Classical and quantization problems in degenerate affine motion

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Classical ideas

Configuration space of an n-dimensional affinely-rigid body is identified with

$$Q = \mathrm{GAf}\,(n, \mathbf{R}) \simeq \mathrm{GL}\,(n, \mathbf{R}) \otimes_s \mathbf{R}^n.$$

$\mathbf{R}^n$ - the center of mass motion, $Q_{\mathrm{int}} = \mathrm{GL}\,(n, \mathbf{R})$ - internal degrees of freedom. In continuum case rather: $Q_{\mathrm{int}} = \mathrm{GL}^+(n, \mathbf{R})$

In more sophisticated terms: (M, V) - the physical affine spaces of dimensional n, and (N, U) - the material space of the same dimension n.

V, U - the linear space of translations in M, N. The fixed, time-independent positive measure μ on N - the material mass distribution.

Lagrange center of mass in N; $v \in N$ such that

$$\int \overrightarrow{va}\ d\mu\left(a\right) = 0$$

If $\Phi \in \mathrm{AfI}\left(N, M\right)$, then $v_\Phi := \Phi\left(v\right)$ - position of the center of mass in M.

Therefore:

$$Q = M \times \mathrm{LI}\left(U, V\right),$$

$$\Phi \colon y^i = x^i + \varphi^i{}_K a^K.$$

The inertia:

- mass:

$$\mathcal{M} = \int d\mu\left(a\right),$$

- inertial tensor:

$$J^{KL} = \int a^K a^L d\mu\left(a\right) = J^{LK}.$$

Kinetic energy - summation over the body elements:

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{M}{2} g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2} g_{ij} \frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt} J^{AB}.$$

Putting $M = N = U = V = \mathbf{R}^n$, $Q_{\mathrm{int}} = \mathrm{GL}\left(n, \mathbf{R}\right)$:

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{M}{2}\frac{dx^T}{dt}\frac{dx}{dt} + \frac{1}{2}\mathrm{Tr}\left(J\frac{d\varphi^T}{dt}\frac{d\varphi}{dt}\right).$$

Lagrangian:

$$L = T - \mathrm{V}\left(x, \varphi\right).$$

Legendre transformation:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i}, \qquad p^A{}_i = \frac{\partial L}{\partial \dot{\varphi}^i_A} = \frac{\partial T}{\partial \dot{\varphi}^i_A}.$$

Explicite:

$$p_i = M g_{ij} \frac{dx^j}{dt}, \qquad p^A{}_i = g_{ij} \frac{d\varphi^j{}_B}{dt} J^{BA}.$$

Hamiltonian:

$$H = T + \mathrm{V} = T_{\mathrm{tr}} + T_{\mathrm{int}} + \mathrm{V} = \frac{1}{2M} g^{ij}\, p_i\, p_j + \frac{1}{2} \widetilde{J}_{AB}\, p^A{}_i\, p^B{}_j\, g^{ij} + \mathrm{V},$$

$$\widetilde{J}_{AC}\, J^{CB} = \delta_A{}^B.$$

Action of $\mathrm{GL}(V)$, $\mathrm{GL}(U)$:

$$\begin{aligned} \mathrm{A} \in \mathrm{GL}(V): &\quad \mathrm{LI}(U,V) \ni \varphi \mapsto A\varphi \in \mathrm{LI}(U,V), \\ \mathrm{B} \in \mathrm{GL}(U): &\quad \mathrm{LI}(U,V) \ni \varphi \mapsto \varphi B \in \mathrm{LI}(U,V). \end{aligned}$$

$$\varphi \mapsto \mathrm{A}\varphi\mathrm{B} \qquad \text{non-effective kernel:}$$

$$\left\{ (\ell Id_V, \ell^{-1} Id_U) : \ell \in \mathbb{R}^+ \right\}.$$

Degenerate dimension, deformable coin

$$m = \dim U < \dim V = n,$$

$$Q = \mathrm{AfM}(N, M) = M \times \mathrm{LM}(U, V),$$

$\mathrm{AfM}(N, M)$, $\mathrm{LM}(U, V)$ - monomorphisms

$$y^i = x^i + \varphi^i{}_K \, a^K$$

The $n \times m$ matrix $\left[\varphi^i{}_K\right]$ has the rank m

$$A \in \mathrm{GL}(V): \quad \mathrm{LM}(U,V) \ni \varphi \mapsto A\varphi \in \mathrm{LM}(U,V), \qquad \text{– transitive action}$$

$$B \in \mathrm{GL}(U): \quad \mathrm{LM}(U,V) \ni \varphi \mapsto \varphi B \in \mathrm{LM}(U,V). \qquad \text{– non-transitive}$$

Only such φ_1, φ_2 for which $\varphi_1(U) = \varphi_2(U)$ may be joined by right action.

Let us fix some $\Psi \in \mathrm{LM}(U,V)$. Then $\mathrm{LM}(U,V)$ may be obtained as follows:

$$\mathrm{LM}(U,V) \ni \Psi \mapsto \varphi = A\Psi \in \mathrm{LM}(U,V), \quad A \in GL(V).$$

What is the stabilizer group $\mathrm{H}[\Psi] \subset \mathrm{GL}[V]$? It preserves both $\Psi(U) \subset V$ but also every element of $\Psi(U)$.

