One million years stability of the solar system

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Consider the Sun, Mercury, Venus, Earth+Moon, Mars, Jupiter, Saturn, Uranus and Neptune as point masses, moving according Newton's inverse–square law of gravitation and the Galilean notion of space and time.

Let the unit of distance is the average distance between the Sun and Earth (that is 1 a.u., $\approx 149,6 \cdot 10^6$ kilometers). Denote by

$$\mu_k := \frac{\text{mass of the } k^{\text{th}} \text{ planet}}{\text{mass of the Sun}} \quad (k = 1 \div 8), \qquad \mu_0 := \mu_\odot = 1 \ .$$

the relative mass of the planet number k; so time $t = 2\pi$ corresponds to about one Julian year, that is 365 days and 6 hours. Also, $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_8$ are rectangular coordinates in $\mathbb{R}^3$ of the Mercury, Venus, etc., Neptune.

The coordinates $\boldsymbol{r}$ and velocity components $\frac{d\boldsymbol{r}}{dt}$ at any instant permit the determination of a unique set of six orbital elements

$$\left(a\,,\,e\,,i\,,t_0\,,g\,,\theta\right)\ ,$$

a being the semi-major axis,

e the eccentricity,

i the inclination of the orbit,

g the argument of perihelion,

θ the longitude of the ascending node,

t_0 an instant when the planet passes through the perihelion $a(1-e)$.

Introduce also the mean anomaly l and the mean motion n,

$$l=\int_{t_0}^{t} n(s)ds, \qquad n^2a^3 = 1+\mu \quad \text{(Third Kepler's law)}.$$

Omitting the index of a planet, its position in $\mathbb{R}^3$ is given by

$$\boldsymbol{r} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos g & -\sin g & 0 \\ \sin g & \cos g & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix},$$

where the planar motion (X, Y) can be expressed by functions of Bessel:

$$\begin{aligned} X &= a\Big[-\frac{3e}{2} + \sum_{m>0} \frac{2}{m} J'_m(me)\cos ml\Big]\ , \\ Y &= a\sqrt{1-e^2}\sum_{m>0} \frac{2}{me} J_m(me)\sin ml\ , \\ J_m(2x) &= \sum_{\beta=0}^{\infty} (-1)^\beta \frac{x^{m+2\beta}}{(m+\beta)!\,\beta!}\ , \end{aligned}$$

see for example Poincaré H., *Leçons de mécanique céleste* (1910).

If we neglect the action of other planets on the motion of the k^{th} planet, then the orbit would be a non-rotating ellipse (with the Sun in one of its foci) and a, e, i, t_0, g, θ would be constants. In fact, μ_k are small but positive; the biggest planet is Jupiter, $\mu_5 = \frac{1}{1047}$. Thus the orbital elements of the planets change over time owing their mutual perturbations and the planets move in slowly varying elliptic orbits.

Denote by $\boldsymbol{r'}$ the coordinates of a disturbing planet and denote its mass and orbital elements by $\mu', a', e', i', l', g', \theta'$. All the perturbations of for the planet $\boldsymbol{r}$ are governed by its disturbing function

$$R = \sum_{\mu' \neq \mu} \frac{\mu'}{\sqrt{1+\mu}} \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} - \frac{\boldsymbol{r}'\boldsymbol{r}}{|\boldsymbol{r}'|^3} \right)$$

with a summation taken over all 7 disturbing planets $\boldsymbol{r}'$. It is seen that to compute R is sufficiently to compute the Euclidean scalar products $\boldsymbol{r}'\boldsymbol{r}$, $\boldsymbol{r}'\boldsymbol{r}'$ and $\boldsymbol{r}\boldsymbol{r}$.

As a consequence of the Newton's inverse–square law of gravitation this variation of the orbital elements is described by a system of 48 ODE's:

$$\frac{da}{dt} = 2\sqrt{a}\,R_l\ , \qquad \frac{dl}{dt} = n - 2\sqrt{a}\,R_a - \frac{1-e^2}{e\sqrt{a}}R_e\ ,$$

$$\frac{de}{dt} = \frac{1-e^2}{e\sqrt{a}}R_l - \frac{\sqrt{1-e^2}}{e\sqrt{a}}R_g\ , \qquad \frac{dg}{dt} = \frac{\sqrt{1-e^2}}{e\sqrt{a}}R_e - \frac{\cot i}{\sqrt{a(1-e^2)}}R_i\ ,$$

$$\frac{di}{dt} = \frac{\cos i\,.\,R_g - R_\theta}{\sqrt{a(1-e^2)}\,\sin i}\ , \qquad \frac{d\theta}{dt} = \frac{R_i}{\sqrt{a(1-e^2)}\,\sin i}\ ,$$

where we have omitted the suffix k of the k^{th} planet but $R_h := \frac{\partial R}{\partial h}$ denotes the partial derivative of the disturbing function R with respect to h.

See for reference: Brower D., Clemence G., *Methods in Celestial Mechanics* (1961), p. 289.

