Star product expression of algebra and star functions

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§1. Introduction

In this short talk, we discuss a basic idea of series of papers "Deformation of expressions for elements of an algebra" (arXiv Omori-Maeda-Miyazaki-Y.)

The idea is

1. We define star products of polynomials by deforming the usual multiplication of functions. The star product algebras contain wide class of algebras, for example, the usual function algebra, Weyl algebra, etc.
2. We show that certain family of star product algebras gives a polynomial representation of the Weyl algebra.
3. Using this polynomial representation, we can consider exponential elements of the Weyl algebra, called star exponentials.
4. Using the star exponentails, we define several functions called star functions in the Weyl algebra.
5. Using the star functions, we can construct several non commutative relations.

§2. Star products

Notice here that we consider on complex domain.

Biderivation

Let $\Lambda = (\Lambda_{kl})$ be an arbitrary $n \times n$ complex matrix. We consider a biderivation

$$\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = (\overleftarrow{\partial_{w_1}}, \cdots, \overleftarrow{\partial_{w_n}})\Lambda(\overrightarrow{\partial_{w_1}}, \cdots, \overrightarrow{\partial_{w_n}}) = \sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}$$

where $(w_1, \cdots, w_n)$ is a generators of polynomials.

The biderivation $\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = \sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}$ is defiend in the following manner:

$$
\begin{aligned}
f\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}g &= f\left(\sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}\right)g \\
&= \sum_{k,l=1}^{n} \Lambda_{kl}\,\partial_{w_k} f\,\partial_{w_l} g
\end{aligned}
$$

Here left arrow means the differential $\overleftarrow{\partial}_{w_j}$ acts on the function on the left hand side and the right arrow on the right.

Star product $*_\Lambda$

For polynomials f, g we consider a product $f *_\Lambda g$ given by the exponential power series of the biderivation $\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}$ such that

$$\begin{aligned} f *_\Lambda g &= f \exp \tfrac{i\hbar}{2}\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right) g = f \sum_{k=0}^{\infty} \tfrac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)^k g \\ &= fg + \tfrac{i\hbar}{2} f\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right) g + \tfrac{1}{2!}\left(\tfrac{i\hbar}{2}\right)^2 f\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)^2 g \\ &\quad + \cdots + \tfrac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k f\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)^k g + \cdots \end{aligned}$$

where $\hbar$ is a positive number.

Remark

The star product

$$f *_{\Lambda} g = fg + \frac{i\hbar}{2} f \left(\overleftarrow{\partial_w} \Lambda \overrightarrow{\partial_w}\right) g + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 f \left(\overleftarrow{\partial_w} \Lambda \overrightarrow{\partial_w}\right)^2 g + \cdots$$

is a deformation of the multiplication of functions. The product is well-defined for polynomials.

We see directly that the product is associative, namely we have

Theorem

For an arbitrary Λ, *the star product* $*_{\Lambda}$ *is a well-defined for complex polynomials and is an associative product .*

As an exercise, we calculate the commutator of the generator functions w_j, w_k, $(j, k = 1, 2, \cdots, n)$. We see

$$w_j *_{\Lambda} w_k = w_j \exp \frac{i\hbar}{2}\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right) w_k = w_j \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)^k w_k$$
$$= w_j w_k + \frac{i\hbar}{2} w_j \left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right) w_k = w_j w_k + \frac{i\hbar}{2}\Lambda_{jk}$$

Similarly we see

$$w_k *_{\Lambda} w_j = w_j w_k + \frac{i\hbar}{2}\Lambda_{kj}$$

Then the functions w_j and w_k satisfy the commutation relation under the commutator of the product $*_{\Lambda}$

$$[w_j, w_k]_* = w_j *_{\Lambda} w_k - w_k *_{\Lambda} w_j = \frac{i\hbar}{2}(\Lambda_{jk} - \Lambda_{kj}) \quad \text{(the skewsymmetic part)}$$

Examples of star product

The star products contain several well-known products.

Example 1. The Moyal product $*_o$

We consider the case where $n = 2m$ and we write the generators such that $w = (q_1, \cdots, q_m, p_1, \cdots, p_m)$.