Let us put: $U = \mathbb{R}^m$, $V = \mathbb{R}^n$ and assume $\Psi(U)$ to have zeros at $(n-m)$ places $\left[a^1, ..., a^m, 0, ..., 0\right]^T$,

$$\Psi\left(a^1, ..., a^m\right) = \begin{bmatrix} a^1 \\ \vdots \\ a^m \\ o \end{bmatrix}, \qquad o - (n-m) \times 1 \text{ zero matrix}$$

Then H is given by:

$$\begin{bmatrix} \mathrm{I}_m & A \\ o & B \end{bmatrix},$$

I_m - $m \times m$ identity matrix, A, B -$m \times (n-m)$ and $(n-m) \times (n-m)$ matrices, o - $(n-m) \times m$ zero matrix. A, B involve $m(n-m) + (n-m)^2 = n(n-m)$ parameters and $\mathrm{GL}(n, \mathbf{R})/H$: $n^2 - n(n-m) = nm$ parameters, the dimension of $\mathrm{L}(m,n)$ on $\mathrm{LM}(m,n)$.

H is indeed a subgroup:

$$\begin{bmatrix} \mathrm{I} & A_1 \\ o & B_1 \end{bmatrix} \begin{bmatrix} \mathrm{I} & A_2 \\ o & B_2 \end{bmatrix} = \begin{bmatrix} \mathrm{I} & A_1 B_2 + A_2 \\ o & B_1 B_2 \end{bmatrix}.$$

Affine velocity:

$$\Omega = \frac{d\varphi}{dt}\varphi^{-1}, \qquad \hat{\Omega} = \varphi^{-1}\frac{d\varphi}{dt} = \varphi^{-1}\Omega\varphi,$$

defined for $m = n$ do not exist when $m < n$. More precisely, the righ inverse ρ such that $\varphi\rho = \mathrm{Id}_V$ does not exist at all. The left inverse λ, $\lambda\varphi = \mathrm{Id}_U$ does exist but is non-unique; various version coincide only on $\varphi(U) \subset V$.

But affine spin do exist; they are momentum mappings (Hamiltonian generators) of the affine group,

$$\Sigma^i{}_j = \varphi^i{}_A p^A{}_j, \qquad \hat{\Sigma}^A{}_B = p^A{}_i \varphi^i{}_B.$$

But remark they are not spatial and material components of any object, because there is no isomorphism between U and V.

The canonical spin and vorticity are also defined:

$$
\begin{aligned}
S^i{}_j &:= \Sigma^i{}_j - g^{ik} g_{jl} \Sigma^l{}_k, \\
\mathcal{V}^A{}_B &:= \hat{\Sigma}^A{}_B - \eta^{AC} \eta_{BD} \hat{\Sigma}^D{}_C.
\end{aligned}
$$

Hamiltonian generators of spatial and material rotations.

Similarly, the translational momentum p_i gives rise to the material one $\hat{p}_A$:

$$\hat{p}_A = p_i \varphi^i{}_A.$$

Poisson brackets are given by structure constants of groups:

$$\left\{\Sigma^{i}{}_{j}, \Sigma^{k}{}_{l}\right\} = \delta^{i}{}_{l}\Sigma^{k}{}_{j} - \delta^{k}{}_{j}\Sigma^{i}{}_{l}, \qquad \left\{\Sigma^{i}{}_{j} + x^{i}p_{j}, p_{k}\right\} = \delta^{i}{}_{k}p_{j},$$

$$\{p_i, p_j\} = 0,$$

$$\left\{\Sigma^{i}{}_{j} + x^{i}p_{j}, \Sigma^{k}{}_{l} + x^{k}p_{l}\right\} = \delta^{i}{}_{l}\left(\Sigma^{k}{}_{j} + x^{k}p_{j}\right) - \delta^{k}{}_{j}\left(\Sigma^{i}{}_{l} + x^{i}p_{l}\right),$$

$$\left\{\Sigma^{i}{}_{j}, p_{k}\right\} = 0.$$

For the material affine spin the following holds:

$$\left\{\Sigma^{A}{}_{B}, \Sigma^{C}{}_{D}\right\} = \delta^{C}{}_{B}\widehat{\Sigma}^{A}_{D} - \delta^{A}{}_{D}\widehat{\Sigma}^{C}_{B},$$

and besides:

$$\left\{\Sigma^{i}{}_{j}, \Sigma^{A}{}_{B}\right\} = 0.$$

Besides, the following Poisson brackets hold:

$$\left\{x^i_j, p_j\right\} = \delta^i_j, \qquad \left\{\varphi^i{}_A, p^B{}_j\right\} = \delta^i_j \delta^B_A.$$

We do not quote similar formulas for $S^i{}_j$, $\mathcal{V}^A{}_B$ in terms of structure constants of $\mathrm{SO}(V,g)$, $\mathrm{SO}(U,\eta)$. If F depends only on the configuration (x,φ), then:

$$\left\{\Sigma^i{}_j, F\right\} = -\varphi^i{}_A \frac{\partial F}{\partial \varphi^j{}_A}, \qquad \left\{\widehat{\Sigma}^A{}_B, F\right\} = -\varphi^k{}_B \frac{\partial F}{\partial \varphi^k{}_A}.$$

Let us also quote the covariant Green tensor and contravariant deformation tensor:

$$G = \varphi^* \cdot g, \qquad \widetilde{C} = \varphi^* \cdot \widetilde{\eta},$$

analytically:

$$G_{AB} = g_{ij}\varphi^i{}_A \varphi^j{}_B, \qquad \widetilde{C}^{ij} = \varphi^i{}_A \varphi^j{}_B \eta^{AB}.$$

In matrix terms:

$$G = \varphi^T \varphi, \qquad \widetilde{C} = \varphi \varphi^T.$$

In the case of non-degenerate affine bodies, $m = n$, we based on the polar and two-polar decompositions:

$$\varphi = RL = \Lambda R, \qquad \varphi = VDU^{-1},$$

$R,\ V,\ U \in \mathrm{SO}(n, \mathbf{R})$, and L, $\Lambda = RLR^{-1}$ are symmetric and positively definite, D is diagonal and positive.