In 1874 Newcomb expressed the equations of planetary motion by series of purely periodic terms having linear functions of time as the argument. After a term–by–term integration in the time t, this merely established the formal stability of the solar system in sense that the obtained series are not uniformly convergent.

However, as pointed out later by Poincaré, even the three–body model cannot be solved analytically. In other words, Newcomb' series diverge.

By **stability** we mean that each semi-major axis a, eccentricity e and inclination i remain limited as shown in the next table

Masses μ, the present and admissible values of a, e and i.

	Mer	**Ven**	**EM**	**Mar**	**Jup**	**Sat**	**Ura**	**Nep**
$1/\mu$	$6 \cdot 10^6$	408523	328900	$3 \cdot 10^6$	1047.3	3497.8	22902	19412
a_{today}	0.387	0.723	1.000	1.523	5.202	9.536	19.18	30.07
$a_{\min}$	0.35	0.69	0.9	1.35	4.7	8.5	17	28
$a_{\max}$	0.42	0.76	1.1	1.7	5.7	10.5	21	32
e_{today}	0.2056	0.0067	0.0167	0.0933	0.0483	0.0538	0.0472	0.0085
$e_{\max}$	0.25	0.08	0.07	0.15	0.07	0.09	0.08	0.02
i_{today}	$6^o20'$	$2^o11'$	$1^o35'$	$1^o40'$	$0^o19'$	$0^o55'$	$1^o01'$	$0^o43'$
$i_{\max}$	12^o	5^o	5^o	8^o	1^o	2^o	2^o	2^o

The simplest way to yield some ‘time of stability’ for a given dynamical variable $w = w(t)$ is to estimate its velocity:

$$\left|\frac{dw}{dt}\right| \ \leq \ \varepsilon \qquad \Rightarrow \qquad |w(t) - w(0)| \ \leq \ \varepsilon\,|t| \ ;$$

if $[w(0) - \varepsilon T, w(0) + \varepsilon T]$ belongs to certain admissible set of values of w, then we say that $w(t)$ is stable at least for time T. Obviously, T is of order ε^{-1}.

As a rule, $\frac{d}{dt}w$ consists of trigonometric expressions and the above estimate appears to be quite unsatisfactory.

A simple analysis of the equations of motion shows the following maximal speeds of changes for the action–variables a, e, i, as well as for the mean longitude $\lambda := l + g + \theta$:

	Mer	**Ven**	**EM**	**Mar**	**Jup**	**Sat**	**Ura**	**Nep**
$\left\|\frac{da}{dt}\right\| \cdot 10^6$	60	60	82	274	529	2650	9462	17427
$\left\|\frac{de}{dt}\right\| \cdot 10^6$	92	118	139	230	141	303	466	571
$\left\|\frac{di}{dt}\right\| \cdot 10^6$	90	120	138	226	140	302	462	575
$\left\|\frac{d\lambda}{dt} - n\right\| \cdot 10^6$	44	59	69	114	70	151	233	286

Since the disturbing masses μ' are small, the changes of our variables have been multiplied by 10^6. For example, the most stable is the motion of Mercury; its semi-major axis satisfies $|\frac{d}{dt}a_1| < 60 \cdot 10^{-6}$ to assure stability for about 100 years.

On the contrary, the most unstable looks the position of Neptune: its perturbation $|\frac{d}{dt}a_8| < 0.01$ guarantees stability for less than 20 years. This is mainly caused by the term

$$\mu_5 \frac{2a_8^{3/2}}{a_5^2} \sin(\lambda_5 - \lambda_8) \approx 0.01 \sin(\lambda_5 - \lambda_8)$$

of the Fourier expansion of $R = R_8$, coming from the indirect part of Jupiter's perturbations.

However, the integral perturbation

$$\begin{aligned} \int_0^T \sin(\lambda_5 - \lambda_8)\, dt &= \int_0^T \sin[n_{58}t + \delta_{58}(t)]\, dt \approx \int_0^T \delta_{58}(t) \cos(n_{58}t)\, dt \\ &< T\|\delta_{58}\| \approx T \cdot \frac{286 + 70}{10^6} \approx \frac{T}{2817} << T\,, \end{aligned}$$

($n_{58} := n_5 - n_8$) and this guaranties stability of a_8 for about 50500 years.

In other words, the maximal value of $|\sin x|$ equals 1, but if x is almost linear in the time, then the integral $|\int_0^T \sin x\, dt| << T$.

More generally, as the eccentricities and inclinations of the actual planets are small, each disturbing function can be expanded in d'Alembert series

$$\begin{aligned} R &= R^{(0)} + R^{(1)} + R^{(2)} + \cdots + R^{(j)} + \cdots \\ &= \sum A \cos \Psi \,, \\ A &= B(a, a')\, e^{j_1}\, e'^{j_2}\, i^{j_3}\, i'^{j_4}, \\ \Psi &= s_1 l + s_2 l' + s_3 g + s_4 g' + s_5 (\theta - \theta') \end{aligned}$$

where the disturbing term $A \cos \Psi$ has amplitude A of degree $j := j_1 + j_2 + j_3 + j_4$ (B is a homogeneous function of degree -1); j_k are non-negative integers; s_k - from the argument Ψ, are integers.