We take

$$\Lambda = J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2m \times 2m \text{ blockwise})$$

Then the biderivation $\overleftarrow{\partial_w} \Lambda \overrightarrow{\partial_w} = \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}$ is the canonical Poisson bracket, namely, we see

$$f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right) g = \partial_p f \, \partial_q g - \partial_q f \, \partial_p g = \{f, g\}$$

Then the star product is the Moyal product $*_O$ such that

$$
\begin{aligned}
f *_O g &= f \exp \tfrac{i\hbar}{2}\left(\overleftarrow{\partial_p}\cdot\overrightarrow{\partial_q} - \overleftarrow{\partial_q}\cdot\overrightarrow{\partial_p}\right) g \\
&= fg + \tfrac{i\hbar}{2}\,\{f,g\} + \tfrac{1}{2!}\left(\tfrac{i\hbar}{2}\right)^2 f\left(\overleftarrow{\partial_p}\cdot\overrightarrow{\partial_q} - \overleftarrow{\partial_q}\cdot\overrightarrow{\partial_p}\right)^2 g \\
&\quad + \cdots + \tfrac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k f\left(\overleftarrow{\partial_p}\cdot\overrightarrow{\partial_q} - \overleftarrow{\partial_q}\cdot\overrightarrow{\partial_p}\right)^k g + \cdots
\end{aligned}
$$

Then the functions p and q satisfy the canonical commutation relation under the commutator of the product $*_O$

$$[p_j, q_k]_* = p_j \ *_O\ q_k - q_k \ *_O\ p_j = \mathrm{i}\,\hbar\,\delta_{jk}$$

$$[p_j, p_k]_* = [q_j, q_k]_* = 0$$

Example 2. Normal product $*_N$ and anti-nomal product $*_A$

Other typical star products are normal product $*_N$, anti-nomal product $*_A$

- if $\Lambda = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, we have the normal product

$$f *_N g = f \exp i\hbar \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} \right) g,$$

- if $\Lambda = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$ then the anti-normal product

$$f *_A g = f \exp -i\hbar \left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right) g$$

Canonical commutation relations

We see that these three typical star products satisfy the same canonical commutation relations

$$[p_j, q_k]_* = p_j * q_k - q_k * p_j = \mathrm{i}\hbar\delta_{jk}$$

$$[p_j, p_k]_* = [q_j, q_k]_* = 0$$

where $* = *_O, *_N, *_A$.

Then we have

Proposition

The algebras $(\mathbb{C}[q, p], *_L)$ $(L = O, N, A)$ are mutually isomorphic and isomorphic to the Wyel algebra, where $\mathbb{C}[q, p]$ is the space of complex polynomials of q, p.

Example 3. Commutative product and usual multiplication

- If Λ is a symmetric matrix, the star product $*_\Lambda$ is commutative.
- Furthermore, if Λ is a zero matrix, then the star product is nothing but a usual multiplication of functions.

§3. Star product representation of the Weyl algebra

For an arbitrary matrix Λ we set

$$\Lambda = J + K$$

where J is the skew symmetric part and K is the symmetric part.

For Λ we have a star product algebra $(\mathbb{C}[w], *_{\Lambda})$ where $\mathbb{C}[w]$ is the space of complex polynomials of $w = (w_1, \cdots, w_n)$.

Algebra isomorphisms

Then by the previous argument, the commutation relations depend only on the skew symmetric part J, namely

$$[w_j, w_k]_* = w_j *_\Lambda w_k - w_k *_\Lambda w_j = \frac{i\hbar}{2}(\Lambda_{jk} - \Lambda_{kj}) = \mathrm{i}\,\hbar J_{jk}$$

Hence

Proposition

The star product algebras $(\mathbb{C}[w], *_\Lambda)$ with the commom skew symmetric part J are mutually isomorphic.

The algebra isomorphisms are given explicitly in the following way.

Let $\Lambda_1 = K_1 + J$, $\Lambda_2 = K_2 + J$ be complex $n \times n$ matrices with the common skew symmetric part J.

Then we have star product algebras $(\mathbb{C}[w], *_{\Lambda_1})$ and $(\mathbb{C}[w], *_{\Lambda_2})$.