There are counterparts in the mechanics of degenerate affine bodies, when $m < n$.

So, we write:

$$\varphi = R \begin{bmatrix} L \\ o \end{bmatrix}$$

where $R \in \mathrm{SO}(n, \mathbb{R})$, $L \in \mathrm{Symm}(m, \mathbb{R})$, o is the $(n - m) \times m$ matrix made up of zeros.

$\mathrm{DimSO}(n, \mathbb{R}) = n(n-1)/2$, $\dim \mathrm{Symm}(\mathrm{m}, \mathbb{R}) = m(m+1)/2$. It is seen that for general values of m, n the total number of these parameters,

$$\frac{n(n-1)}{2} + \frac{m(m+1)}{2}$$

does not equal the number of internal degrees of freedom, i.e., to (nm). Because of some redundant variables the configuration space cannot be identified with the Cartesian product $\mathrm{SO}(n, \mathbb{R}) \times \mathrm{Sym}(m, \mathbb{R})$. Because the subgroup $\mathrm{SO}(n-m, \mathbf{R})$ acting on the $(n-m)$-tuple of the last variables in $\mathbf{R}^n$ does not affect $\begin{bmatrix} L \\ o \end{bmatrix}$ when multiplying it on the left.

Let us take the subgroup $K \subset \mathrm{SO}(n, \mathbf{R})$ composed of,

$$R = \begin{bmatrix} \mathrm{I}_m & \underline{o}^T \\ o & u \end{bmatrix},$$

where I_m is an $m \times m$ identity matrix, o is $(n-m) \times m$ zero matrix and $u \in \mathrm{SO}(n-m, \mathbb{R})$ is an arbitrary $(n-m) \times (n-m)$ rotation matrix. The subgroup K, isomorphic with $\mathrm{SO}(n-m, \mathbb{R})$ is $(n-m)(n-m-1)/2$-dimensional. The quotient manifold of left cosets, $\mathrm{SO}(n, \mathbf{R})/K$, has the dimension $n(n-1)/2 - (n-m)(n-m-1)/2 = mn - m(m+1)/2$. The configuration space of internal (relative) degrees of freedom Q_{int} is diffeomorphic with $(\mathrm{SO}(n, \mathbf{R})/K) \times \mathrm{Sym}(m, \mathbf{R})$. And the Cartesian product is an mn-dimensional manifold, just as $Q_{\mathrm{int}} = \mathrm{LM}(m, n, \mathbf{R})$ itself.

Let $\Psi \in \mathrm{LM}(U, V)$ be a reference configuration, $\Psi(U) \subset V$ a linear subspace and $K(\Psi) \subset \mathrm{SO}(V, g)$ - a subgroup preserving every point of $\Psi(U)$, the more $\Psi(U)$ itself. It acts trivially on $\Psi(U)$ and is the group of rotations on $\Psi(U)^{\perp}$. The quotient manifold $\mathrm{SO}(V, g)/K(\Psi)$ describes rotational degrees of freedom. Without using Ψ: this manifold is $\mathrm{F}(V, g; m)$, Stiefel manifold. When $V = \mathbf{R}^n$, $U = \mathbf{R}^m$, $\mathrm{F}(V, g; m) = \mathrm{SO}(n, \mathbf{R})/\mathrm{SO}(n - m, \mathbf{R})$.

Remark:

Stiefel manifold differs from Grassmann manifold,

$$\mathrm{SO}\left(n, \mathbf{R}/\mathrm{SO}\left(\mathbf{n} - \mathbf{m}, \mathbf{R}\right)\right) \times \mathrm{SO}\left(m, \mathbf{R}\right).$$

Grassmann dimension equals $m(n - m)$.

The "polar" decomposition identifies the internal configuration space with:

$$(\mathrm{SO}\,(n,\mathbf{R})\,/\mathrm{SO}\,(n-m,\mathbf{R}))\times\mathrm{Sym}\,(m,\mathbf{R})\,.$$

Fortunately when $m=n-1$ (physically 2), then $\mathrm{SO}(1,\mathbf{R})=\{1\}$ and $\mathrm{SO}(3,\mathbf{R})/\mathrm{SO}(1,\mathbf{R})=\mathrm{SO}(3,\mathbf{R})$, and

$$Q_{\mathrm{int}}=\mathrm{SO}(3,\mathbf{R})\times\mathrm{Sym}(2,\mathbf{R}).$$

The “two-polar” decomposition has the form:

$$\varphi=V\begin{bmatrix}D\\o\end{bmatrix}U^{-1},$$

$V\in\mathrm{SO}(n,\mathbf{R})$, $D=\mathrm{diag}(D_1,\ldots,D_m)$, $U\in\mathrm{SO}(m,\mathbf{R})$.

Physically,when $n = 3$, $m = 2$ the configurations are correctly represented by the triples (V, D, U). The "polar" and "two-polar" decompositions have the forms:

$$\varphi = R \begin{bmatrix} \xi & \alpha \\ \alpha & \zeta \\ 0 & 0 \end{bmatrix}, \qquad \varphi = V \begin{bmatrix} \lambda & 0 \\ 0 & \mu \\ 0 & 0 \end{bmatrix} U^{-1}(\theta),$$

where $R \in \mathrm{SO}(3, \mathbf{R})$, $U[\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathrm{SO}(2, \mathbf{R})$, and

$$\xi > 0, \qquad \xi\zeta - \alpha^2 > 0, \qquad \lambda > 0, \qquad \mu > 0.$$

$\mathrm{SO}(3, \mathbf{R})$ is parametrized by local coordinates like e.g., Euler angles, rotation vector etc.

