*i*) if $|m| > |l|$, then $m^2 > l^2$ and

$$E = \hbar\omega\,(2n \pm m)\,, \tag{25}$$

(*ii*) if $|m| < |l|$, then $m^2 < l^2$ and

$$E = \hbar\omega\,(2n \pm l)\,, \tag{26}$$

(*iii*) if $|m| = |l|$, then $m^2 = l^2$ and

$$E = \hbar\omega\,(2n \pm m) = \hbar\omega\,(2n \pm l)\,. \tag{27}$$

And similarly, on the purely classical level of the action variables we have the following formulas:

(*i*) in the phase space region where $|J_\varphi| > |J_\psi|$:

$$E = \frac{\omega}{2\pi}\,(2J \pm J_\varphi) = \frac{\omega}{2\pi}\,(2J_\alpha + 2J_\beta \pm J_\varphi)\,, \tag{28}$$

(*ii*) in the region where $|J_\varphi| < |J_\psi|$:

$$E = \frac{\omega}{2\pi}\,(2J \pm J_\psi) = \frac{\omega}{2\pi}\,(2J_\alpha + 2J_\beta \pm J_\psi)\,, \tag{29}$$

(*iii*) on the submanifold where $J_\varphi = J_\psi$:

$$E = \frac{\omega}{2\pi}\,(2J \pm J_\varphi) = \frac{\omega}{2\pi}\,(2J \pm J_\psi)\,. \tag{30}$$

The total degeneracy of the doubly invariant model with the potential (22) is a priori obvious because in coordinates (x, y, z, u) it is explicitly seen that we deal with four-dimensional isotropic harmonic oscillator (equivalently–with the quadruple of identical non-interacting oscillators). If we use coordinates $(D_1, D_2, \varphi, \psi)$, or equivalently $(\alpha, \beta, \varphi, \psi)$, then the total degeneracy is visualized by the fact that the action variables J_α, J_β, J_φ, J_ψ enter (28) with integer coefficients, J_ψ with the vanishing one. Similarly in (29) they are also combined with integer coefficients, but now the coefficient at J_φ does vanish. The third case (30) is, so-to-speak, the seven-dimensional "separatrice" submanifold. The existence of those regions with various expressions for the functional dependence of energy on the action variables is due to the fact that the coordinate system $(D_1, D_2, \varphi, \psi)$ is not regular in the global sense and has some very peculiar singularities. Nevertheless, it is just those coordinates that are more natural and physically lucid in dynamical problems.

The quasiclassical degeneracy of the Bohr-Sommerfeld energy levels is due to the fact that the quantum numbers may be combined in a single one, although in slightly different ways in three possible ranges. Let us observe that in (25) the quantum number l still does exist although does not explicitly occur in the formula for E. It runs the range $|l| < |m|$ and labels quasiclassical states within the same energy levels. And analogously in the remaining cases (26), (27). The action variables J_φ, J_ψ and the corresponding quantum numbers m, l take symmetrically the positive and negative values, thus, as a matter of fact, the ambiguity of signs in the above formulas (25)–(27) does not matter when the values of energy in stationary states are concerned. Nevertheless, this ambiguity is essential for classical trajectories, namely, for different signs the orbits or rather their angular cycles are "swept" in opposite directions.

Let us observe that the formulas (28)–(30) resemble the action-angle description of the two-dimensional isotropic harmonic oscillator in terms of usual polar coordinates (r, φ) on $\mathbb{R}^2$. Namely, the Cartesian formula

$$E = \omega(J_x + J_y) \tag{31}$$

is then alternatively reformulated as

$$E = \omega(2J_r + J_\varphi). \tag{32}$$

The ratio $2 : 1$ of coefficients is due to the fact that the total angular rotation in the φ-variable is accompanied by the exactly two total cycles of "libration" in the r-variable. The analogy is neither accidental nor superficial. For the deformative motion the deformation invariants D_1, D_2, i.e., stretchings, are analogues to the radial variable r, whereas the two-polar angles φ, ψ describing the spatial and material orientation of stretchings play a role similar to the polar angle φ in material point dynamics on $\mathbb{R}^2$ (do not confuse–the symbol φ is used in two different meanings). This is just the reason for the $2 : 1$ ratio in (20) and (28)–(30).

Let us now review certain still isotropic, but anharmonic modifications of the harmonic model of affine vibrations (11) and (22). They are based on the use of variables $(\alpha, \beta, \varphi, \psi)$ or $(\rho, \vartheta, \varphi, \psi)$. The corresponding potentials are given by

$$V(\alpha, \beta) = \frac{C}{2}\left(\alpha^2 + \frac{4}{\alpha^2}\right) + \frac{C}{2}\beta^2 = \frac{C}{2}\left(\alpha^2 + \beta^2\right) + \frac{2C}{\alpha^2}, \tag{33}$$

$$V(\rho, \vartheta) = \frac{C}{2}\left(\rho^2 + \frac{4}{\rho^2}\right) + \frac{2C}{\rho^2}\tan^2\frac{\vartheta}{2} = \frac{C}{2}\rho^2 + \frac{2C}{\rho^2}\frac{1}{\cos^2\frac{\vartheta}{2}}, \tag{34}$$

where in both formulas C denoting some positive constant.

Using the former symbols we have

$$V_\alpha = \frac{C}{2}\left(\alpha^2 + \frac{4}{\alpha^2}\right), \; V_\beta = \frac{C}{2}\beta^2, \; V_r = \frac{C}{2}r, \; V_\vartheta = \frac{2C}{\cos^2\frac{\vartheta}{2}}.$$

An important peculiarity of these models is that they have the stable equilibria in the natural configuration $D_1 = D_2 = 1$, so they are viable from the elastic point of view. And both of them are separable ((97) in the obvious additive sense), therefore, the corresponding Hamiltonian systems are integrable.

One can calculate explicitly the action variables $(J_\alpha, J_\beta, J_\varphi, J_\psi)$ and $(J_r, J_\vartheta, J_\varphi, J_\psi)$ that correspond to (97) and (34). They are some functions of the separation constants (one of them is the energy E). Eliminating other constants one obtains the expression of E, or more precisely, of the Hamiltonian H, as a function of action variables.

For the model (97) one obtains

$$E = \frac{\omega}{4\pi}\left(4(J_\alpha + J_\beta) + |J_\varphi + J_\psi| + \sqrt{64\mu\pi^2 C + (J_\varphi - J_\psi)^2}\right),$$

where, as usually, we denote

$$\omega = \sqrt{\frac{C}{\mu}}.$$

It is seen that the collapse-preventing term C/α^2 in V_α partially removes the degeneracy. Evidently, there is no longer resonance between φ and ψ. The resonance between α and β obviously survives; their conjugate actions J_α, J_β enter the energy formula through the rational combination $J = J_\alpha + J_\beta$ and the corresponding frequencies are equal:

$$\nu_\alpha = \nu_\beta = \frac{\omega}{\pi}.$$

Obviously, we use here the standard formulas:

$$\nu_\alpha = \frac{\partial E}{\partial J_\alpha}, \quad \nu_\beta = \frac{\partial E}{\partial J_\beta}, \quad \nu_\varphi = \frac{\partial E}{\partial J_\varphi}, \quad \nu_\psi = \frac{\partial E}{\partial J_\psi}.$$

There are two phase-space regions given respectively by $J_\varphi + J_\psi > 0$ and $J_\varphi + J_\psi < 0$. In any of these regions there is a resonance between $\gamma = \varphi + \psi$ and α, β. This is seen from the formulas

$$J_\varphi = J_\eta + J_\gamma, \quad J_\psi = -J_\eta + J_\gamma.$$

In the mentioned regions we have respectively

$$E = \frac{\omega}{4\pi}\left(4J_\alpha + 4J_\beta \pm 2J_\gamma + \sqrt{16\mu\pi^2 C + {J_\eta}^2}\right).$$

This implies the following independent resonances:

$$\nu_\alpha - \nu_\beta = 0, \quad \nu_\alpha \mp 2\nu_\gamma = 0$$

or, equivalently,

$$\nu_\alpha - \nu_\beta = 0, \quad \nu_\beta \mp 2\nu_\gamma = 0.$$

Therefore, in any of the mentioned regions, where $J_\gamma > 0$ or $J_\gamma < 0$, the system is twice degenerate and the closures of its trajectories are two-dimensional isotropic tori in the eight-dimensional phase space.

Using the primary variables φ, ψ, we have the following expressions for ν_φ, ν_ψ:

$$\begin{aligned}
\nu_\varphi &= \frac{\omega}{4\pi}\left(\pm 1 + \frac{2(J_\varphi - J_\psi)}{\sqrt{64\mu\pi^2 C + (J_\varphi - J_\psi)^2}}\right), \\
\nu_\psi &= \frac{\omega}{4\pi}\left(\pm 1 + \frac{2(J_\psi - J_\varphi)}{\sqrt{64\mu\pi^2 C + (J_\psi - J_\varphi)^2}}\right),
\end{aligned}$$

the $\pm$ signs respectively in the regions where $J_\varphi + J_\psi > 0$ or $J_\varphi + J_\psi < 0$. Then, taking into account that

$$\omega = \pi\nu_\alpha = \pi\nu_\beta = \pi\nu = \frac{\partial E}{\partial J},$$

we have the following degeneracy conditions:

$$\nu_\alpha - \nu_\beta = 0, \quad \nu_\alpha \mp 2\nu_\varphi \mp 2\nu_\psi = 0,$$

respectively in the regions where $J_\alpha + J_\beta > 0$ or $J_\alpha + J_\beta < 0$. Obviously, in the second equation, ν_α may be equivalently replaced by ν_β.

The corresponding quasiclassical Bohr-Sommerfeld spectrum is given by

$$E = \frac{1}{2}\hbar\omega\left(4n + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right). \tag{35}$$

Another interesting model is (34), separable in the variables (ρ, ϑ), i.e., equivalently (r, ϑ). Then we obtain

$$\begin{aligned} E &= \frac{\omega}{4\pi}\left(4(J_r + J_\vartheta) + |J_\varphi + J_\psi| + \sqrt{64\mu\pi^2 C + (J_\varphi - J_\psi)^2}\right) \\ &= \frac{\omega}{4\pi}\left(4(2J_\rho + J_\vartheta) + |J_\varphi + J_\psi| + \sqrt{64\mu\pi^2 C + (J_\varphi - J_\psi)^2}\right). \end{aligned}$$

Again there is only a two-fold degeneracy and the system is not periodic. Trajectories are dense in two-dimensional isotropic tori. Degeneracy is described by the following pair of independent equations:

$$\nu_\rho - 2\nu_\vartheta = 0, \quad \nu_\vartheta \mp 2\nu_\varphi \mp 2\nu_\psi = 0,$$

respectively in the phase-space regions where $J_\varphi + J_\psi > 0$ or $J_\varphi + J_\psi < 0$. Obviously, the second equation may be alternatively replaced by

$$\nu_\rho \mp 4\nu_\varphi \mp 4\nu_\psi = 0.$$

The corresponding quasiclassical Bohr-Sommerfeld spectrum is given by

$$E = \frac{1}{2}\hbar\omega\left(4n + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right),$$

where the quantum numbers n, m, l, refer respectively to the action variables J, J_φ, J_ψ, and the system is twice degenerate. Quasiclassical energy levels are labelled by two effective quantum numbers, namely, $(4n+m+l)$ and $(m-l)$, and there is also an obvious degeneracy with respect to the simultaneous change of signs of m and l.