We consider a second order differential operator

$$\partial_w (K_2 - K_1) \partial_w = \sum_{jk} (K_2 - K_1)_{jk} \, \partial_{w_j} \partial_{w_k}.$$

We define a linear map $I_{K_1}^{K_2} : \mathbb{C}[w] \to \mathbb{C}[w]$ by exponentiating the differential operator $\partial_w(K_2 - K_1)\partial_w$ such that

$$
\begin{aligned}
I_{K_1}^{K_2}(f) &= \exp\left(\frac{i\hbar}{4}\partial_w(K_2 - K_1)\partial_w\right)(f) \\
&= \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{i\hbar}{4}\right)^l \left(\partial_w(K_2 - K_1)\partial_w\right)^l (f)
\end{aligned}
$$

Then by a direct calculation we have

Theorem

1. *The linear map* $I_{K_1}^{K_2}$ *gives an algebra isomorphism* $I_{K_1}^{K_2} : (\mathbb{C}[w], *_{\Lambda_1}) \rightarrow (\mathbb{C}[w], *_{\Lambda_2})$
2. *such that* $\left(I_{K_1}^{K_2}\right)^{-1} = I_{K_2}^{K_1}$
3. *satisfying the chain rule:* $I_{K_3}^{K_1} I_{K_2}^{K_3} I_{K_1}^{K_2} = Id$

Star product representation of the Weyl algebra

In what follows, we fix $n = 2m$ and also fix the antisymmetric part of Λ as J_0 below in order to represent the Weyl algebra.

Let K be an arbitrary $2m \times 2m$ complex symmetric matrix. We put a completx matrix

$$\Lambda = J_0 + K$$

where J_0 is a fixed matrix such that

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Since Λ is determined by the complex symmetric matrix K, we denote the star product by $*_K$ instead of $*_\Lambda$.

We consider polynomials of variables
$(w_1, \cdots, w_{2m}) = (u_1, \cdots, u_m, v_1, \cdots, v_m)$.
Due to the previous arguments, we have

Theorem

For an arbitrary K, it holds

(i) *For a star product $*_K$, the generators $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ satisfy the canoical commutation relations*

$$[u_k, v_l]_* = -i\hbar\delta_{kl}, \; [u_k, u_l]_* = [v_k, v_l]_* = 0, \quad (k, l = 1, 2, \ldots, m)$$

(ii) *Then the algebra $(\mathbb{C}[u, v], *_K)$ is isomorphic to the Weyl algebra, and the star product algebra $(\mathbb{C}[u, v], *_K)$ is regarded as a polynomial representation of the Weyl algebra.*

§4. Star exponentials

Here we consider the general case $(\mathbb{C}[w], *_\Lambda)$ again.

Using polynomial algebra $(\mathbb{C}[w], *_\Lambda)$, we can consider exponential elements.

The point

Star product algebras $(\mathbb{C}[w], *_\Lambda)$ with common skew symmetric part J are mutually isomorphic.

Then the family $\{(\mathbb{C}[w], *_\Lambda)\}_\Lambda$ of common J defines an associative algebra and each algebra $(\mathbb{C}[w], *_\Lambda)$ is regarded as a polynomial representation of this algebra. Hence each of these is the same as an algebra.

However, when we consider exponential elements of this algebra, the difference of expression is important and crucial.

This is the fundamental idea of our work.

Definition of star exponential

For a polynomial H_* of the algebra $(\mathbb{C}[w], *_\Lambda)$, we want to define a star exponential $e_{*_\Lambda}^{t\frac{H_*}{i\hbar}} = \sum_n \frac{t^n}{n!} \underbrace{\left(\frac{H_*}{i\hbar}\right) *_K \cdots *_K \left(\frac{H_*}{i\hbar}\right)}_{n}$.

However, except special cases, the expansion $\sum_n \frac{t^n}{n!} \left(\frac{H_*}{i\hbar}\right)^n$ is not convergent in general.

Then we define a star exponential by means of the differential equation as follows.

Definition

The star exponential $e_{*_\Lambda}^{t\frac{H_*}{i\hbar}}$ *is given as a solution* $F_t(u,v)$ *of the following differential equation*

$$\frac{d}{dt}F_t = H_* *_\Lambda F_t, \quad F_0 = 1.$$

Examples

We are interested in the star exponentials of linear, and quadratic plonomials. For these, we have explicit solutions.