The convergence of the resulting series depend on how close the orbits are to intersection. A sufficient condition for convergence (if, say, $|\boldsymbol{r}'| > |\boldsymbol{r}|$) is:

$$a(1 + e) < a'(1 - e')\,,$$

or that the apocentric distance of the inner orbit is less than the pericentric distance of the outer orbit.

Our main formula to decrease the integral influence of a monomial perturbation term $A \cos \Psi$ will be

$$A \cos \Psi \quad = \quad \underbrace{\frac{d}{dt}\big[\gamma \sin \Psi\big]}_{\text{exact derivative}} \; - \quad \underbrace{\frac{d\gamma}{dt} \sin \Psi}_{\text{new perturbation term}_1} \quad + \; \underbrace{\left[A - \gamma \frac{d\Psi}{dt}\right] \cos \Psi}_{\text{new perturbation term}_2} \; ,$$

where $\gamma = \gamma(t)$ is an *arbitrary* differentiable function.

An exact derivative is not important, provided it is small enough:

$$\frac{dw}{dt} = \frac{df}{dt} + F \qquad \Rightarrow \qquad |w(t) - w(0)| \;\leq\; |f(t) - f(0)| + t \sup |F| \; .$$

Thus we replace the old amplitude A with two new amplitudes $\frac{d\gamma}{dt}$ and $A - \gamma \frac{d\Psi}{dt}$.

Remark that is possible to apply again the above formula but for the two new perturbations.

Our first choice

$$\gamma \;=\; constant \;\approx\; \frac{A}{\frac{d\Psi}{dt}}\,, \qquad A\cos\Psi = \underbrace{\frac{d}{dt}\big[\gamma\sin\Psi\big]}_{\text{exact derivative}} \;+\; \underbrace{\Big[A-\gamma\frac{d\Psi}{dt}\Big]\cos\Psi}_{\text{new perturbation term}_2}\;,$$

works very effectively when $|\frac{d\Psi}{dt}| > \frac{1}{100}$, say. It decreases the new perturbation amplitude about 1000 times. Recall that

$$\frac{d\Psi}{dt} \;=\; s_1 n + s_2 n' + \delta\Psi\,.$$

since g, θ, g', θ' are slow angle–variables and $\delta\Psi < 0.001$.

We also choose the approximate mean motions $n_1, \ldots, n_8$ of the planets from the solar system as square roots

$$\frac{\sqrt{52}}{\sqrt{3}}\,,\ \frac{\sqrt{8}}{\sqrt{3}}\,,\ 1\,,\ \frac{\sqrt{2}}{\sqrt{7}}\,,\ \frac{1}{\sqrt{140}}\,,\ \frac{\sqrt{6}}{\sqrt{140\cdot 37}}\,,\ \frac{\sqrt{3}}{\sqrt{560\cdot 37}}\,,\ \frac{\sqrt{19}}{7\sqrt{280\cdot 37}}$$

and thus $s_1 n + s_2 n'$ would not be a small divisor:

$$\frac{1}{|s_1 n + s_2 n'|} = \frac{|s_1 n - s_2 n'|}{|s_1^2 n^2 - s_2^2 n'^2|} > 0 \qquad if\ (s_1, s_2) \neq (0,0)\,.$$

Such a constant choice of γ reduces to 10^{-8} the sum of almost all amplitudes.

It remains to consider the cases with $s_1 = s_2 = 0$. Then we choose

$$\gamma \ := \ \frac{A}{\frac{d\Psi}{dt}}, \quad A\cos\Psi = \underbrace{\frac{d}{dt}\big[\gamma\sin\Psi\big]}_{\text{exact derivative}} + \underbrace{\left[\frac{\frac{dA}{dt}}{\frac{d\Psi}{dt}} - \frac{A\,\frac{d^2\Psi}{dt^2}}{\left(\frac{d\Psi}{dt}\right)^2}\right]\sin\Psi}_{\text{new perturbation term}_1},$$

and use that $|\frac{dA}{dt}| \ll |\frac{d\Psi}{dt}|$ and $\left|A\,\frac{d^2\Psi}{dt^2}\right| \ll \left(\frac{d\Psi}{dt}\right)^2$ to prove that the sum of all amplitudes with $s_1 = s_2 = 0$ (i.e. so-called seqular perturbations) and $j < 4$ is also less than 10^{-7}.

Finally, the influence of the degree ≥ 4 perturbations

$$|R^{(4)}| + |R^{(5)}| + \cdots \quad < \quad 10^{-8},$$

since the eccentricities and inclinations are small.

We have proved the following

Theorem. Each semi-major axis a_k, eccentricity e_k and inclination i_k will remain well–bounded at least for one million years.

THANK YOU