Let us mention that some anharmonic potentials independent of ϑ, e.g., the first term in (34), are also of some practical utility as models of a bounded motion. The point is that, as seen in formula (15), the variable r depends both on the area of the body (its "two-dimensional volume") and on the shear parameter. Therefore, to be bounded in r implies to be bounded both in the "volume" and shear degrees of freedom. Due to the separability, the motion in $(\varphi, \vartheta, \psi)$-variables is geodetic in the sense of invariant metric tensors on $\mathrm{SO}(3, \mathbb{R})$ or $\mathrm{SU}(2)$. And this problem is mathematically isomorphic with the motion of the free spherically-symmetric rigid body in the three-dimensional space (purely rotational one, without translations in $\mathbb{R}^3$).

Another helpful model would be one with $V_\vartheta(\vartheta) = A \cos \vartheta$, where A denotes some constant. The resulting problem is isomorphic with that of the three-dimensional heavy top.

It is not excluded that some more general problems from the realm of three-dimensional gyroscopic dynamics, e.g., symmetric top, might be also of some mathematical usefulness when studying the two-dimensional affine motion.

Quantized problems

Classical dynamical models described above may be easily quantized in the sense of Schrödinger wave mechanics on manifolds. And those rigorously solvable on the classical level are so as well on the quantum level.

Let us fix some notation. Let Q be a differential manifold of dimension n with the metric tensor G. The components of G with respect to some local coordinates $q^1, \ldots, q^n$ will be denoted by G_{ij} and the components of the contravariant inverse of G will be denoted by G^{ij}; obviously, $G_{ik}G^{kj} = \delta_i{}^j$. The determinant of the matrix $[G_{ij}]$ will be briefly denoted by the symbol $|G|$ (no confusion between two its meanings); obviously, this determinant is an analytic representation of some scalar density of weight two; the square root $\sqrt{|G|}$ is a scalar density of weight one. The invariant measure induced by G will be denoted by $\widetilde{\mu}$; analytically its element is given by

$$d\widetilde{\mu}(q) = \sqrt{|G(q)|}dq^1 \cdots dq^n.$$

Operators of the covariant differentation induced in the Levi-Civita sense by G will be denoted by ∇_i. The corresponding Laplace-Beltrami operator Δ is analytically given by

$$\Delta = G^{ij}\nabla_i\nabla_j$$

or explicitly, when acting on scalar fields,

$$\Delta\mathbf{\Psi} = \frac{1}{\sqrt{|G|}}\sum_{i,j}\frac{\partial}{\partial q^i}\left(\sqrt{|G|}G^{ij}\frac{\partial\mathbf{\Psi}}{\partial q^j}\right),$$

$\mathbf{\Psi}$ denoting a twice differentiable complex function on Q.

Wave mechanics is formulated in $L^2(Q, \widetilde{\mu})$, the space of square-integrable functions on Q with the scalar product meant as follows:

$$\langle\mathbf{\Psi}|\mathbf{\Phi}\rangle := \int \mathbf{\Psi}(q)\mathbf{\Phi}(q)d\widetilde{\mu}(q).$$

The operator Δ is symmetric with respect to this product, and ∇_i are skew-symmetric. The metric G underlies the classical kinetic energy, therefore, the classical energy/Hamiltonian function

$$H = \frac{\mu}{2} G_{ij}(q) \frac{dq^i}{dt} \frac{dq^j}{dt} + V(q) = \frac{1}{2\mu} G^{ij}(q) p_i p_j + V(q)$$

becomes the operator

$$\widehat{H} = -\frac{\hbar}{2\mu} \Delta + V.$$

Then, denoting and ordering our coordinates q^i as $(\varphi, \psi, \alpha, \beta)$ in the Cartesian case we have for explicitly separable isotropic potentials:

$$[G_{ij}] = \begin{bmatrix} \alpha^2 + \beta^2 & \beta^2 - \alpha^2 & 0 & 0 \\ \beta^2 - \alpha^2 & \alpha^2 + \beta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{36}$$

$$\widehat{H} = \widehat{H}_\alpha + \widehat{H}_\beta = -\frac{\hbar^2}{2\mu} \Delta + V(\alpha, \beta), \tag{37}$$

where

$$\widehat{H}_\alpha = \frac{1}{2\mu} \left(\frac{1}{\alpha^2} \left(\widehat{S} - \widehat{\Sigma} \right)^2 - \hbar^2 \left(\frac{\partial^2}{\partial \alpha^2} + \frac{1}{\alpha} \frac{\partial}{\partial \alpha} \right) \right) + V_\alpha(\alpha), \tag{38}$$

$$\widehat{H}_\beta = \frac{1}{2\mu} \left(\frac{1}{\beta^2} \left(\widehat{S} + \widehat{\Sigma} \right)^2 - \hbar^2 \left(\frac{\partial^2}{\partial \beta^2} + \frac{1}{\beta} \frac{\partial}{\partial \beta} \right) \right) + V_\beta(\beta), \tag{39}$$

and $\widehat{S} = (\hbar/i)\partial/\partial\varphi$ is the spin operator, the generator of spatial rotations about the current spatial position of the center of mass, whereas $\widehat{\Sigma} = (\hbar/i)\partial/\partial\psi$ is the "vorticity" operator, the generator of material rotations. Operators $\widehat{H}_\alpha$, $\widehat{H}_\beta$, $\widehat{S}$, $\widehat{\Sigma}$ are the quantum constants of motion. They also commute with each other (they represent co-measurable physical quantities).

Those formulas follow from the expression of Δ in coordinates $(\varphi, \psi, \alpha, \beta)$

$$\begin{aligned}\Delta \mathbf{\Psi} &= \frac{\partial^2 \mathbf{\Psi}}{\partial \alpha^2} + \frac{\partial^2 \mathbf{\Psi}}{\partial \beta^2} + \frac{1}{\alpha}\frac{\partial \mathbf{\Psi}}{\partial \alpha} + \frac{1}{\beta}\frac{\partial \mathbf{\Psi}}{\partial \beta} + \left(\frac{1}{4\alpha^2} + \frac{1}{4\beta^2}\right)\left(\frac{\partial^2 \mathbf{\Psi}}{\partial \varphi^2} + \frac{\partial^2 \mathbf{\Psi}}{\partial \psi^2}\right) \\ &+ \left(\frac{1}{2\beta^2} - \frac{1}{2\alpha^2}\right)\frac{\partial^2 \mathbf{\Psi}}{\partial \varphi \partial \psi}. \end{aligned} \tag{40}$$

Separable solutions of the stationary Schrödinger equation $\hat{H}\mathbf{\Psi} = E\Psi$ have the form:

$$\mathbf{\Psi}(\varphi, \psi, \alpha, \beta) = f_\varphi(\varphi) f_\psi(\psi) f_\alpha(\alpha) f_\beta(\beta), \tag{41}$$

where $f_\varphi(\varphi) = e^{im\varphi}$, $f_\psi(\psi) = e^{il\psi}$ (m, l are integers) and $f_\alpha(\alpha)$, $f_\beta(\beta)$ are the deformative wave functions.

Hence, the stationary Schrödinger equation with an arbitrary potential $V(\alpha, \beta) = V_\alpha(\alpha) + V_\beta(\beta)$ leads after the standard separation procedure to the following system of one-dimensional eigenequations:

$$\frac{d^2 f_\alpha(\alpha)}{d\alpha^2} + \frac{1}{\alpha}\frac{df_\alpha(\alpha)}{d\alpha} - \frac{(m-l)^2}{4\alpha^2} f_\alpha(\alpha) + \frac{2\mu}{\hbar^2}\left(E_\alpha - V_\alpha(\alpha)\right) f_\alpha(\alpha) = 0, \tag{42}$$

$$\frac{d^2 f_\beta(\beta)}{d\beta^2} + \frac{1}{\beta}\frac{df_\beta(\beta)}{d\beta} - \frac{(m+l)^2}{4\beta^2} f_\beta(\beta) + \frac{2\mu}{\hbar^2}\left(E_\beta - V_\beta(\beta)\right) f_\beta(\beta) = 0. \tag{43}$$

It is natural to expect that for potentials (5) the resulting Schrödinger equations should be rigorously solvable in terms of some standard special functions. The most convenient way of solving them is to use the Sommerfeld polynomial method. In this method the solutions are expressed by the usual or confluent Riemann P-functions. They are deeply related to the hypergeometric functions (respectively usual F_1 or confluent F_2). If the usual convergence demands are imposed, then the hypergeometric functions become polynomials and our solutions are expressed by elementary functions. At the same time the energy levels are expressed by the eigenvalues of the corresponding operators. There exists some special class of potentials to which the Sommerfeld polynomial method is applicable. The restriction to solutions expressible in terms of Riemann P-functions is reasonable, because this class of functions is well investigated and many special functions used in physics may be expressed by them. There is also an intimate relationship between these functions and representations of Lie groups.

Let us now quote some formulas for quantized problems separable in coordinates (r, Φ, Θ, Ψ) (equivalently $(\rho, \Phi, \Theta, \Psi)$), namely, the quantum counterparts of classical models (18). One can easily show that the Laplace operators take on the form:

$$\begin{aligned}\Delta\boldsymbol{\Psi} &= 4r\frac{\partial^2\boldsymbol{\Psi}}{\partial r^2} + 8\frac{\partial\boldsymbol{\Psi}}{\partial r} + \frac{1}{r\sin^2\vartheta}\left(\frac{\partial^2\boldsymbol{\Psi}}{\partial\varphi^2} + 2\cos\vartheta\frac{\partial^2\boldsymbol{\Psi}}{\partial\varphi\partial\psi} + \frac{\partial^2\boldsymbol{\Psi}}{\partial\psi^2}\right)\\ &+ \frac{4}{r}\left(\frac{\partial^2\boldsymbol{\Psi}}{\partial\vartheta^2} + \cot\vartheta\frac{\partial\boldsymbol{\Psi}}{\partial\vartheta}\right),\end{aligned}$$

i.e.,

$$\begin{aligned}\Delta\boldsymbol{\Psi} &= \frac{\partial^2\boldsymbol{\Psi}}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial\boldsymbol{\Psi}}{\partial\rho} + \frac{4}{\rho^2\sin^2\Theta}\left(\frac{\partial^2\boldsymbol{\Psi}}{\partial\Phi^2} + 2\cot\Theta\frac{\partial^2\boldsymbol{\Psi}}{\partial\Phi\partial\Psi} + \frac{\partial^2\boldsymbol{\Psi}}{\partial\Psi^2}\right)\\ &+ \frac{4}{\rho^2}\left(\frac{\partial^2\boldsymbol{\Psi}}{\partial\Theta^2} + \cot\Theta\frac{\partial\boldsymbol{\Psi}}{\partial\Theta}\right).\end{aligned}$$

We assume the doubly isotropic separable potential energy (21), i.e.,

$$V = V_r(r) + \frac{V_\vartheta(\vartheta)}{r} = V_\rho(\rho) + \frac{V_\vartheta(\vartheta)}{\rho^2}.$$

The corresponding Schrödinger equation separates and, taking into account the cyclic character of angular variables φ, ψ, we put

$$\mathbf{\Psi}(\varphi, \psi, r, \vartheta) = e^{im\varphi} e^{il\psi} f_r(r) f_\vartheta(\vartheta) = e^{im\varphi} e^{il\psi} f_\rho(\rho) f_\vartheta(\vartheta), \tag{44}$$

where m, l are integers.