Lienar case

We consider a linear polynomial $l = \sum_{j=1}^{n} a_j w_j,\ a_j \in \mathbb{C}$. We see

Proposition

For $l = \sum_j a_j w_j =< \boldsymbol{a}, \boldsymbol{w} >$, the star exponential with respect to the product $*_\Lambda$ is

$$e_{*_\Lambda}^{t(l/i\hbar)} = e^{t^2 \boldsymbol{a} K \boldsymbol{a}/4i\hbar} e^{t(l/i\hbar)}$$

where K is the symmetric part of Λ.

Quadratic case

Proposition

For a quadratic polynomial $Q_* = \langle \boldsymbol{w}A, \boldsymbol{w}\rangle_*$ where A is a $2m \times 2m$ complex symmetric matrix,

$$e_{*_\Lambda}^{t(Q_*/i\hbar)} = \frac{2^m}{\sqrt{\det(I-\kappa+e^{-2t\alpha}(I+\kappa))}} e^{\frac{1}{i\hbar}\langle \boldsymbol{w}\frac{1}{I-\kappa+e^{-2t\alpha}(I+\kappa)}(I-e^{-2t\alpha})J, \boldsymbol{w}\rangle}$$

where $\kappa = KJ$ and $\alpha = AJ$.

§5. Star functions

Using star exponentials, we can consider several star functions following the standard method in text books.

First, we consider the star product for the simplest case where the polynomial of one variable $w = w_1$ and $\Lambda = \rho \in \mathbb{C}$.

Then Λ is symmetric and the star product is commutative and is written explicitly as

$$p_1 *_{\Lambda} p_2 = p_1 \exp\left(\frac{i\hbar\rho}{2}\overleftarrow{\partial_{w_1}}\overrightarrow{\partial_{w_1}}\right) p_2$$

For simplicity, we write $i\hbar\rho = \tau$ and $w_1 = w$ in what follows.

Star Hermite function

Recall the identity

$$\exp\left(\sqrt{2}tw - \frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} H_n(w)\frac{t^n}{n!}$$

where $H_n(w)$ is an Hermite polynomial. By applying the explicit formula of linear case $e_{*_K}^{t(l/i\hbar)} = e^{t^2 \boldsymbol{a}K\boldsymbol{a}/4i\hbar}e^{t(l/i\hbar)}$ to $l = w$, we see

$$\exp_*(\sqrt{2}tw_*)_{\tau=-1} = \exp\left(\sqrt{2}tw - \frac{1}{2}t^2\right)$$

Since $\exp_*(\sqrt{2}tw_*) = \sum_{n=0}^{\infty}(\sqrt{2}tw_*)^n \frac{t^n}{n!}$ we have

$$H_n(w) = (\sqrt{2}tw_*)^n_{\tau=-1}$$

We define $*$-Hermite function by

$$H_n(w,\tau) = (\sqrt{2}tw_*)^n, \quad (n = 0, 1, 2, \cdots)$$

with respect to $*_\tau$ product. Then we have

$$\exp_*(\sqrt{2}tw_*) = \sum_{n=0}^{\infty} H_n(w,\tau)\,\frac{t^n}{n!}$$

Identities

Trivial identity $\frac{d}{dt}\exp_*(\sqrt{2}tw_*) = \sqrt{2}w * \exp_*(\sqrt{2}tw_*)$ for the product $*_\tau$ yields the identity

$$\frac{\tau}{\sqrt{2}}H_n'(w,\tau) + \sqrt{2}wH_n(w,\tau) = H_{n+1}(w,\tau), \quad (n = 0, 1, 2, \cdots).$$

for every $\tau \in \mathbb{C}$.

The exponential law

$$\exp_*(\sqrt{2}sw_*) * \exp_*(\sqrt{2}tw_*) = \exp_*(\sqrt{2}(s+t)w_*)$$

for the product $*_\tau$ yields the identity

$$\sum_{k+l=n} \frac{n!}{k!l!} H_k(w,\tau) *_\tau H_l(w,\tau) = H_n(w,\tau).$$

for every $\tau \in \mathbb{C}$.