We introduce angular velocities in co-moving representation:

$$\omega = R^{-1}\frac{dR}{dt} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix},$$

$$\chi = V^{-1}\frac{dV}{dt} = \begin{bmatrix} 0 & \chi_3 & -\chi_2 \\ -\chi_3 & 0 & \chi_1 \\ \chi_2 & -\chi_1 & 0 \end{bmatrix}$$

$$\vartheta = U^{-1}\frac{dU}{dt} = \frac{dU}{dt}U^{-1} = \frac{d\theta}{dt}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The internal kinetic energy in the "polar" representation has the form:

$$T = -\frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} LJL & o^T \\ o & O_{n-m} \end{bmatrix}\omega^2\right) + \mathrm{Tr}\left(\begin{bmatrix} LJ\dot{L} & o^T \\ o & O_{n-m} \end{bmatrix}\omega\right) + \frac{1}{2}\mathrm{Tr}\left(J\dot{L}^2\right).$$

The three terms are interpreted as:

1. T_{rot} - rotational part coupled to the deformation matrix L.
2. $T_{\mathrm{rot-def}}$ - Coriolis term - coupling between angular velocity and the deformation velocity.
3. T_{def} - the kinetic energy of deformation.

Explicite:

$$T = T_{\text{rot}} + T_{\text{rot-def}} + T_{\text{def}},$$

where

$$\begin{aligned}
T_{\text{rot}} &= \frac{J_1\alpha^2 + J_2\zeta^2}{2}\,\omega_1^2 + \frac{J_1\xi^2 + J_2\alpha^2}{2}\,\omega_2^2 \\
&+ \frac{J_1\xi^2 + J_2\zeta^2 + (J_1 + J_2)\,\alpha^2}{2}\,\omega_3^2 - (J_1\xi + J_2\zeta)\,\alpha\omega_1\omega_2, \\
T_{\text{rot-def}} &= \left(J_1\alpha\frac{d\xi}{dt} + (J_2\zeta - J_1\xi)\,\frac{d\alpha}{dt} - J_2\alpha\frac{d\zeta}{dt}\right)\omega_3, \\
T_{\text{def}} &= \frac{J_1}{2}\left(\frac{d\xi}{dt}\right)^2 + \frac{J_2}{2}\left(\frac{d\zeta}{dt}\right)^2 + \frac{J_1 + J_2}{2}\left(\frac{d\alpha}{dt}\right)^2.
\end{aligned}$$

In the “two-polar” case, when $J^{AB} = J\delta^{AB}$,

$$T = \frac{J}{2}\left(\mu^2\chi_1^2 + \lambda^2\chi_2^2 + \left(\lambda^2 + \mu^2\right)\chi_3^2\right) + 2J\lambda\mu\chi_3\frac{d\theta}{dt} + $$
$$+ \frac{J\left(\lambda^2 + \mu^2\right)}{2}\left(\frac{d\theta}{dt}\right)^2 + \frac{J}{2}\left(\left(\frac{d\lambda}{dt}\right)^2 + \left(\frac{d\mu}{dt}\right)^2\right).$$

Legendre transformation in the "polar” case:

$$p_\alpha = (J_1 + J_2)\,\frac{d\alpha}{dt} - (J_1\xi - J_2\zeta)\,\omega_3,$$

$$p_\xi = J_1\left(\frac{d\xi}{dt} + \alpha\omega_3\right), \qquad p_\zeta = J_2\left(\frac{d\zeta}{dt} - \alpha\omega_3\right),$$

$$s_1 = \left(J_1\alpha^2 + J_2\zeta^2\right)\omega_1 - \alpha\,(J_1\xi + J_2\zeta)\,\omega_2,$$

$$s_2 = -\alpha\,(J_1\xi + J_2\zeta)\,\omega_1 + \left(J_1\xi^2 + J_2\alpha^2\right)\omega_2,$$

$$s_3 = \alpha\left(J_1\frac{d\xi}{dt} - J_2\frac{d\zeta}{dt}\right) - \frac{d\alpha}{dt}\,(J_1\xi - J_2\zeta) + \left(J_1\xi^2 + (J_1 + J_2)\,\alpha^2 + J_2\zeta^2\right)\omega_3.$$

In the "two-polar" case:

$$p_\lambda = J\frac{d\lambda}{dt}, \quad p_\mu = J\frac{d\mu}{dt}, \quad p_\theta = J\left(\lambda^2 + \mu^2\right)\frac{d\theta}{dt} + 2J\lambda\mu\chi_3,$$

$$s_1 = J\mu^2\chi_1, \quad s_2 = J\lambda^2\chi_2, \quad s_3 = 2J\lambda\mu\frac{d\theta}{dt} + J\left(\lambda^2 + \mu^2\right)\chi_3.$$