Quantum integration constants responsible for this separability are given by operators:

- $\widehat{p}_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial\varphi} = \widehat{S}-$ spin,
- $\widehat{p}_\psi = \frac{\hbar}{i}\frac{\partial}{\partial\psi} = \widehat{V}-$ vorticity,
- $\widehat{h}_\vartheta = \frac{1}{2\mu\sin^2\vartheta}\left(\widehat{p}_\varphi{}^2 + 2\cos\vartheta\widehat{p}_\varphi\widehat{p}_\psi + \widehat{p}_\psi{}^2\right) - \frac{4\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\vartheta^2} + \cot\vartheta\frac{\partial}{\partial\vartheta}\right) + V_\vartheta,$
- $\widehat{H} = \widehat{H}_r + \widehat{H}_\vartheta = \widehat{H}_r + \frac{1}{r}\widehat{h}_\vartheta = \widehat{H}_\rho + \frac{1}{\rho^2}\widehat{h}_\vartheta-$ energy,

where the "radial energy" is given by

$$\widehat{H}_r = \widehat{H}_\rho = -\frac{\hbar^2}{2\mu}\left(4r\frac{\partial^2}{\partial r^2} + 8\frac{\partial}{\partial r}\right) + V_r(r) = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho}\right) + V_\rho(\rho).$$

The four mentioned constants of motion $\widehat{p}_\varphi$, $\widehat{p}_\psi$, $\widehat{h}_\vartheta$, $\widehat{H}$ are pairwise commuting and therefore they represent co-measurable physical quantities.

Warning: the two indicated contributions to $\widehat{H}$, i.e., $\widehat{H}_r$ and $\widehat{H}_\vartheta = \widehat{h}_\vartheta/r$ are not constants of motion.

The stationary Schrödinger equation for the factorized wave function (44) reduces to the following pair of ordinary Schrödinger equations (Sturm-Lioville equations) for the factors depending only on one variable, respectively ϑ and r (or ρ):

$$\hat{h}_\vartheta f_\vartheta = e_\vartheta f_\vartheta, \tag{45}$$

$$\hat{H}_r f_r + \frac{e_\vartheta}{r} f_r = E f_r, \quad \text{i.e.,} \quad \hat{H}_\rho f_\rho + \frac{e_\vartheta}{\rho^2} f_\rho = E f_\rho. \tag{46}$$

The procedure is first to solve the ϑ-equation and then to substitute the resulting eigenvalues e_ϑ to the r/ρ-equation. Then one obtains (at least in principle) the energy levels E.

It was mentioned that there exists some strange relationship between the two-polar parametrization of $\mathrm{GL}(2,\mathbb{R})$ and the Euler angles and scale parameters of rigid body with dilatations. There is some very interesting aspect of this link, which we noticed first quite accidentally, on the purely analytical level, before the trivial geometric meaning of this surprise became evident to us. This artificial detour (wandering about) was due to the fact that by chance we invented our separating coordinates (r,ϑ) better (ρ,ϑ) just where they are rather obscurely hidden, namely as polar parametrization of the pair of quantities $(2D_1D_2, {D_1}^2 - {D_2}^2)$ (14)–(16).

Namely, differential eigenequations (45), (46) may be explicitly written down as follows:

$$\frac{d^2 f_\vartheta}{d\vartheta^2} + \cot\vartheta \frac{df_\vartheta}{d\vartheta} - \left(\frac{m^2 + 2ml\cos\vartheta + l^2}{4\sin^2\vartheta} + \frac{\mu}{2\hbar^2}(V_\vartheta - e_\vartheta)\right) f_\vartheta = 0, \tag{47}$$

$$4r\frac{d^2 f_r}{dr^2} + 8\frac{df_r}{dr} + \frac{2\mu}{\hbar^2}\left(E - \left(V_r + \frac{e_\vartheta}{r}\right)\right) f_r = 0, \tag{48}$$

where m, l are integers in $\mathbf{\Psi}$ as coefficients at the angles φ, ψ in complex exponential functions (eigenfunctions of $\widehat{p}_\varphi$, $\widehat{p}_\psi$). Let us now divide by 4 the nominator and denominator in the bracket expression (47) and formally admit half-integer coefficients.

We can rewrite our equations as follows:

$$\frac{d^2 f_\vartheta}{d\vartheta^2} + \cot\vartheta \frac{df_\vartheta}{d\vartheta} - \left(\frac{m^2 + 2ml\cos\vartheta + l^2}{\sin^2\vartheta} + \frac{\mu}{2\hbar^2}(V_\vartheta - e_\vartheta) \right) f_\vartheta = 0, \tag{49}$$

$$\frac{d^2 f_\rho}{d\rho^2} + \frac{3}{\rho}\frac{df_\rho}{d\rho} + \frac{2\mu}{\hbar^2}\left(E - \left(V_\rho + \frac{e_\vartheta}{\rho^2} \right) \right) f_\rho = 0, \tag{50}$$

where now the numbers m, l are assumed to run over the set of non-negative integers and half-integers, i.e., $m, l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$.

Let us notice that when there is no purely shear-like potential, i.e., $V_\vartheta = 0$, then the ϑ-equation is just nothing else but the eigenequation for the nutation ϑ-factor of the stationary states of the spherical top:

$$\frac{d^2 f_\vartheta}{d\vartheta^2} + \cot\vartheta \frac{df_\vartheta}{d\vartheta} - \left(\frac{m^2 + 2ml\cos\vartheta + l^2}{\sin^2\vartheta} - \frac{\mu}{2\hbar^2} e_\vartheta \right) f_\vartheta = 0. \tag{51}$$

The history of this equation traces back to the Reiche-Rademacher theory of quantum top and to the Wigner theory of irreducible unitary representations of the group SU(2), i.e., roughly speaking, to the one-valued and two-valued irreducible unitary representations of the rotation group $SO(3, \mathbb{R})$. Then the quantized eigenvalues e_Θ are given by the expression

$$e_{\Theta j} = \frac{2\hbar^2}{\mu} j(j+1)$$

labelled by non-negative half-integer and integer numbers, $j = 0, 1/2, 1, 3/2,$ $\ldots$, i.e., $j \in \{0\} \cup (\mathbb{N}/2)$, $\mathbb{N}$ denoting the set of naturals.

The corresponding eigenfunctions $d^j{}_{ml}(\Theta)$ were found by Wigner as factors in expressions for the matrix elements of unitary irreducible representations of SU(2),

$$D^j{}_{ml}(\Phi, \Theta, \Psi) = e^{im\Phi} d^j{}_{ml}(\Theta) e^{il\Psi}.$$

Here, as mentioned, Φ, Θ, Ψ denote the Euler angles parametrization of SU(2). Their range is twice larger than the range of Euler angles on the quotient group $SO(3, \mathbb{R})$; this is the reason why the half-integer quantum numbers do appear.

The celebrated functions $D^j{}_{ml}$ appear also as stationary states of the quantized spherical free top. Energy levels are then given by

$$E_j = \frac{\hbar^2}{2I} j(j+1), \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots,$$

I denoting the main moment of inertia, and of course they are $(2j+1)^2$-fold degenerate. The labels of basic j-states, m, l, are quantum numbers of projections of the angular momentum respectively on the space-fixed and body-fixed z-axes:

$$\frac{\hbar}{i}\frac{\partial}{\partial \Phi} D^j{}_{ml} = \hbar m D^j{}_{ml}, \quad \frac{\hbar}{i}\frac{\partial}{\partial \Psi} D^j{}_{ml} = \hbar l D^j{}_{ml}.$$

Obviously, m, l run over the range $-j, -j+1, \ldots, j-1, j$, jumping by one. Strictly speaking, in applications concerning the rotational spectra of mole-cules, one has to restrict ourselves to integer values of j, m and l. There are however some arguments that perhaps the half integer values might be also acceptable.

Let us also mention that m, l are good quantum numbers also for a more general free symmetric top, not necessarily the spherical one. If I, K are two main moments of inertia, I doubly degenerate one, then $D^j{}_{ml}$ are still basic eigenfunctions corresponding to the energy levels

$$E_{j,l} = \frac{\hbar^2}{2I} j(j+1) + \hbar^2 \left(\frac{1}{2I} - \frac{1}{2K} \right) l^2.$$

They are $2(2j+1)$-fold degenerate, namely with respect to the quantum number m and to the sign of l.

One can wonder whether such a symmetric free top in three dimensions, or more general three-dimensional top with some external potential, first of all one of the shape $U(\Theta)$ (e.g., heavy top), might be useful as a tool for analyzing the two-dimensional affinely-rigid body. This is just a question worth to be analyzed.

Quantized harmonic and anharmonic vibrations

First let us consider the model of the harmonic oscillator potential (22). Applying the Sommerfeld polynomial method we obtain the energy levels $E = E_\alpha + E_\beta$ as follows:

$$E = \frac{1}{2}\hbar\omega\left(4n + 4 + |m-l| + |m+l|\right), \tag{52}$$

where

$$E_\alpha = \frac{\hbar\omega}{2}\left(4n_\alpha + 2 + |m-l|\right), \quad E_\beta = \frac{\hbar\omega}{2}\left(4n_\beta + 2 + |m+l|\right), \tag{53}$$

and $\omega = \sqrt{C/\mu}$, $n = n_\alpha + n_\beta$, $n = 0, 1, \ldots$, $m, l = 0, \pm 1, \ldots$. We may write:

(i) if $|m| > |l|$, then $m^2 > l^2$ and

$$E = \hbar\omega\left(2n + 2 \pm m\right),$$

(ii) if $|m| < |l|$, then $m^2 < l^2$ and

$$E = \hbar\omega\left(2n + 2 \pm l\right),$$

(iii) if $|m| = |l|$, then $m^2 = l^2$ and

$$E = \hbar\omega\left(2n + 2 \pm m\right) = \hbar\omega\left(2n + 2 \pm l\right).$$

After some calculations we obtain the deformative wave functions $f_\alpha(\alpha)$ and $f_\beta(\beta)$ in the form:

$$f_\alpha(\alpha) = \alpha^\sigma \kappa^{\frac{1}{4}+\frac{\sigma}{2}} e^{-\frac{\kappa}{2}\alpha^2} F_2\left(-n_\alpha; 1+\sigma; \kappa\alpha^2\right), \tag{54}$$

$$f_\beta(\beta) = \beta^\gamma \kappa^{\frac{1}{4}+\frac{\gamma}{2}} e^{-\frac{\kappa}{2}\beta^2} F_2\left(-n_\beta; 1+\gamma; \kappa\beta^2\right), \tag{55}$$

where $\sigma = \frac{1}{2}|m-l|$, $\kappa = \sqrt{C\mu/\hbar^2}$, $\gamma = \frac{1}{2}|m+l|$.

The constant term 4 occuring in the rigorous quantum formula (52) and absent in the quasiclassical one (24) was in principle expected. This resembles the difference between Schrödinger and Bohr-Sommerfeld-quantized harmonic oscillators. This is an essentially quantum effect.

In the classical part we mentioned that the harmonic oscillator model, in spite of its academic character, may have some practical utility, and besides, it suggests some reasonable anharmonic corrections well suited to certain of its degeneracy properties. The mentioned corrections reduce degeneracy in some characteristic way and at the same time the model becomes more realistic. On the classical and quasiclassical level we discussed the potential (97), i.e.,

$$V(\alpha,\beta) = \frac{C}{2}\left(\alpha^2 + \frac{4}{\alpha^2}\right) + \frac{C}{2}\beta^2.$$

The model may be rigorously solved on the quantum level and one obtains the following formula for the energy levels:

$$E = \frac{1}{2}\hbar\omega\left(4n + 4 + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right). \tag{56}$$

The energy in (56) depends on an integer combination of the quantum numbers, i.e., $n = n_\alpha + n_\beta$. The wave functions are as follows:

$$f_\alpha(\alpha) = \alpha^\chi \kappa^{\frac{1}{4}+\frac{\chi}{2}} e^{-\frac{\kappa}{2}\alpha^2} F_2\left(-n_\alpha; 1+\chi; \kappa\alpha^2\right), \tag{57}$$

$$f_\beta(\beta) = \beta^\gamma \kappa^{\frac{1}{4}+\frac{\gamma}{2}} e^{-\frac{\kappa}{2}\beta^2} F_2\left(-n_\beta; 1+\gamma; \kappa\beta^2\right), \tag{58}$$

where

$$\chi = \frac{1}{2}\sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}.$$

It is seen that the formula for the energy levels is structurally "almost" identical with the quasiclassical one (35), i.e.,

$$E = \frac{1}{2}\hbar\omega\left(4n + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right).$$

This is rather typical for systems invariant under "large" symmetry groups and based on interesting geometric structures. There is a characteristic shift of energy levels, corresponding to the "null vibrations" of the harmonic part of the system. Just like on the classical and quasiclassical levels, the system is twice degenerate and its energy levels are essentially controlled by two effective quantum numbers: $n_\alpha + n_\beta + |m+l|$ and $|m-l|$.