Star theta function

We can express the Jacobi's theta functions by using star exponentials.

Using the formula of linear case, a direct calculation gives

$$\exp_{*_\tau} i\,tw = \exp(i\,tw - (\tau/4)t^2)$$

Hence for $\mathrm{Re}\,\tau > 0$, the star exponential $\exp_{*_\tau} ni\,w = \exp(ni\,w - (\tau/4)n^2)$ is rapidly decreasing with respect to integer n and then the summation converges to give

$$\sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2ni\,w = \sum_{n=-\infty}^{\infty} \exp\left(2ni\,w - \tau\,n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2ni\,w}, \quad (q = e^{-\tau})$$

This is Jacobi's theta function $\theta_3(w,\tau)$.
Similarly we have expression of theta functions as

$$\theta_{1*_\tau}(w) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n \exp_{*_\tau}(2n+1)i\,w, \quad \theta_{2*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau}(2n+1)i\,w$$

$$\theta_{3*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau} 2ni\,w, \quad \theta_{4*_\tau}(w) = \sum_{n=-\infty}^{\infty}(-1)^n \exp_{*_\tau} 2ni\,w$$

where $\theta_{k*_\tau}(w)$ is the Jacobi's theta function $\theta_k(w,\tau)$, $k = 1,2,3,4$ respectively.

The exponential law of the star exponential yields trivial identities such that

$$\exp_{*_\tau} 2i\, w *_\tau \theta_{k*_\tau}(w) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$\exp_{*_\tau} 2i\, w *_\tau \theta_{k*_\tau}(w) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

Then using $\exp_{*_\tau} 2i\, w = e^{-\tau} e^{2i\, w}$ and the product formula directly we see the above identities are just

$$e^{2i\, w - \tau} \theta_{k*_\tau}(w + i\,\tau) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$e^{2i\, w - \tau} \theta_{k*_\tau}(w + i\,\tau) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

$*$-delta functions

Since the $*_\tau$-exponential $\exp_*(itw_*) = \exp(itw - \frac{\tau}{4}t^2)$ is raidly decreasing with respect to t when $\mathrm{Re}\ \tau > 0$. Then the integral of $*_\tau$-exponential

$$\int_{-\infty}^{\infty} \exp_*(it(w-a)_*)\,dt = \int_{-\infty}^{\infty} \exp_*(it(w-a)_*)dt = \int_{-\infty}^{\infty} \exp(it(w-a)-\tfrac{\tau}{4}t^2)dt$$

converges for any $a \in \mathbb{C}$. We put a star δ-function

$$\delta_*(w-a) = \int_{-\infty}^{\infty} \exp_*(it(w-a)_*)dt$$

which has a meaning at τ with $\mathrm{Re}\ \tau > 0$. It is easy to see for any element $p_*(w) \in \mathcal{P}_*(\mathbb{C})$,

$$p_*(w) * \delta_*(w-a) = p(a)\delta_*(w-a),\ w_* * \delta_*(w) = 0.$$

Using the Fourier transform we have

Proposition

$$\theta_{1*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w+\tfrac{\pi}{2}+n\pi)$$

$$\theta_{2*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w+n\pi)$$

$$\theta_{3*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+n\pi)$$

$$\theta_{4*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+\tfrac{\pi}{2}+n\pi).$$

Now, we consider the τ with the condition $\mathrm{Re}\,\tau > 0$. Then we calcultate the integral and obtain $\delta_*(w-a) = \frac{2\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w-a)^2\right)$ Then we have

$$\theta_3(w,\tau) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta_*(w+n\pi) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w+n\pi)^2\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \sum_{n=-\infty}^{\infty} \exp\left(-2n\frac{1}{\tau}w - \frac{1}{\tau}n^2\tau^2\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \theta_{3*}\left(\frac{2\pi w}{i\tau}, \frac{\pi^2}{\tau}\right).$$

We also have similar identities for other $*$-theta functions by the similar way.

Clifford algebra

We show that the clifford algebra is constructed by means of star exponential of Weyl algebra in its certain representation.