Kinetic energy in the "polar" case:

$$
\begin{aligned}
T \;=\; & \frac{J_1(\xi s_1+\alpha s_2)^2+J_2(\alpha s_1+\zeta s_2)^2}{2(\alpha^2-\xi\zeta)^2 J_1 J_2}+\frac{\left(\xi^2 J_1+\zeta^2 J_2\right)p_\alpha{}^2}{2(\xi+\zeta)^2 J_1 J_2}+\\
+\; & \frac{\left(\alpha^2 J_1+\left(\alpha^2+(\xi+\zeta)^2\right)J_2\right)p_\xi{}^2+\left(\left(\alpha^2+(\xi+\zeta)^2\right)J_1+\alpha^2 J_2\right)p_\zeta{}^2}{2(\xi+\zeta)^2 J_1 J_2}+\\
+\; & \frac{(J_1+J_2)\left(s_3\left(s_3-2\alpha p_\zeta\right)+2\alpha p_\xi\left(s_3-\alpha p_\zeta\right)\right)}{2(\xi+\zeta)^2 J_1 J_2}+\\
+\; & \frac{2\left(\xi J_1-\zeta J_2\right)p_\alpha\left(\alpha p_\zeta-\alpha p_\xi+s_3\right)}{2(\xi+\zeta)^2 J_1 J_2}.
\end{aligned}
$$

Kinetic energy in the "two-polar" case:

$$
\begin{aligned}
T = & \frac{1}{2J}p_\lambda^2+\frac{1}{2J}p_\mu^2+\frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)^2}p_\theta^2+\frac{1}{2J\mu^2}s_1^2\\
& +\frac{1}{2J\lambda^2}s_2^2+\frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)^2}s_3^2-\frac{2\lambda\mu}{J\left(\lambda^2-\mu^2\right)^2}p_\theta s_3.
\end{aligned}
$$

Poisson brackets:

$$\frac{dq^i}{dt} = \left\{q^i, H\right\}, \quad \frac{ds_i}{dt} = \{s_i, H\}, \quad \frac{dp_i}{dt} = \{p_i, H\}.$$

$$\{f, g\} = -\{g, f\}, \quad \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0,$$

$$\{f, F(g)\} = \{f, g\} \frac{dF}{dg}.$$

$$\left\{q^i, p_j\right\} = \delta^i{}_j, \quad \{s_i, s_j\} = -\varepsilon_{ijk} s_k, \quad \{p_i, s_j\} = 0, \quad \left\{q^i, s_j\right\} = 0.$$

The "two-polar" equations of motion:

$$\frac{d\lambda}{dt} = \frac{p_\lambda}{J}, \quad \frac{d\mu}{dt} = \frac{p_\mu}{J}, \quad \frac{d\theta}{dt} = \frac{\left(\lambda^2 + \mu^2\right) p_\theta}{J\left(\lambda^2 - \mu^2\right)^2} - \frac{2\lambda\mu p_\theta}{J\left(\lambda^2 + \mu^2\right)^2},$$

$$\frac{ds_1}{dt} = \frac{\lambda\left(2\lambda^3 p_\theta + \lambda\left(\mu^2 - 3\lambda^2\right) s_3\right) s_2}{J(\lambda^3 - \lambda\mu^2)^2}, \quad \frac{ds_2}{dt} = \frac{\lambda\left(2\mu^3 p_\theta + \lambda\left(\lambda^2 - 3\mu^2\right) s_3\right) s_1}{J(\mu^3 - \mu\lambda^2)^2},$$

$$\frac{ds_1}{dt} = \frac{\left(\mu^2 - \lambda^2\right) s_1 s_2}{J\lambda^2\mu^2}, \quad \frac{dp_\mu}{dt} = 0,$$

$$\frac{dp_\lambda}{dt} = K + \frac{\lambda\left(\lambda^2 + 3\mu^2\right) p_\theta^2}{J(\lambda^2 - \mu^2)^3} + \frac{s_2^2}{J\lambda^3} - \frac{2\left(3\lambda^2\mu + \mu^3\right) p_\theta s_3}{J(\lambda^2 - \mu^2)^3} + \frac{\lambda\left(\lambda^2 + 3\mu^2\right) s_3^2}{J(\lambda^2 - \mu^2)^3},$$

$$\frac{dp_\mu}{dt} = P - \frac{\left(3\lambda^2\mu + \mu^2\right) p_\theta^2}{J(\lambda^2 - \mu^2)^3} + \frac{s_1^2}{J\mu^3} + \frac{2\left(3\lambda\mu^2 + \lambda^3\right) p_\theta s_3}{J(\lambda^2 - \mu^2)^3} - \frac{\left(3\mu\lambda^2 + \mu^3\right) s_3^2}{J(\lambda^2 - \mu^2)^3}.$$

The resulting equations are terribly complicated. But there are stationary ellipses as solutions on which the Green deformation tensor and angular velocities are constant.

We have shown this for the potentials:

$$\mathrm{V} = k\left(\lambda^2 + \mu^2\right)/2.$$

$$\mathrm{V} = c\left(\frac{1}{\lambda^2} + \lambda^2\right) + c\left(\frac{1}{\mu^2} + \mu^2\right)$$

$$\mathrm{V} = k\left(\frac{1}{\lambda\mu} + \frac{\lambda^2 + \mu^2}{2}\right)$$

where

$$x = \varrho\cos\varepsilon, \quad y = \varrho\sin\varepsilon.$$

$$x = \frac{1}{\sqrt{2}}\left(\lambda + \mu\right), \quad y = \frac{1}{\sqrt{2}}\left(\lambda - \mu\right)$$

Quantization ideas

Applications:

- internal degrees of freedom of molecules
- microobjects with almost “flat” core
- “Schwungrad” model used in molecular dynamics in the pioneering days of quantum theory
- convolution of “classical” and “quantum” in nanophysics