Using the formulas (47), (48), i.e., (49), (50), we can also quantize the model (34), i.e.,

$$V(r,\vartheta) = \frac{C}{2}\left(r + \frac{4}{r}\right) + \frac{2C}{r} tg^2 \frac{\vartheta}{2}.$$

The expression for the energy levels E is as follows:

$$E = \frac{1}{2}\hbar\omega\left(4n + 4 + |m + l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right), \tag{59}$$

where $n = n_r + n_\vartheta$. The functions $f_r(r)$, $f_\vartheta(\vartheta)$ have the form:

$$f_r(r) = r^{-\frac{1}{2}+\varepsilon}\kappa^{\frac{1}{2}+\varepsilon}e^{-\frac{\kappa}{2}r}F_2\left(-n_r; 1 + 2\varepsilon; \kappa r\right), \tag{60}$$

$$f_\vartheta(\vartheta) = \left(\cos\frac{\vartheta}{2}\right)^\chi \left(\sin\frac{\vartheta}{2}\right)^\gamma F_1\left(-n_\vartheta, 1 + n_\vartheta + \gamma + \chi; 1 + \chi; \cos^2\frac{\vartheta}{2}\right), \tag{61}$$

where

$$\varepsilon = \frac{1}{2}\sqrt{1 + \frac{2\mu}{\hbar^2}e_\vartheta + \frac{2C\mu}{\hbar^2}},$$

$$e_\vartheta = \frac{\hbar^2}{8\mu}\left(\left(4n_\vartheta + 2 + |m + l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right)^2 - 4 - \frac{16C\mu}{\hbar^2}\right).$$

For many physical reasons it would be interesting to discuss the model (21), however, we were not yet successful in solving explicitly the corresponding Schrödinger equation.

Rigorous solutions for two-dimensional problems may be useful in microscopic physical problems (vibrations of planar molecules such as S_8, C_6H_6) and in macroscopic elasticity (cylinders with homogeneously-deformable cross-sections). Applications in dynamics of nanotubes seem to be possible.

The next important thing to be done is a more comprehensive analysis of the status of analogy with Euler angles and the related complexification problems. This will be done in a subsequent paper. Some introductory analysis is outlined below.

Planar affine body versus spatial rigid body

It was mentioned above about certain interesting links between mechanics of isotropic affine body in two dimension and the dynamics of three-dimensional rigid body, more precisely, rigid body with imposed dilatations. Only certain analytical aspects, useful in calculations, were stressed there. However, the problem is geometrically interesting in itself and has to do with certain complexification procedures on Lie groups used as configuration spaces. We shall analyze this problem in more detail in a forthcoming paper; here we mention only a few simple analytical relationships.

Let us remind that the metric tensor underlying kinetic energy of the planar isotropic affine body was given by

$$ds^2 = \operatorname{Tr}\left(d\phi^T d\phi\right) = dx^2 + dy^2 + dz^2 + du^2; \tag{62}$$

the corresponding kinetic energy form reads

$$T = \frac{\mu}{2}\operatorname{Tr}\left(\frac{d\phi^T}{dt}\frac{d\phi}{dt}\right) = \frac{\mu}{2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + \left(\frac{du}{dt}\right)^2\right), \tag{63}$$

where μ denotes the scalar inertial moment.

For certain reasons it is convenient to use some modified parametrization of the two-polar decomposition

$$\phi = ODR^{-1}, \tag{64}$$

where O, R are proper orthogonal and D is diagonal, namely,

$$O = \begin{bmatrix} \cos\frac{\Phi}{2} & -\sin\frac{\Phi}{2} \\ \sin\frac{\Phi}{2} & \cos\frac{\Phi}{2} \end{bmatrix}, \ D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \ R = \begin{bmatrix} \cos\frac{\Psi}{2} & \sin\frac{\Psi}{2} \\ -\sin\frac{\Psi}{2} & \cos\frac{\Psi}{2} \end{bmatrix}$$

and $D_1 = \exp(a + b/2)$, $D_2 = \exp(a - b/2)$.

It is convenient and instructive from the point of view of our analogies to write these matrices as:

$$O = \exp\left(\Phi\frac{1}{2i}\sigma_2\right), \; R^{-1} = \exp\left(\Psi\frac{1}{2i}\sigma_2\right), \; D = \exp\left(a\frac{1}{2}\sigma_0\right)\exp\left(b\frac{1}{2}\sigma_3\right),$$

where σ_ν $(\nu = 0, 1, 2, 3)$ are Pauli matrices; more precisely, σ_{a} $(\mathrm{a} = 1, 2, 3)$ are "true" Pauli matrices, so

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{65}$$

The crucial point for our analogies and links is that the matrices

$$\tau_{\mathrm{a}} = \frac{1}{2i}\sigma_{\mathrm{a}}, \quad \mathrm{a} = 1, 2, 3, \tag{66}$$

are generators of the group $\mathrm{SU}(2)$, the universal covering of $\mathrm{SO}(3, \mathbb{R})$, with standard commutation rules

$$[\tau_1, \tau_2] = \tau_3, \quad [\tau_2, \tau_3] = \tau_1, \quad [\tau_3, \tau_1] = \tau_2. \tag{67}$$

And similarly, the matrices

$$\widetilde{\tau}_1 = i\tau_1, \quad \widetilde{\tau}_2 = i\tau_2, \quad \widetilde{\tau}_3 = i\tau_3 \tag{68}$$

are generators of $\mathrm{SL}(2, \mathbb{R})$ with the standard structure constants,

$$[\widetilde{\tau}_1, \widetilde{\tau}_2] = \widetilde{\tau}_3, \quad [\widetilde{\tau}_2, \widetilde{\tau}_3] = \widetilde{\tau}_1, \quad [\widetilde{\tau}_3, \widetilde{\tau}_1] = -\widetilde{\tau}_2. \tag{69}$$

Obviously, the matrix

$$\tau_0 = \widetilde{\tau}_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{70}$$

generates real dilatations. So, the matrices $\widetilde{\tau}_\nu$ generate the group $\mathrm{GL}(2, \mathbb{R})$, the configuration space of the planar affine body, and τ_ν generate $\mathbb{R}^+\mathrm{SU}(2)$, the $2:1$ covering of the configuration space of rigid body with admitted dilatations ("breathing top").

In our models of the doubly isotropic planar affine body, with the metric element (62) we were used rather to parametrize the plane of deformation invariants (D_1, D_2) by $r = \rho^2 = (D_1)^2 + (D_2)^2$ and the angle ϑ such that $\sin\vartheta = {D_1}^2 - {D_2}^2/{D_1}^2 + {D_2}^2$ so that the relationships (14)–(16) and those following them are satisfied. However, in models with affinely-invariant kinetic energies the variables a, b as deformation invariants are more convenient. As mentioned, one can show that

$$\begin{aligned} ds^2 &= d\rho^2 + \frac{1}{4}\rho^2\left(d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2\right) \\ &= \frac{1}{4r}\left(dr^2 + r^2\left(d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2\right)\right). \end{aligned} \tag{71}$$

We easily recognize the term characteristic for the spherical top described in terms of the "Euler angles" (Φ, Θ, Ψ) and the term corresponding to the evolution of the invariant r, a kind of "dilatation" (not in a rigorous sense). Using the more geometric variables a, b and the auxiliary, literally dilatational variable

$$\delta = \sqrt{D_1 D_2} = \exp\left(a/2\right), \tag{72}$$

we express (71) as follows:

$$\begin{aligned} ds^2 &= \cosh b\ d\delta^2 + \delta\sinh b\ d\delta db + \frac{1}{4}\delta^2\cosh b\ db^2 \\ &+ \frac{1}{4}\delta^2\cosh b\left(d\Phi^2 + \frac{2}{\cosh b}d\Phi d\Psi + d\Psi^2\right). \end{aligned} \tag{73}$$

This is an ugly non-diagonal form; the reason is that ds^2 is not affinely-invariant, but only isotropic. The "Euler angles" term is readable, because, as we saw, $(\cosh b)^{-1} = \cos\Theta$. There are no essential geometric arguments against modifying (71) by some extra term proportional to $d\rho^2$.

Let us compare these formulas with those for the spherical three-dimen-sional rigid body with dilatations. More precisely, we write down the formulas on the group $\mathbb{R}^{+}\mathrm{SU}(2)$, the universal $(2:1)$ covering group of $\mathrm{SO}(3,\mathbb{R})$ (roughly speaking, the spinorial breathing-rigid-body). Then $\phi \in \mathbb{R}^{+}\mathrm{SU}(2)$ is "Euler-parametrized" as follows:

$$\phi = \exp(a\tau_0)\exp(\Phi\tau_2)\exp(\Theta\tau_3)\exp(\Psi\tau_2). \tag{74}$$

More precisely, historical term "Euler angles" is used when the following convention is used:

$$\widetilde{\phi}' = \exp(a\tau_0)\exp(\Phi\tau_3)\exp(\Theta\tau_1)\exp(\Psi\tau_3), \tag{75}$$

or similarly, (more popular in textbooks),

$$\widetilde{\phi}'' = \exp(a\tau_0)\exp(\Phi\tau_3)\exp(\Theta\tau_2)\exp(\Psi\tau_3). \tag{76}$$

If (74)–(76) are identified, then, obviously, (Φ, Θ, Ψ) in those formulas denote numerically different functions on $\mathrm{SU}(2)$. Nevertheless, there is no essential difference between them. What matters is that the $SU(2)$-matrices are factorized into products of three elements taken from two orthogonal one-parameter subgroups. This is only the question how those three one-parameter subgroups are called (ordered). The non-historical, apparently exotic convention (74) is optimally adapted to our programme of exhibiting some links between planar affine body and spatial rigid body.

Namely, let us take the following metric on $\mathbb{R}^{+}\mathrm{SU}(2)$, underlying the kinetic energy of the spherical breathing top:

$$ds^2 = \mathrm{Tr}\left(d\phi^{\dagger} d\phi\right), \tag{77}$$

where, obviously, the "$\dagger$ - symbol" denotes Hermitian conjugation of matrices.