In this subsection, we fix $n = 2$ for simplicity and consider the polynomial representation of the Weyl algebra.

Namely, let K be an arbitrary 2×2 complex symmetric matrix. We put a completx matrix

$$\Lambda = J_0 + K$$

and we consider star product algebras $(\mathbb{C}[u, v], *_K)$.

We will construct a Clifford algebra by means of the star exponential $\exp_* t(\frac{2u*v}{i\hbar})$ of the star product algebra $(\mathbb{C}[u, v], *_K)$ for certain K.

Let us denote a complex symmetric matrix by

$$K = \begin{pmatrix} \tau' & \kappa \\ \kappa & \tau \end{pmatrix}$$

In the star product algebra $(\mathbb{C}[u, v], *_K)$, the generators u, v satisfy the canonical commutation relation

$$[u, v]_{*_K} = -i\hbar$$

Then the star product algebra $(\mathbb{C}[u, v], *_K)$ is a polynomial representation of the Weyl algebra at $K = \begin{pmatrix} \tau' & \kappa \\ \kappa & \tau \end{pmatrix}$.

Applying the previous formula to $H = 2u * v$, we have the explicit form as

$$\exp_{*_K} t\left(\frac{2u * v}{i\hbar}\right) = \frac{2e^{-t}}{\sqrt{D}} \exp\left[\frac{e^t - e^{-t}}{i\hbar D}\Big((e^t - e^{-t})\tau u^2 + 2\triangle uv + (e^t - e^{-t})\tau' v^2\Big)\right]$$

where

$$D = \triangle^2 - (e^t - e^{-t})\tau'\tau, \quad \triangle = e^t + e^{-t} - \kappa(e^t - e^{-t})$$

Bamping lemma and commutation relation

The canonical commutation relation gives a so-called a bumping lemma such that

$$v *_K (u *_K v) = (v *_K u) *_K v = (u *_K v + i\hbar) *_K u$$

and then we have a relation

$$\begin{aligned} v *_K \exp_{*_K} t\left(\frac{2u * v}{i\hbar}\right) &= \exp_{*_K} t\left(\frac{2v * u}{i\hbar}\right) *_K v \\ &= \exp_{*_K} t\left(\frac{2u * v + 2i\hbar}{i\hbar}\right) *_K v \\ &= e^{2t} \exp_{*_K} t\left(\frac{2u * v}{i\hbar}\right) *_K v \end{aligned}$$

And also we have $\exp_{*_K} t(\frac{2v * u}{i\hbar}) = e^{2t} \exp_{*_K} t\left(\frac{2u * v}{i\hbar}\right)$.

Special representation

In the sequel, **we assum** $\tau' = 0$, that is, we take a point

$$K = \begin{pmatrix} \tau' & \kappa \\ \kappa & \tau \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ \kappa & \tau \end{pmatrix}$$

Then we see $D = \triangle^2 - (e^t - e^{-t})\tau'\tau = \triangle^2$ and hence we have $\sqrt{D} = \triangle$ and the star exponential at this point becomes

$$\exp_{*_K} t\left(\frac{2u * v}{i\hbar}\right) = \frac{2e^{-t}}{\triangle} \exp\left[\frac{e^t - e^{-t}}{i\hbar\triangle^2}\Big((e^t - e^{-t})\tau u^2 + 2\triangle uv\Big)\right]$$

Vacuum

At this sepcial representaion $*_K$ we can take a limit

$$\varpi_{00} = \lim_{t\to-\infty} \exp_{*_K} t\left(\frac{2u*v}{i\hbar}\right) = \frac{2}{1+\kappa}\exp\left(-\frac{1}{i\hbar(1+\kappa)}(2uv - \frac{\tau}{1+\kappa}u^2)\right)$$

which we call a **vacuum**.