The quantum operator of the internal kinetic energy has the form proportional to the Laplace-Beltrami operator:

$$\mathbf{T} = -\frac{\hbar^2}{2J}\Delta_\Gamma,$$

and $\Gamma_{\mu\nu}(Q)$ is given by the underlying classical kinetic energy:

$$T_{\mathrm{int}} = \frac{1}{2}\Gamma_{\mu\nu}(Q)\frac{dQ^\mu}{dt}\frac{dQ^\nu}{dt}.$$

Laplace-Beltrami operator is given by:

$$\Delta\Psi = \sum_{\mu\nu} \frac{1}{\sqrt{|\Gamma|}} \frac{\partial}{\partial Q^{\mu}} \left(\sqrt{|\Gamma|} \Gamma^{\mu\nu} \frac{\partial \Psi}{\partial Q^{\nu}} \right),$$

where

$$|\Gamma| = \left| \det \left[\Gamma_{\mu\nu} \right] \right| .$$

More geometrically:

$$\Delta\Psi = \Gamma^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \Psi,$$

The Hilbert space is $L^2(Q_{\text{int}}, \nu)$, where ν is the induced Γ-Riemannian measure,

$$d\nu(Q) = \sqrt{\Gamma(Q)}\, dQ^1 \dots dQ^6.$$

The scalar product is given by:

$$\langle \Psi, \Phi \rangle = \int \overline{\Psi}(Q)\Phi(Q) d\nu(Q).$$

To calculate anything in detail is difficult, but one can replace the spin variables s_a by the quantum operators $\mathbf{S}_a$ generating right rotations of V:

$$f\left(V(I+\varepsilon)\right) = f(V) + \varepsilon^i \mathcal{R}_i f(V) = f(V) + \frac{i}{\hbar}\,\varepsilon^i \mathbf{S}_i f(V) + \hbar\mathfrak{o}(\varepsilon)$$

where $\mathcal{R}_i$ are generators and

$$\varepsilon = \begin{bmatrix} 0 & \varepsilon_3 & -\varepsilon_2 \\ -\varepsilon_3 & 0 & \varepsilon_1 \\ \varepsilon_2 & -\varepsilon_1 & 0 \end{bmatrix}$$

and $\frac{\mathfrak{o}(\varepsilon)}{\varepsilon} \to 0$ when $\varepsilon \to 0$. Quantum spin operators $\mathbf{S}_i$ satisfy the quantum Poisson brackets:

$$\frac{1}{\hbar i}\left[\mathbf{S}_a, \mathbf{S}_b\right] = -\varepsilon_{abc}\mathbf{S}_c.$$

The classical quantity p_θ will be replaced by the operator:

$$\mathbf{p}_\theta = \frac{\hbar}{i}\frac{\partial}{\partial\theta}.$$

It is possible to show that the operator $\mathbf{T}_{\mathrm{int}} = -\frac{\hbar^2}{2J}\Delta$ may be expressed as:

$$\begin{aligned}
\mathbf{T}_{\mathrm{int}} &= \frac{{\mathbf{S}_1}^2}{2J\mu^2} + \frac{{\mathbf{S}_2}^2}{2J\lambda^2} + \frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)^2}{\mathbf{S}_3}^2 + \frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)^2}{\mathbf{p}_\theta}^2 - \frac{2\lambda\mu}{J\left(\lambda^2-\mu^2\right)^2}\mathbf{p}_\theta\mathbf{S}_3 \\
&- \frac{\hbar^2}{2J}\frac{1}{\mathcal{P}}\left[\frac{\partial}{\partial\lambda}\mathcal{P}\frac{\partial}{\partial\lambda} + \frac{\partial}{\partial\mu}\mathcal{P}\frac{\partial}{\partial\mu}\right],
\end{aligned}$$

where the weight factor $\mathcal{P}$ is given by

$$\mathcal{P} = \lambda\mu\left|\lambda^2-\mu^2\right|.$$

The kinetic energy operator becomes:

$$\begin{aligned}\mathbf{T}_{\mathrm{int}} &= \frac{{\mathbf{S}_1}^2}{2J\mu^2} + \frac{{\mathbf{S}_2}^2}{2J\lambda^2} + \frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)^2}{\mathbf{S}_3}^2 + \frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)^2}{\mathbf{p}_\theta}^2 - \frac{2\lambda\mu}{J\left(\lambda^2-\mu^2\right)^2}\mathbf{p}_\theta\mathbf{S}_3 \\ &- \frac{\hbar^2}{2J}\frac{\partial^2}{\partial\lambda^2} - \frac{\hbar^2}{2J}\frac{\partial\ln\mathcal{P}}{\partial\lambda}\frac{\partial}{\partial\lambda} - \frac{\hbar^2}{2J}\frac{\partial^2}{\partial\mu^2} - \frac{\hbar^2}{2J}\frac{\partial\ln\mathcal{P}}{\partial\mu}\frac{\partial}{\partial\mu}.\end{aligned}$$

The total operator of the kinetic energy is obviously given by:

$$\mathbf{T} = \mathbf{T}_{\mathrm{tr}} + \mathbf{T}_{\mathrm{int}},$$

where the translational part is given as usual by:

$$\mathbf{T}_{\mathrm{tr}} = -\frac{\hbar^2}{2m}\,\Delta_{\mathrm{tr}} = -\frac{\hbar^2}{2m}\,g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} = -\frac{\hbar^2}{2m}\sum_i\frac{\partial^2}{\partial x^{i2}}.$$

The total volume element in $Q = Q_{\mathrm{tr}} \times Q_{\mathrm{int}}$ is given by:

$$dv_{vol}(x, V, \lambda, \mu, \varphi) = d_3x \; d\nu(V, \lambda, \mu, U(\theta)).$$

When the body is isotropic $J^{AB} = \mathrm{I}\eta^{AB}$ and the potential depends only on invariants λ, μ, then the solving procedure of Schrödinger equation may be partially algebraized. Namely, one can perform the Fourier analysis on $\mathrm{SO}(3, \mathbb{R})$, $\mathrm{SO}(2, \mathbb{R})$.