Denoting again:

$$\delta = \exp\left(a/2\right), \quad \lambda = \delta^2 = \exp(a), \tag{78}$$

we obtain:

$$ds^2 = d\delta^2 + \frac{1}{4}\delta^2\left(d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2\right), \tag{79}$$

i.e., equivalently,

$$ds^2 = \frac{1}{4\lambda}\left(d\lambda^2 + \lambda^2\left(d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2\right)\right), \tag{80}$$

or,

$$ds^2 = \frac{1}{4}e^a\left(da^2 + d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2\right). \tag{81}$$

Obviously, the $\mathbb{R}^+$-factor in $\mathbb{R}^+\mathrm{SU}(2)$ is a normal divisor and from the purely geometrical point of view of two-side invariant metrics on $\mathbb{R}^+\mathrm{SU}(2)$, there are no obstacles against modifying ds^2 by adding an arbitrary correction term ${ds^2}_{corr} = c\, d\delta^2$, c being a constant. This means that (79)–(81) may be replaced by

$$ds^2 = (1+c)d\delta^2 + \frac{1}{4}\delta^2\left(d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2\right), \tag{82}$$

$$ds^2 = \frac{1}{4\lambda}\left((1+c)d\lambda^2 + \lambda^2\left(d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2\right)\right), \tag{83}$$

$$ds^2 = \frac{1}{4}e^a\left((1+c)da^2 + d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2\right). \tag{84}$$

Concerning the extra dilatational term in dynamics of the breathing top. Replacing the real parameter a in (77), (78) by imaginary one ia, one obtains instead (84) the following arc element for the two-side invariant Riemannian metric on the unitary group $\mathrm{U}(2)$:

$$ds^2 = \frac{1}{4}\left((1+c)da^2 + d\Theta^2 + d\Phi^2 + 2\cos\Theta d\Phi d\Psi + d\Psi^2\right). \tag{85}$$

For some application or just comparison purposes one can admit in (77), (78) the general complex parameter a. This results in the doubly-invariant Riemannian metric on $(\mathbb{C} \,/\, \{0\})\mathrm{SU}(2) = \mathbb{R}^+\mathrm{U}(2)$.

This was, so-to-speak, "one side" of injecting geometry and dynamics of the "breathing top" into those of planar affine body (or conversely). There is also another aspect, namely one based on affinely-invariant metric tensors on $\mathrm{GL}^+(2,\mathbb{R})$. Such metric tensors, of non-definite signature ($\mathrm{SL}(2,\mathbb{R}), \mathrm{GL}^+(2,\mathbb{R})$ are non-compact, $\mathrm{SL}(2,\mathbb{R})$ is semisimple, and $\mathrm{GL}^+(2,\mathbb{R})$ is the direct product of $\mathbb{R}^+\mathrm{SL}(2,\mathbb{R})$) are linear combinations of those given by the arc element

$$ds^2 = \mathrm{Tr}\left(\Omega^2\right) = \mathrm{Tr}\left(\widehat{\Omega}^2\right) \tag{86}$$

and the purely dilatational correction term

$$ds^2{}_{corr} = \mathrm{Tr}\left(\Omega\right)^2 = \mathrm{Tr}\left(\widehat{\Omega}\right)^2, \tag{87}$$

where the Lie-algebraic Cartan one-forms Ω, $\widehat{\Omega}$ on $\mathrm{GL}(2,\mathbb{R})$ are given by the usual formulas:

$$\Omega = (d\phi)\phi^{-1}, \quad \widehat{\Omega} = \phi^{-1}d\phi = \phi^{-1}\Omega\phi. \tag{88}$$

Obviously, (86) is the main, non-degenerate term of signature $(+ + + -)$. Killing tensor on $\mathrm{GL}(2,\mathbb{R})$ is degenerate; the singular direction is that of the one-dimensional center $\mathbb{R}^+\mathrm{Id}_2$. This Killing case corresponds to the ratio $4 : (-2)$ of coefficients at (86), (87).

For calculations we need the following parametrization of $\phi \in \mathrm{GL}^{+}(2,\mathbb{R})$, analogous to (74)

$$\begin{aligned} \phi &= \exp(a\widetilde{\tau}_0)\exp(\Phi\widetilde{\tau}_2)\exp(b\widetilde{\tau}_3)\exp(\Psi\widetilde{\tau}_2) \\ &= \delta\exp(\Phi\widetilde{\tau}_2)\exp(b\widetilde{\tau}_3)\exp(\Psi\widetilde{\tau}_2), \end{aligned} \tag{89}$$

where, obviously,

$$\delta = \exp\left(a/2\right) = \sqrt{\lambda}. \tag{90}$$

Combining (86), (87) with appropriate coefficients (that at the main term (86) must be non-vanishing), we finally obtain:

$$\begin{aligned} ds^2 &= (1+c)d\delta^2 + \frac{1}{4}\delta^2\left(db^2 - d\Phi^2 - 2\cosh b\; d\Phi d\Psi - d\Psi^2\right) \\ &= \frac{1}{4\lambda}\left((1+c)d\lambda^2 + \lambda^2\left(db^2 - d\Phi^2 - 2\cosh b\; d\Phi d\Psi - d\Psi^2\right)\right) \\ &= \frac{1}{4}e^a\left((1+c)da^2 + db^2 - d\Phi^2 - 2\cosh b\; d\Phi d\Psi - d\Psi^2\right). \end{aligned} \tag{91}$$

The relationship between these formulas (as matter of fact, one formula written in three alternative forms) and (71), (73), (79)–(81) is obvious. Namely, the last four terms in any form of (91) become the "minus" terms of the spherical top, when some complexification procedure is performed, i.e., when we put $b = i\Theta$, Θ being real. Then, obviously, the last four terms become the spherical top expression with reversed sign,

$$-\,d\Theta^2 - d\Phi^2 - 2\cos\Theta d\Phi d\Psi - d\Psi^2, \tag{92}$$

and no wonder, because $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ are two different (and is a sense, having opposite properties) real forms of the same complex Lie group $\mathrm{SL}(2,\mathbb{C})$. The over-all minus term of the Killing metric on $\mathrm{SU}(2)$ is due to its compactness. Performing a similar "imaginarization" of a, we obtain just the "minus" expression (85), the doubly invariant metric on $\mathrm{U}(2)$. This also expresses the fact that $\mathrm{GL}^{+}(2,\mathbb{R})$, $\mathrm{U}(2)$ are two different real forms of $\mathrm{GL}(2,\mathbb{C})$.

Two-dimensional models in Riemannian manifolds

In generic Riemannian manifold (M, g) there is no isometry concept obviously, except for the trivial isometry (the identity transformation). So, there is no concept of an extended rigid body. Similarly, in general there are no finite affine transformations (with an exception of the trivial one), and therefore, there is no concept of extended affine bodies (homogeneously deformable gyroscopes). But we can consider some models of infinitesimal affinely-rigid body and metrically-rigid body.

The treatment consists in replacing extended bodies by structured material points, i.e., by material points with attached linear frames (affine body) or orthonormal frames (gyroscope). These bases describe internal degrees of freedom. This means that degrees of freedom are analytically described by the spatial coordinates x^i $(i = 1, \cdots, n)$ and the components $e^i{}_A$ of the attached co-moving bases e_A $(A = 1, \cdots, n)$. In gyroscopic case, the quantities $e^i{}_A$ are constrained by the orthogonality condition:

$$g_{ij} e^i{}_A e^j{}_B = \delta_{AB}. \tag{93}$$

Obviously, the metric tensor g_{ij} is always taken at the point $x \in M$, where the body is instantaneously placed, and the basis $(\cdots, e_A, \cdots)$ is attached, so $e_A \in T_x M$. Therefore, the quantities $e^i{}_A$ are then functionally constrained by (93), and they are not generalized coordinates. So, they are not very suitable for analytical calculations.

The configuration space Q of infinitesimal rigid body in (M, g) may be identified with $F(M, g)$, i.e., the manifold of all g-orthonormal frames in all tangent spaces of M. Obviously, $F(M, g)$ is $n(n+1)/2$-dimensional manifold; there is n-translational degrees of freedom and $n(n-1)/2$ rotational ones

$$\dim Q = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

To obtain an effective analytical description, one fixes some, usually non-holonomic field of frames E_A, $A = 1, \cdots, n$, usually somehow distinguished by the geometry of (M, g). Then we take the expansion:

$$e_A(t) = E_B(x(t)) R^B{}_A(t), \tag{94}$$

where $R(t)$ is a time-dependent orthogonal matrix, i.e.,

$$\delta_{CD} R^C{}_A R^D{}_B = \delta_{AB}. \tag{95}$$

The angular velocity ω in the co-moving representation is defined by

$$\frac{De_B}{Dt} := e_A \omega^A{}_B. \tag{96}$$

One can show that

$$\omega^A{}_B = \rho^A{}_B + d^A{}_B, \quad \rho^A{}_B = (R^{-1})^A{}_C \frac{d}{dt} R^C{}_B,$$
$$d^A{}_B = (R^{-1})^A{}_F \Gamma^F{}_{DC} R^D{}_B R^C{}_G v^G,$$

we omit the simple proof. Here ρ is the "relative" angular velocity of the co-moving frame e with respect to the fixed reference frame E, the object d ("drive") describes the angular velocity with which E itself rotates along the trajectory of motion. The symbols

$$v^G = e^G{}_i \frac{dx^i}{dt}$$

are the co-moving components of the translational velocity,

$$\Gamma^A{}_{BC} = E^A{}_i \Gamma^i{}_{jk} E^j{}_B E^k{}_C - E^A{}_{i,j} E^i{}_B E^j{}_C$$

are the anholonomic components of the Levi-Civita affine connection with respect to E_A.

We have the following expression for the total kinetic energy:

$$T = T_{tr} + T_{int} = \frac{m}{2} g_{ij} v^i v^j + \frac{1}{2} \delta_{AB} \omega^A{}_C \omega^B{}_D J^{CD}. \tag{97}$$

In this formula the descriptors "*tr*" and "*int*" refer obviously to the translational and internal parts, m denotes the mass, and $J^{CD} = J^{DC}$ are co-moving components of the tensor of internal inertia.

Here we are interested mainly in the two-dimensional gyroscope, however this procedure is also convenient when dealing with infinitesimal affinely-rigid body. The reason for this is that also in the case of affine motion there is a distinction between the compact $n(n-1)/2$ - dimensional subgroup of rotations and the $n(n+1)/2$ - dimensional quotient manifold. Therefore, even in this case it may be convenient to distinguish between analytical formulas for the rotations and deformations.

The formulas above, first of all (97), are very convenient, almost indispensible in the technical procedures of solving equations of motion. However, their disadvantage is that some geometric aspects are rather hidden.

Let us repeat some of them. In a more general case of affine motion, i.e., one without constraints (93), the expression for the kinetic energy has the form

$$T = T_{tr} + T_{int} = \frac{m}{2} g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2} g_{ij} \frac{De^i{}_A}{Dt}\frac{De^j{}_B}{Dt} J^{AB}. \tag{98}$$

Obviously, it remains also true when (93) is imposed. For Lagrangians of the potential form

$$L = T - V(x, e)$$

the resulting equations of motion read:

$$\begin{aligned} m\frac{Dv^a}{Dt} &= \frac{1}{2} S^k{}_l R^l{}_k{}^a{}_j v^j + F^a \\ e^a{}_K \frac{D^2 e^b{}_L}{Dt^2} J^{KL} &= N^{ab}. \end{aligned} \tag{99}$$

The meaning of symbols is as follows:

$$\begin{aligned} v^a = \frac{dx^a}{dt}, \quad S^{kl} = S^k{}_m g^{ml} = K^{kl} - K^{lk}, \quad K^{ab} = e^a{}_A \frac{De^b{}_B}{Dt} J^{AB}, \\ F^a = g^{ab} F_b = -g^{ab}\left(\frac{\partial V}{\partial x^b} - \Gamma^i{}_{jb} e^j{}_B \frac{\partial V}{\partial e^i{}_B}\right), \\ N^{ab} = N^a{}_c g^{cb} = -g^{bc} e^a{}_K \frac{\partial V}{\partial e^c{}_K} \end{aligned} \tag{100}$$

and $R^i{}_{jkl}$ is the curvature tensor of (M, g).