Using the exponential law and the bumping relation, we easily see

Lemma

i) $\varpi_{00} *_K \varpi_{00} = \varpi_{00}$
ii) $v *_K \varpi_{00} = \varpi_{00} *_K u = 0.$

Since $\exp_{*_K} t(\frac{2v*u}{i\hbar}) = e^{2t}\exp_{*_K} t\left(\frac{2u*v}{i\hbar}\right)$, the integral $\frac{1}{2}\int_{-\infty}^{0}\exp_{*_K} t(\frac{2v*u}{i\hbar})dt$ converges. Then we define

$$\frac{1}{2}\int_{-\infty}^{0}\exp_{*_K} t\left(\frac{2v*u}{i\hbar}\right)dt = (v *_K u)_+^{-1}$$

and

$$\overset{\circ}{v} = u *_K (v *_K u)_+^{-1}.$$

Then we have

Lemma

The element $\overset{\circ}{v}$ is the right inverse of v satisfying

$$v *_K \overset{\circ}{v} = 1, \quad \overset{\circ}{v} *_K v = 1 - \varpi_{00}$$

Putting $t = \pi i$ in the star exponentia, we have the identity

$$\exp_{*_K} \pi i\left(\frac{2u*v}{i\hbar}\right) = \frac{2e^{-t}}{\triangle}\exp\left[\frac{e^t - e^{-t}}{i\hbar\triangle^2}\left((e^t - e^{-t})\tau u^2 + 2\triangle uv\right)\right] = 1$$

Now we fix an integer l.

By putting

$$t = t_l = \frac{\pi i}{2^l}$$

we obtain 2^l roots of the unity

$$\Omega_l = \exp_{*K} \frac{\pi i}{2^l}\left(\frac{2u*v}{i\hbar}\right), \ \varpi_l = \exp 2\left(\frac{\pi i}{2^l}\right)$$

such that

$$\Omega_{l*_K}^{2^l} = \underbrace{\Omega_l *_K \cdots *_K \Omega_l}_{2^l} = 1, \ \varpi_l^{2^l} = 1$$

because of the exponential law.

Then using the bumping relations we have

Lemma

These satisfy

$$\Omega^{k}_{l*_K} *_K u^m_{*_K} *_K \varpi_{00} *_K v^m_{*_K} = \varpi^{km}_{l} u^m_{*_K} *_K \varpi_{00} *_K v^m_{*_K}$$

Now we take appropriate complex numbers $a_0, a_1, \cdots, a_{2^l-1}$ so that an element

$$E = \sum_{k=0}^{2^l-1} a_k \Omega^k_{l *_K}$$

satisfies the identies

$$E *_K u^m_{*_K} *_K \varpi_{00} *_K v^m_{*_K} = \begin{cases} u^m_{*_K} *_K \varpi_{00} *_K v^m_{*_K} & \cdots 0 \le m \le 2^{l-1} - 1 \\ 0 & \cdots 2^{l-1} \le m \le 2^l - 1 \end{cases}$$

We see this is equivalent to

$$\sum_{k=0}^{2^l-1} a_k \varpi_l^{km} = \begin{cases} 1 \cdots 0 \leq m \leq 2^{l-1}-1 \\ 0 \cdots 2^{l-1} \leq m \leq 2^l - 1 \end{cases}$$

The complex numbers $a_0, a_1, \cdots, a_{2^l-1}$ are uniquely determined by these equations. Then we have

Lemma

The element E satisfies

$$E *_K E = 1$$

and the element $F = 1 - E$ satisfies

$$F *_K F = 1, \; E *_K F = F *_K E = 0$$

Further we have

Lemma

$$E *_K (v)^{2^{l-1}}_{*_K} = (v)^{2^{l-1}}_{*_K} *_K F, \quad (\mathring{v})^{2^{l-1}}_{*_K} *_K F = E * (\mathring{v})^{2^{l-1}}_{*_K}$$

where $(v)^{2^{l-1}}_{*_K} = \underbrace{v *_K \cdots *_K v}_{2^{l-1}}$ *and* $(\mathring{v})^{2^{l-1}}_{*_K} = \underbrace{\mathring{v} *_K \cdots *_K \mathring{v}}_{2^{l-1}}$

Now we set

$$\xi = E *_K (v)^{2^{l-1}}_{*_K}, \ \eta = (\mathring{v})^{2^{l-1}}_{*_K} *_K F$$

Then we have

Theorem

The elements ξ and η of the $*_K$ product algebra satisfies the identities

$$\xi *_K \xi = \eta *_K \eta = 0$$
$$\xi *_K \eta + \xi *_K \eta = 1$$