Parametrizing:

$$V(\bar{k}) = \exp\left(k^a E_a\right), \qquad \left(E_a\right)^b{}_c = -\varepsilon_a{}^b{}_c$$

on $\mathrm{SO}(3,\mathbb{R})$, and

$$\mathrm{SU}(2) \ni u(\bar{k}) = \exp\left(k^a e_a\right) = \cos\frac{k}{2}\, I_2 - \frac{k^a}{k}\sin\frac{k}{2}\, i\sigma_a,$$

we find the j-th irreducible representation of $G = \mathrm{SU}(2) \times \mathrm{SO}(3,\mathbb{R})$ is given by:

$$\mathfrak{D}^j(\bar{k}) = \exp\left(\frac{i}{\hbar} k^a S^j{}_a\right),$$

where $S^j{}_a$ are the Wigner matrices of the angular momentum with the Casimir quantum number j and the square of magnitudes $\hbar^2 j\,(j+1)$). In any case, $\mathfrak{D}^j$ are unitary $(2j+1) \times (2j+1)$ matrices.

We introduce the operators:

$$\mathfrak{S}_a = \frac{\hbar}{i}\mathcal{L}_a, \qquad \mathfrak{s}_a = \frac{\hbar}{i}\mathcal{R}_a,$$

where for „small” values of $\bar{\varepsilon}$ we have:

$$\begin{aligned} f\left(V(\bar{k})V(\bar{\varepsilon})\right) &= f\left(V(\bar{k})\right) + \varepsilon^i \mathcal{R}_i f\left(V(\bar{k})\right) + \mathfrak{o}(\varepsilon), \\ f\left(V(\bar{\varepsilon})V(\bar{k})\right) &= f\left(V(\bar{k})\right) + \varepsilon^i \mathcal{L}_i f\left(V(\bar{k})\right) + \mathfrak{o}(\varepsilon). \end{aligned}$$

They are formally self-adjoint and:

$$\begin{aligned} \mathcal{L}_a &= \frac{k}{2}\cot\frac{k}{2}\frac{\partial}{\partial k^a} + \left(1 - \frac{k}{2}\cot\frac{k}{2}\right)\frac{k_a}{k}\frac{k^b}{k}\frac{\partial}{\partial k^b} + \frac{1}{2}\varepsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c}, \\ \mathcal{R}_a &= \frac{k}{2}\cot\frac{k}{2}\frac{\partial}{\partial k^a} + \left(1 - \frac{k}{2}\cot\frac{k}{2}\right)\frac{k_a}{k}\frac{k^b}{k}\frac{\partial}{\partial k^b} - \frac{1}{2}\varepsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c}. \end{aligned}$$

Using the operator:

$$\mathcal{D}_a = \mathcal{L}_a - \mathcal{R}_a$$

we have:

$$\begin{aligned} \mathcal{L}_a &= \frac{k_a}{2}\frac{\partial}{\partial k} - \frac{1}{2}\varepsilon_{ab}{}^{c}k^{b}\mathcal{D}_c + \frac{1}{2}\mathcal{D}_a, \\ \mathcal{R}_a &= \frac{k_a}{2}\frac{\partial}{\partial k} - \frac{1}{2}\varepsilon_{ab}{}^{c}k^{b}\mathcal{D}_c - \frac{1}{2}\mathcal{D}_a. \end{aligned} \tag{1}$$

The Casimir invariants have the form:

$$\mathcal{L}^2 = \mathcal{R}^2 = {\mathcal{L}_1}^2 + {\mathcal{L}_2}^2 + {\mathcal{L}_3}^2 = {\mathcal{R}_1}^2 + {\mathcal{R}_2}^2 + {\mathcal{R}_3}^2 =$$

$$= \left(\frac{\partial^2}{\partial k^2} + \cot \frac{k}{2} \frac{\partial}{\partial k} \right) + \frac{1}{4 \sin^2 \frac{k}{2}} \mathcal{D}^2,$$

where

$$\mathcal{D}^2 = {\mathcal{D}_1}^2 + {\mathcal{D}_2}^2 + {\mathcal{D}_3}^2.$$

$$\mathfrak{S}_a = \frac{\hbar}{i} \mathcal{L}_a, \qquad \mathfrak{s}_a = \frac{\hbar}{i} \mathcal{R}_a$$

are respectively operators of the internal angular momentum (spin) in the laboratory representation and in the system of axes connected with the moving top.

$$\frac{1}{\hbar i}\left[\mathfrak{S}_a, \mathfrak{S}_b\right] \;=\; \varepsilon_{ab}{}^{c}\mathfrak{S}_c,$$

$$\frac{1}{\hbar i}\left[\mathbf{S}_a, \mathbf{S}_b\right] \;=\; -\varepsilon_{ab}{}^{c}\mathbf{S}_c.$$

Obviously

$$\mathfrak{S}^2 = \sum_a (\mathfrak{S}_a)^2 = \mathbf{S}^2 = \sum_a (\mathbf{S}_a)^2.$$