Therefore, v^a are components of the translational velocity, S^{kl} are components of spin (intrinsic angular momentum), F^a are coordinates of the translational force, and N^{ab} are components of the affine torque. It is important that the covariant components of F in general differ from $-\partial V/\partial x^b$; moreover, the latter ones are not covector components in M. It is only the total F, the covariant exterior differential of V that is a good M-covector. When the metrical constraints are imposed, i.e., when we deal with the metrically-rigid body, (99) becomes:

$$m\frac{Dv^a}{Dt} = \frac{1}{2}S^k{}_l R^l{}_k{}^a{}_j v^j + F^a$$

$$\frac{DS^{ab}}{Dt} = e^a{}_K \frac{D^2 e^b{}_L}{Dt^2} J^{KL} - e^b{}_K \frac{D^2 e^a{}_L}{Dt^2} J^{KL} = \mathcal{N}^{ab} = N^{ab} - N^{ba} \tag{101}$$

obviously, with algebraically substituted (93). This is a nice balance of linear momentum and spin, geometrically suggestive, but computationally not so effective as equations derived from (97). Nevertheless, (101) presents a nice description of the mutual interaction between the translational and the attitude motion.

Now we shall consider some special two-dimensional cases. Therefore, for the infinitesimal rigid body (infinitesimal gyroscope) we are dealing with three degrees of freedom: two translational ones and one internal, rotational. The resulting models are interesting in themselves from the point of view of pure analytical mechanics, in particular, some integrability and hyperintegrability (degeneracy) problems may be effectively studied. Obviously, the explicit analytical results exist only in Riemann manifolds (M, g) with some peculiar structure, first of all (but not only) in constant-curvature spaces. Some practical applications of classical two-dimensional models also seem to be possible, e.g., in geophysical problems, in mechanics of structured micropolar and micromorphic shells, etc. What concerns geophysics, we mean, e.g., motion of continental plates. Motion of pollutions like oil spots on the oceanic surface is an another suggestive example.

Let us now quote some instructive special examples, namely, the two-dimensional rigid body moving in constant-curvature spaces like the spherical space $S^2(0, R)$ and pseudo-spherical Lobachevsky space $H^{2,2,+}(0, R)$. The corresponding metric elements are given respectively by

$$ds^2 = dr^2 + R^2 \sin^2 \frac{r}{R} d\varphi^2, \quad ds^2 = dr^2 + R^2 \sinh^2 \frac{r}{R} d\varphi^2, \tag{102}$$

with the proviso that in the spherical case all situations with $r = \pi R$ and arbitrary values of φ correspond to the same point (the "southern" pole, or if $r = 0$ - the "northern" pole). The range of r is respectively $[0, \pi R]$, and $[0, \infty]$.

The most convenient choice of the reference frame is

$$E_r = \frac{\partial}{\partial r}; \quad E_\varphi = \frac{1}{R \sin \frac{r}{R}} \frac{\partial}{\partial \varphi}, \quad E_\varphi = \frac{1}{R \sinh \frac{r}{R}} \frac{\partial}{\partial \varphi},$$

respectively, in the spherical and pseudospherical case.

In two dimensions the angular velocity matrix has only one essential component, namely

$$\omega^1{}_2 = -\omega_2{}^1 = \omega, \quad \rho^1{}_2 = -\rho_2{}^1 = \rho, \quad d^1{}_2 = -d_2{}^1 = d,$$

the diagonal entries obviously vanish.

After some calculations the above formulae give:

(i) sphere:

$$\rho = \frac{d\psi}{dt}, \quad d = \cos\frac{r}{R}\frac{d\varphi}{dt}, \quad \omega = \frac{d\psi}{dt} + \cos\frac{r}{R}\frac{d\varphi}{dt}, \tag{103}$$

(ii) pseudosphere:

$$\rho = \frac{d\psi}{dt}, \quad d = \cosh\frac{r}{R}\frac{d\varphi}{dt}, \quad \omega = \frac{d\psi}{dt} + \cosh\frac{r}{R}\frac{d\varphi}{dt}. \tag{104}$$

Therefore, using the formula (97), we obtain for the kinetic energy $T = T_{tr} + T_{int}$ the expression below. Depending on the considered manifold, it has the following form:

(i) sphere:

$$T = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2\sin^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right) + \frac{I}{2}\left(\frac{d\psi}{dt} + \cos\frac{r}{R}\frac{d\varphi}{dt}\right)^2, \tag{105}$$

where I is the inertial moment. In the absence of deformations the internal inertia is controlled only by this single scalar. This is the peculiarity of the ”two-dimensional world”.

Even for the purely translational motion some interesting questions arise, e.g., what are spherically symmetric potentials $V(r)$ for which all orbits are closed? Obviously we mean problems based on Lagrangians

$$L_{\rm tr} = T_{\rm tr} - V(r).$$

This is a counterpart of the famous Bertrand problem in $\mathbb{R}^2$. And it may be shown that the answer is similar, i.e., the possible potentials are as follows:

(a) oscillatory potentials:

$$V(r) = \frac{\gamma}{2} R^2 \tan^2 \frac{r}{R}, \tag{106}$$

(b) Kepler-Coulomb potentials:

$$V(r) = -\frac{\alpha}{R} \cot \frac{r}{R}. \tag{107}$$

Obviously, with the spherical topology also the geodetic problem belongs here:

(c) $V(r) = 0$, i.e., (in a sense) the special case of (a) or (b) when $\gamma = 0$, $\alpha = 0$.

There is an obvious correspondence with the flat-space Bertrand problem; it is suggested by the very asymptotics for $r \approx 0$, i.e.,

$$V(r) \approx \frac{\gamma}{2} r^2, \qquad V(r) \approx -\frac{\alpha}{r}.$$

Obviously, this is a rough argument, but it may be shown that there exists a rigorous isomorphism based on the projective geometry.

The mentioned Bertrand models lead to completely integrable and maximally degenerate (hyperintegrable) problems. But even for the simplest, i.e., geodetic, models with the internal degrees of freedom the situation drastically changes. There exist interesting and practically applicable integrable models, but as a rule interaction with internal degrees of freedom reduces or completely removes degeneracy.

For certain reasons it will be convenient to rewrite the formula (105) in terms of the new variable $\vartheta = r/R$ - modified "geographic latitude", i.e.,

$$T = \frac{mR^2}{2} \left(\left(\frac{d\vartheta}{dt} \right)^2 + \sin^2 \vartheta \left(\frac{d\varphi}{dt} \right)^2 \right) + \frac{I}{2} \left(\frac{d\psi}{dt} + \cos \vartheta \frac{d\varphi}{dt} \right)^2. \tag{108}$$

It is seen that if formally $(\varphi, \vartheta, \psi)$ are interpreted as Euler angles (respectively the precession, nutation and rotation), the above expression is formally identical with the kinetic energy of the three-dimensional symmetric rigid body (without translations) with the main moments of inertia given by

$$I_1 = I_2 = mR^2, \quad I_3 = I.$$

If $I = mR^2$ one obtains the expression for the spherical top in three dimensions.

There is nothing surprising in the mentioned isomorphism because the quotient manifold $SO(3,\mathbb{R})/SO(2,\mathbb{R})$ may be in a natural way identified with $S^2(0,1)$ (or with any $S^2(0,R)$). Projecting the motion of the three-dimensional symmetric top onto the quotient sphere-manifold we obtain two-dimensional translational motion; the one dimensional subgroup of rotations about z-axis refers to the internal motion of the two-dimensional rotator.

The projection procedure is exactly compatible with the mentioned correspondence between Euler angles in $SO(3,\mathbb{R})$ and our generalized coordinates $(\varphi,\ \vartheta = r/R,\ \psi)$ of the infinitesimal rotator in $S^2(0,R)$.

Let $U(\varphi, \vartheta, \psi) \in SO(3,\mathbb{R})$ be just the element labelled by the Euler angles $(\varphi, \vartheta, \psi)$, thus

$$U(\varphi, \vartheta, \psi) = U_z(\varphi) U_x(\vartheta) U_z(\psi), \tag{109}$$

where U_z, U_x are rotations respectively around the z- and x-axes; angles of rotations are indicated as arguments and

$$U(\varphi, \vartheta, \psi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{110}$$

Calculating the "co-moving angular velocity"

$$\widehat{\varkappa} = U^{-1} \frac{dU}{dt} \tag{111}$$

of this fictitious three-dimensional top one obtains that

$$\widehat{\varkappa} = \widehat{\varkappa}_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \widehat{\varkappa}_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \widehat{\varkappa}_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{112}$$

where

$$\widehat{\varkappa}_1 = \sin\vartheta \sin\psi \frac{d\varphi}{dt} + \cos\psi \frac{d\vartheta}{dt}, \tag{113}$$

$$\widehat{\varkappa}_2 = \sin\vartheta \cos\psi \frac{d\varphi}{dt} - \sin\psi \frac{d\vartheta}{dt}, \tag{114}$$

$$\widehat{\varkappa}_3 = \cos\vartheta \frac{d\varphi}{dt} + \frac{d\psi}{dt}. \tag{115}$$

In expression (115) we easily recognize ω in (103), i.e., the expression for the one-component angular velocity of the two-dimensional rotator. Calculating formally the kinetic energy of the three-dimensional symmetric $SO(3,\mathbb{R})$-top, i.e.,

$$T = \frac{K}{2}\left(\widehat{\varkappa}_1\right)^2 + \frac{K}{2}\left(\widehat{\varkappa}_2\right)^2 + \frac{I}{2}\left(\widehat{\varkappa}_3\right)^2, \tag{116}$$

and substituting $K = mR^2$, $\vartheta = r/R$, we obtain exactly (105), i.e., (108).

As usual in analytical mechanics, the kinetic energy (105), (108) may be identified with some Riemannian structure on the configuration space. Let us write down our kinetic energy in the following form with the explicitly separated mass factor:

$$T = \frac{m}{2} G_{ij}(q) \frac{dq^i}{dt}\frac{dq^j}{dt}. \tag{117}$$

Just as above, our generalized coordinates q^i, $i = 1, 2, 3$, are the variables (r, φ, ψ) written just in this direction. After some calculations we obtain that

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 \sin^2 \frac{r}{R} + \frac{I}{m}\cos^2 \frac{r}{R} & \frac{I}{m}\cos\frac{r}{R} \\ 0 & \frac{I}{m}\cos\frac{r}{R} & \frac{I}{m} \end{bmatrix}. \tag{118}$$

In the special case $I = mR^2$ one obtains that G simplifies to $\breve{G}$, where

$$\left[\breve{G}_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & R^2 \cos\frac{r}{R} \\ 0 & R^2 \cos\frac{r}{R} & R^2 \end{bmatrix}. \tag{119}$$

In analogy to (105), (108) we obtain that:

(*ii*) pseudosphere:

$$T = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2 \sinh^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right) + \frac{I}{2}\left(\frac{d\psi}{dt} + \cosh\frac{r}{R}\frac{d\varphi}{dt}\right)^2, \tag{120}$$

i.e.,

$$T = \frac{mR^2}{2}\left(\left(\frac{d\vartheta}{dt}\right)^2 + \sinh^2\vartheta\left(\frac{d\varphi}{dt}\right)^2\right) + \frac{I}{2}\left(\frac{d\psi}{dt} + \cosh\vartheta\frac{d\varphi}{dt}\right)^2. \tag{121}$$

The kinetic energy is based on the metric tensor G_{ij} with components:

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2\sinh^2\frac{r}{R} + \frac{I}{m}\cosh^2\frac{r}{R} & \frac{I}{m}\cosh\frac{r}{R} \\ 0 & \frac{I}{m}\cosh\frac{r}{R} & \frac{I}{m} \end{bmatrix}. \tag{122}$$

It is seen that the spherically very special case $I = mR^2$ here, in the pseudospherical case also leads to some simplification of $[G_{ij}]$, but not so striking one as previously. This fact has deep geometric reasons which will be explained in the sequel.

Namely, in the spherical space an essential point is the natural identification between the quotient manifold $SO(3,\mathbb{R})/SO(2,\mathbb{R})$ and the spheres $S^2(0,R)$, $S^2(0,1)$. And this has to do with the formal identification between two-dimensional rigid body moving over the spherical surface and the three-dimensional symmetrical top without translational degrees of freedom. The special case $I = mR^2$ corresponds to the spherical top.

In general the kinetic energy is then invariant under $SO(3,\mathbb{R})\times SO(2,\mathbb{R})$. In the three-dimensional top analogy $SO(3,\mathbb{R})$ is acting as left regular translations and $SO(2,\mathbb{R})$ as right regular translations corresponding to the group of rotations around the body-fixed z-axis. If $I = mR^2$ we have the full invariance under $SO(3,\mathbb{R}) \times SO(3,\mathbb{R})$.

In the hyperbolic pseudospherical geometry the problem is isomorphic with the three-dimensional Lorentzian (Minkowskian) top on $\mathbb{R}^3$. The rotation group $SO(3,\mathbb{R})$ is replaced by the three-dimensional Lorentz group $SO(1,2)$. And still an important role is played by $SO(2,\mathbb{R})$ interpreted again as the group of usual rotations in Euclidean space of (x,y)-variables (thus, not affecting the z-direction). The above kinetic energy (120), (121) is invariant under $SO(1,2) \times SO(2,\mathbb{R})$. But it is never invariant under $SO(1,2) \times SO(1,2)$, i.e., under left and right Lorentz regular translations in the $SO(1,2)$-sense. The spherical special case $I = mR^2$ does not help here. Indeed, the underlying metric G (and the kinetic energy itself) is positively definite. But the doubly-invariant ($SO(1,2) \times SO(1,2)$-invariant) metric on $SO(1,2)$, i.e., its Killing metric is not positively definite. Instead it has the normal-hyperbolic signature $(++-)$. The reason is that it is semi-simple (even simple) non-compact group. This brings about the question about non-positive kinetic energies (metric tensors) on our configuration space. As the negative contribution to the Killing metric tensor on $SO(1,2)$ comes from its compact subgroup $SO(2,\mathbb{R})$ of (x,y)-rotations, i.e., from the gyroscopic degree of freedom in the language of $H^{2,2,+}(0,R)$, there is a natural suggestion to invert the sign of the gyroscopic contribution to (105), (108), i.e., to make it negative. One is naturally reluctant to indefinite kinetic energies but there are examples when they are just convenient and very useful as tools for describing some kinds of physical interactions, just encoding them even without any use of potentials.

So, we can try to use, or at least mathematically analyze, the "Lorentz-type kinetic energies" T_L of the form

$$\begin{aligned} T_L &= \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2\sinh^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right) - \frac{I}{2}\left(\frac{d\psi}{dt} + \cosh\frac{r}{R}\frac{d\varphi}{dt}\right)^2 \\ &= \frac{mR^2}{2}\left(\left(\frac{d\vartheta}{dt}\right)^2 + \sinh^2\vartheta\left(\frac{d\varphi}{dt}\right)^2\right) - \frac{I}{2}\left(\frac{d\psi}{dt} + \cosh\vartheta\frac{d\varphi}{dt}\right)^2. \end{aligned} \tag{123}$$

Thus, it is so as if the extra rotation diminished effectively the kinetic energy of translational motion. If we write as usual that

$$T_L = \frac{m}{2}\,{}_LG_{ij}(q)\frac{dq^i}{dt}\frac{dq^j}{dt},$$

then, with the same as previously convention concerning the ordering of coordinates (r, φ, ψ), we have that

$$[{}_L G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 \sinh^2 \frac{r}{R} - \frac{I}{m} \cosh^2 \frac{r}{R} & -\frac{I}{m} \cosh \frac{r}{R} \\ 0 & -\frac{I}{m} \cosh \frac{r}{R} & -\frac{I}{m} \end{bmatrix} \tag{124}$$

(compare this with (122)).

And now, obviously, the remarkable simplification occurs in the very special case $I = mR^2$ just as in the spherical symmetry. This has to do "as usual" with the enlarging of the symmetry group from $SO(1,2) \times SO(2,\mathbb{R})$ to $SO(1,2) \times SO(1,2)$ (two additional parameters of symmetry). And namely, ${}_L G$ becomes then ${}_L \breve{G}$, i.e.,

$$\left[{}_L \breve{G}_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R^2 & -R^2 \cosh \frac{r}{R} \\ 0 & -R^2 \cosh \frac{r}{R} & -R^2 \end{bmatrix} \tag{125}$$

(compare this with (119) and notice the characteristic sign differences).

Obviously, if we use the above isomorphism between the two-dimensional top sliding over the Lobachevsky plane with the three-dimensional Lorentz top without translational motion in $\mathbb{R}^3$, then it is clear that ${}_L G$ is, up to normalization, identical with the Killing metric tensor of $SO(1,2)$. Let us quote some formulas and concepts analogous to three-dimensional angular velocities, i.e., to (111), (112). And then the kinetic energy will be expressed like in (116).

First of all we parameterize $SO(1,2)$ with the help of what we call the "pseudo-Euler angles". So, let us write that

$$SO(1,2) \ni L(\varphi, \vartheta, \psi) = U_z(\varphi) L_x(\vartheta) U_z(\psi),$$

where the meaning of U_z is like in (109) and L_x denotes some Lorentz transformation in $\mathbb{R}^3$, namely, the "boost" along the x-axis, i.e.,

$$L_x(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \vartheta & \sinh \vartheta \\ 0 & \sinh \vartheta & \cosh \vartheta \end{bmatrix}.$$

During the motion all these quantities are functions of time and we can calculate the corresponding Lie-algebraic element

$$\widehat{\lambda} = L^{-1}\frac{dL}{dt},$$

i.e., the co-moving pseudo-angular velocity. After some calculations we obtain formulas analogous to (112)-(115), and namely,

$$\widehat{\lambda} = \widehat{\lambda}_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \widehat{\lambda}_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \widehat{\lambda}_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned}
\widehat{\lambda}_1 &= \sinh\vartheta \sin\psi \frac{d\varphi}{dt} + \cos\psi \frac{d\vartheta}{dt}, \\
\widehat{\lambda}_2 &= -\sinh\vartheta \cos\psi \frac{d\varphi}{dt} + \sin\psi \frac{d\vartheta}{dt}, \\
\widehat{\lambda}_3 &= \cosh\vartheta \frac{d\varphi}{dt} + \frac{d\psi}{dt}.
\end{aligned}$$

Both the similarities and differences in comparison with the corresponding spherical formulas are easily seen.

And now we can write two formulas analogous to (116), i.e.,

$$T = \frac{K}{2}\left(\widehat{\lambda}_1\right)^2 + \frac{K}{2}\left(\widehat{\lambda}_2\right)^2 + \frac{I}{2}\left(\widehat{\lambda}_3\right)^2, \tag{126}$$

$$T = \frac{K}{2}\left(\widehat{\lambda}_1\right)^2 + \frac{K}{2}\left(\widehat{\lambda}_2\right)^2 - \frac{I}{2}\left(\widehat{\lambda}_3\right)^2, \tag{127}$$

where $K > 0$ and $I > 0$. This is the symmetric $SO(1,2)$-top in $\mathbb{R}^3$. The indefinite expression (127) is structurally suited to the normal-hyperbolic signature of $SO(1,2)$. When $K = I$, then it becomes the spherical Lorentz top in $\mathbb{R}^3$ in the indefinite version based on the Killing metric.

It is easily seen that both expressions (126) and (127) are invariant under $SO(1,2) \times SO(2,\mathbb{R})$, where $SO(1,2)$ and $SO(2,\mathbb{R})$ acts on $SO(1,2)$ through respectively the left and right regular translations.

The form (127) with $K = I$ is invariant under all regular translations (both left and right), i.e., under $SO(1,2) \times SO(1,2)$. And specifying $K = mR^2$ in (126) and (127), we obtain respectively (120) and (123).

And again there are two Bertrand-type potentials, i.e.,

(a) the "harmonic oscillator"-type potential:

$$V(r) = \frac{\gamma}{2} R^2 \tan^2 \frac{r}{R}, \qquad \gamma > 0, \tag{128}$$

(b) the "attractive Kepler-Coulomb"-type one:

$$V(r) = -\frac{\alpha}{R} \cot \frac{r}{R}, \qquad \alpha > 0. \tag{129}$$

With these and only these potentials all bounded orbits are closed. And now the term "bounded" is essential because the "physical space" is now not compact. And indeed, there exist unbounded motions corresponding to energy values exceeding some thresholds. It is interesting that unlike in the spherical world, in Lobachevsky space the isotropic degenerate oscillator has an open subset of unbounded trajectories because the potential (128) has a finite upper bound, i.e.,

$$\text{Sup } V = \frac{\gamma}{2} R^2.$$

For energy values above this threshold all trajectories are unbounded, the motion is infinite. Below this threshold all trajectories are not only bounded but also periodic.

The existence of threshold in (129) is not surprising, it is like in the usual Kepler in $\mathbb{R}^2$. But the threshold for the isotropic degenerate oscillator is a very interesting feature of the Lobachevsky "world".

Quantized problems

We formulate now the quantized version of our models. Before doing this, let us remain briefly the general ideas of quantization in Riamannian configuration spaces. Considered is a classical geodetic system in a Riemannian manifold (Q, G), where Q denotes the configuration space, and G is the "metric" tensor field on Q underlying the kinetic energy form. In terms of generalized coordinates we have

$$T = \frac{1}{2} G_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt}.$$

As usual, the metric tensor G gives rise to the natural measure μ on Q,

$$d\mu(q) = \sqrt{|\det[G_{ij}]|} dq^1 \cdots dq^f,$$

where f denotes the number of degrees of freedom, i.e., $f = \dim Q$. For simplicity the square-root expression will be denoted by $\sqrt{|G|}$. The mathematical framework of Schrödinger quantization is based on $L^2(Q, \mu)$, i.e., the Hilbert space of complex-valued wave functions on Q, which are square-integrable in the μ-sense. Their scalar product is given by the usual formula:

$$\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi}_1(q) \Psi_2(q) d\mu(q).$$

The classical kinetic energy expression is replaced by the operator

$$\hat{T} = -\frac{\hbar^2}{2} \Delta,$$

where $\hbar$ denotes the ("crossed") Planck constant, and Δ is the Laplace-Beltrami operator corresponding to G,

$$\Delta = \frac{1}{\sqrt{|G|}} \sum_{i,j} \partial_i \sqrt{|G|} G^{ij} \partial_j = G^{ij} \nabla_i \nabla_j,$$

where ∇ denotes the Levi-Civita covariant differentiation in the G-sense.

If the problem is non-geodetic and some potential $V(q)$ is admitted, the corresponding Hamilton (energy) operator is given by:

$$\hat{H} = \hat{T} + \hat{V},$$

where the operator $\hat{V}$ acts on wave functions simply multiplying them by V,

$$(\hat{V}\Psi)(q) = V(q)\Psi(q).$$

This is the reason why very often one does not distinguish graphically between $\hat{V}$ and V.

Remark: There is a delicate problem concerning quantization which cannot be discussed here, and, fortunately, does not interfere directly with the main subjects of our analysis. Strictly speaking, wave functions are not scalars but complex densities of the weight $1/2$ so that the bilinear expression $\overline{\Psi}\Psi$ is a real scalar density of weight one, thus, a proper object for describing probability distributions. But in all realistic models, and the our one is not an exception, the configuration space is endowed with some Riemannian structure. And this enables one to factorize scalar (and tensor) densities into products of scalars (tensors) and some standard densities built of the metric tensor. Therefore, the wave function may be finally identified with the complex scalar field (multicomponent one when there are internal degrees of freedom).