It is clear that:

$$
\begin{aligned}
\mathfrak{S}_a \mathfrak{D}^j &= {S^j}_a \mathfrak{D}^j, \\
\mathbf{S}_a \mathfrak{D}^j &= \mathfrak{D}^j {S^j}_a, \\
\mathfrak{S}^2 \mathfrak{D}^j &= \mathbf{S}^2 \mathfrak{D}^j = \hbar^2 j(j+1) \mathfrak{D}^j, \\
\mathfrak{S}_3 \mathfrak{D}^j_{m\ m'} &= \hbar m \mathfrak{D}^j_{m\ m'}, \\
\mathbf{S}_3 \mathfrak{D}^j_{m\ m'} &= \hbar m' \mathfrak{D}^j_{m\ m'}.
\end{aligned}
$$

Let us make the afore-mentioned Weyl-Peter expansion:

$$
\Psi(V; \lambda, \mu; \theta) = \sum_{j,m,m',k} f^{j,k}_{m',m}(\lambda, \mu)\ \mathfrak{D}^j_{m\ m'}(V)\ e^{ik\theta}.
$$

In the compact matrix form:

$$\Psi(V;\lambda,\mu;\theta)=\sum_{j,k}\mathrm{Tr}\left(f^{j,k}(\lambda,\mu)\ \mathfrak{D}^j(V)\right)\ e^{ik\theta}.$$

The action of the operator $\mathfrak{S}_a$ and $\mathbf{S}_a$ on Ψ is algebraically represented in such a way that the reduced amplitudes $f^{j,k}_{m',m}$ interpreted with the fixed values of j,k as $(2j+1)\times(2j+1)$ matrices with indices m',m are transformed as follows:

$$\begin{aligned} f^{j,k} &\mapsto S^j{}_a f^{j,k}, \\ f^{j,k} &\mapsto f^{j,k} S_a{}^j. \end{aligned}$$

And $\mathbf{p}_\theta$ acts on Ψ:

$$f^{j,k}\mapsto \hbar k f^{j,k}.$$

If we use the isotropic internal Hamiltonian:

$$\mathbf{H} = \mathbf{T} + \mathcal{V}(\lambda, \mu).$$

the stationary Schrödinger equation:

$$\mathbf{H}\Psi = E\Psi$$

becomes reduced to the system of independent equations for the matrix amplitudes $f^{j,k}(\lambda, \mu)$:

$$H^{j,k} f^{j,k} = E^{j,k} f^{j,k},$$

where

$$
\begin{aligned}
H^{j,k} f^{j,k} \;=\;& \frac{{S_1^j}^2}{2J\mu^2} f^{j,k} + \frac{{S_2^j}^2}{2J\lambda^2} f^{j,k} + \frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)^2} {S_3^j}^2 f^{j,k} \\
&+ \frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)^2} \hbar^2 k^2 f^{j,k} - \frac{2\lambda\mu}{J\left(\lambda^2-\mu^2\right)^2} {S_3^j}^2 \hbar k f^{j,k} \qquad (2) \\
&- \frac{\hbar^2}{2J}\frac{1}{\mathcal{P}}\frac{\partial}{\partial\lambda}\left(\mathcal{P}\frac{\partial}{\partial\lambda} f^{j,k}\right) - \frac{\hbar^2}{2J}\frac{1}{\mathcal{P}}\frac{\partial}{\partial\mu}\left(\mathcal{P}\frac{\partial}{\partial\mu} f^{j,k}\right) + V f^{j,k}.
\end{aligned}
$$

We obtain, the family of reduced Schrödinger equations for the system of matrix-valued amplitudes $f^{j,k}(\lambda,\mu)$. These amplitudes are dependent on deformation invariants.

In this way the number of degrees of freedom of internal motion of our model is effectively reduced from six to two. The price we pay is that we obtain the system of Schrödinger equations for multicomponent complex amplitudes, however, depending only on two variables.

This reduction is possible only for models with high symmetries, when both the inertial tensor and the potential energy are isotropic.

$$\begin{aligned}
\mathcal{V} &= \frac{k}{2}\left(\lambda^2+\mu^2\right), \\
\mathcal{V} &= c\left(\frac{1}{\lambda^2}+\lambda^2\right)+c\left(\frac{1}{\mu^2}+\mu^2\right), \\
\mathcal{V} &= \mathcal{V}_\rho(\rho)+\frac{\mathcal{V}_\varepsilon(\varepsilon)}{\rho^2},
\end{aligned}$$

where:

$$\begin{aligned}
x &= \frac{1}{\sqrt{2}}(\lambda+\mu)=\rho\cos\varepsilon, \\
y &= \frac{1}{\sqrt{2}}(\lambda-\mu)=\rho\sin\varepsilon.
\end{aligned}$$

Comments:

In non-degenerate three-dimensional models it appeared in a natural way by taking instead of $\mathrm{GL}(3,\mathbb{R})$ its covering group $\overline{\mathrm{GL}(3,\mathbb{R})}$. In the two-polar decomposition the orthogonal group $\mathrm{SO}(3,\mathbb{R})$ had to be replaced by the universal covering group $\mathrm{SU}(2)$. The same may be done here. Namely, the $\mathrm{SO}(3,\mathbb{R})$-factor of the decomposition must be replaced by $\mathrm{SU}(2)$. And the resulting wave functions must satisfy a condition that they combine expressions $\mathfrak{D}^{j}{}_{mm'}(u)e^{ik\theta}$ in such a way that either both j and k in the admissible superposition are integers, or both of them are half-integers.

Thank you !