From now on we concentrate ourselves on the special case of test rigid body moving on the two-dimensional sphere and pseudosphere. The previous notations are used to denote the variables. The Hamiltonian operator is given by the expression:

$$\hat{H} = \hat{T} + V(r) = -\frac{\hbar^2}{2m}\Delta + V(r). \tag{130}$$

The variables φ, ψ have the cyclic character in T. This focuses our attention on dynamical models where the potential energy is also independent of the angles φ, ψ.

After some calculations we obtain for the Laplace-Beltrami operator the expression below. Depending on the considered manifold, it has the following form:

(*i*) sphere:

$$\begin{aligned}\Delta \;=\; & \frac{\partial^2}{\partial r^2} + \frac{1}{R}\cot\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\psi} \\ & + \frac{mR^2\sin^2\frac{r}{R} + I\cos^2\frac{r}{R}}{IR^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2}.\end{aligned} \tag{131}$$

In the special case, when $I = mR^2$, we obtain

$$\breve{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{R}\cot\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{1}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2}, \tag{132}$$

as was mentioned above, the problem is isomorphic with the three-dimensional spherical top (without translations) and similarly on the pseudosphere:

(*ii*) pseudosphere:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{R}\coth\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{mR^2\sinh^2\frac{r}{R} + I\cosh^2\frac{r}{R}}{IR^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2}. \tag{133}$$

If we assume the rotational kinetic energy to contribute with the negative sign, then the expression (133) becomes

$${}_L\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{R}\coth\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{-mR^2\sinh^2\frac{r}{R} + I\cosh^2\frac{r}{R}}{IR^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2}. \tag{134}$$

In particular, in the very special case $I = mR^2$, these operators have the following form

$$\breve{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{R}\coth\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{1}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2}. \tag{135}$$

$$
\begin{aligned}
{}_L\check{\Delta} &= \frac{\partial^2}{\partial r^2} + \frac{1}{R}\coth\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\psi} \\
&+ \frac{1}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2}.
\end{aligned} \tag{136}
$$

Let us observe that Δ and therefore, also the kinetic energy operator $\hat{T}$ for (131) are left-invariant under the action of the group $SO(3,\mathbb{R})$, while (134) and the corresponding $\hat{T}$ are left-invariant under $SO(1,2)$. On the right they are invariant only under $SO(2,\mathbb{R})$, the subgroup of rotations about the "material" z-axis. In the special case $I = mR^2$, they are invariant also under the total right actions of $SO(3,\mathbb{R})$, $SO(1,2)$. Obviously, even in the case $I = mR^2$, (133) fails to be right-Lorentz-invariant, but of course (136) is right-invariant. But in applications, when some potential V is admitted, the invariance is lost. In any case, it is seen that the quantum operators of the kinetic energy have invariance properties quite analogous to the corresponding classical ones. Obviously, the symmetry operations are meant in the sense of the argument-wise action of the unitary operators representing the transformations of wave functions. Infinitesimal generators of the corresponding one-parameter subgroups simply do commute with the kinetic energy operator.

A basis of solutions of the stationary Schrödinger equation $\hat{H}\Psi = E\Psi$ has the form:

$$
\Psi(r,\varphi,\psi) = f_r(r)f_\varphi(\varphi)f_\psi(\psi). \tag{137}
$$

It is convenient to use the variable $\vartheta = r/R$ for our calculations, then we put

$$
\Psi(\vartheta,\varphi,\psi) = f_\vartheta(\vartheta)e^{in\varphi}e^{il\psi}, \qquad \text{n, l} \; - \; \text{integers}. \tag{138}
$$

The true quantum dynamics is contained in the factor f_ϑ. The separation of variables and the procedure of Sommerfeld polynomials guarantee that our wave functions are proper global solutions. However, some comments are necessary here. Namely, in a sense one can admit some additional "solutions". The point is that the configuration space, in the case, e.g., of the motion on sphere, being isomorphic with the rotation group $SO(3,\mathbb{R})$ is doubly connected. Its universal covering is $SU(2)$, obviously the homomorphism ratio is 2 : 1. Therefore, according to certain ideas of Pauli one can try to use the covering manifold as a modified configuration space. The same holds in principle in the hyperbolic case. This "covering space-philosophy" seems to suggest us to admit the numbers n, l in our product formula for Ψ to be simultaneously integers or simultaneously half-integers. Let us mention that there are systematically returning ideas of spin as the "internal angular momentum" of a quantized rigid (or deformable) body.

Nevertheless, we were dealing there with the three-dimensional rigid body without translational motion. One can quite reasonably expect the corresponding "half-integer" solutions to be then realistic. It is not clear if this is the case also for the body moving on the sphere $S^2(0, R)$ with one internal/rotational degree of freedom, although the configuration space is then the same, i.e., $SO(3, \mathbb{R})$.

Hence, the stationary Schrödinger equation with an arbitrary potential $V(\vartheta)$ leads after the standard separation procedure to the following one-dimensional "radial" eigenequations:

(i) sphere:

$$\frac{d^2 f_\vartheta(\vartheta)}{d\vartheta^2} + \cot\vartheta \frac{df_\vartheta(\vartheta)}{d\vartheta} - \left(\frac{\left(\frac{m}{I} R^2 \sin^2\vartheta + \cos^2\vartheta\right) n^2 + l^2 - 2nl\cos\vartheta}{\sin^2\vartheta} - \frac{2mR^2}{\hbar^2}(E - V(\vartheta)) \right) f_\vartheta(\vartheta) = 0. \tag{139}$$

(ii) pseudosphere:

$$\frac{d^2 f_\vartheta(\vartheta)}{d\vartheta^2} + \coth\vartheta \frac{df_\vartheta(\vartheta)}{d\vartheta} - \left(\frac{\left(\pm\frac{m}{I} R^2 \sinh^2\vartheta + \cosh^2\vartheta\right) n^2 + l^2 - 2nl\cosh\vartheta}{\sinh^2\vartheta} - \frac{2mR^2}{\hbar^2}(E - V(\vartheta)) \right) f_\vartheta(\vartheta) = 0 \tag{140}$$

with the above-mentioned meaning of the $\pm$ signs.

It is natural to expect that for Bertrand potentials the resulting Schrödinger equations should be rigorously solvable in terms of some standard special functions. The most convenient way of solving them is to use the Sommerfeld polynomial method. In this method the solutions are expressed by the usual or confluent Riemann P-functions. They are deeply related to the hypergeometric functions (respectively usual F or confluent F_1). If the usual convergence demands are imposed, then the hypergeometric functions become polynomials and our solutions are expressed by elementary functions. At the same time the energy levels and separation constants are expressed by the eigenvalues of the corresponding operators. There exists some special class of potentials to which the Sommerfeld polynomial method is applicable.

Examples

The equations (139) and (140) may be solved only when the explicit form of potential is specified. We consider a special case, when the translational part of the potential energy $V(\vartheta)$ ($V(r)$) has the Bertrand structure, i.e. with the "frozen" rotations all orbits would be closed.

(i) sphere:

Here we consider the following model of the oscillatory potential (106):

$$V(r) = \frac{\gamma}{2} R^2 \tan^2 \frac{r}{R}.$$

Let us mention, it is a kind of the anharmonic potential which in the neighbourhood of the equilibrium resembles some properties of the harmonic oscillator. This is the reason we are interested in it.

Applying the Sommerfeld polynomial method we obtain the energy levels E as follows:

$$\begin{aligned} E &= \frac{1}{2}\hbar\Omega \left(\left(2k + 1 + |n-l| + \sqrt{(n+l)^2 + \frac{\gamma m R^4}{\hbar^2}} \right)^2 \right. \\ &+ \left. 4n^2 \left(\frac{m}{I} R^2 - 1 \right) - \frac{4\gamma m R^4}{\hbar^2} - 1 \right), \end{aligned} \tag{141}$$

where $\Omega = \hbar\omega/4mR^2$, $\omega = \sqrt{\gamma/m}$ and $k = 0, 1, \ldots$. After some calculations we obtain the function $f_r(r)$ in the form:

$$f_r(r) = \left(\cos \frac{r}{R} \right)^{\kappa} \left(\sin \frac{r}{R} \right)^{\nu} F \left(-k, k + 1 + \kappa + \nu; 1 + \kappa; \cos^2 \frac{r}{R} \right), \tag{142}$$

where

$$\kappa = \sqrt{(n+l)^2 + \frac{\gamma m R^4}{\hbar^2}}, \quad \nu = |n - l|.$$

(*ii*) pseudosphere:

We take the "harmonic oscillator" - type potential (128):

$$V(r) = \frac{\gamma}{2} R^2 \tanh^2 \frac{r}{R}, \quad \gamma > 0.$$

We find the energy levels E in the form:

$$\begin{aligned} E &= \frac{1}{2}\hbar\Omega \left(\left(2k + 1 + |n - l| + \sqrt{(n + l)^2 + \frac{\gamma m R^4}{\hbar^2}} \right)^2 \right. \\ &- \left. 4n^2 \left(\pm \frac{m}{I} R^2 - 1 \right) - \frac{4\gamma m R^4}{\hbar^2} - 1 \right). \end{aligned} \tag{143}$$

The function $f_r(r)$ is as follows:

$$f_r(r) = \left(\cosh \frac{r}{R} \right)^{\kappa} \left(\sinh \frac{r}{R} \right)^{\nu} F \left(-k, k + 1 + \kappa + \nu; 1 + \kappa; \cosh^2 \frac{r}{R} \right). \tag{144}$$

Let us notice that

$$\lim_{r \to \infty} \frac{\gamma}{2} R^2 \tanh^2 \frac{r}{R} = \frac{\gamma}{2} R^2.$$

This is the upper bound of the potential V (128). Therefore, the formula (143) is correct only for such quantum numbers that

$$E < \mathrm{Sup}\ V = \frac{\gamma}{2} R^2.$$

Above this threshold we are dealing with the continuous spectrum and the classically non-restricted motion.

The considered systems are completely non-degenerate. On the quantum level this fact is reflected by the existence of three quantum numbers labelling the energy levels. They cannot be combined into a single quantum number, i.e., there is no total quantum degeneracy, i.e., hyperintegrability, with respect to them. The interaction between translational and rotational degrees of freedom completely removes degeneracy. As yet it is not clear for us if some weaker degeneracy does occur for some relationships between constants m, I, R, γ.

Nevertheless, in the spherical, resonanse $I = mR^2$ model

$$\frac{d^2 f_\vartheta(\vartheta)}{d\vartheta^2} + \cot\vartheta \frac{df_\vartheta(\vartheta)}{d\vartheta} - \left(\frac{n^2 + l^2 - 2nl\cos\vartheta}{\sin^2\vartheta} - \frac{2I}{\hbar^2}(E - V(\vartheta)) \right) f_\vartheta(\vartheta) = 0 \tag{145}$$

some special case of the total degeneracy is seen, namely $\gamma = 0$. This is the geodetic motion $V = 0$. The point is that this problem is isomorphic, as we mentioned above, with the quantum mechanics of the spherical rigid body without translational motion in three dimensions, i.e., with evidently completely degenerate model.

In the pseudospherical case, when $I = mR^2$

$$\frac{d^2 f_\vartheta(\vartheta)}{d\vartheta^2} + \coth\vartheta \frac{df_\vartheta(\vartheta)}{d\vartheta} - \left(\frac{n^2 + l^2 - 2nl\cosh\vartheta}{\sinh^2\vartheta} - \frac{2I}{\hbar^2}(E - V(\vartheta)) \right) f_\vartheta(\vartheta) = 0 \tag{146}$$

the situation is not clear, because in the geodetic problem $\gamma = 0$ one deals with the continuous spectrum, where the Sommerfeld polynomial method is not literally applicable.

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Thank you for your attention

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