ON SOME SEMIPARALLEL SURFACES IN EUCLIDEAN SPACES

Betül BULCA

Uludağ University, Art and Science Faculty, Department of Mathematics, Bursa-TURKEY

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1. Introduction

Let M a submanifold of a $(n+d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. Denote by $\overline{R}$ the curvature tensor of the Vander Waerden-Bortoletti connection $\overline{\nabla}$ of M and h is the second fundamental form of M in $\mathbb{E}^{n+d}$.

The submanifold M is called **semi-parallel** (or semi-symmetric (Ferus, 1980)) if $\overline{R} \cdot h = 0$ (Decruyenaere et. al, 1994). This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which $R \cdot R = 0$ and a direct generalization of parallel submanifolds, i.e. submanifolds for which $\overline{\nabla} h = 0$.

2. Basic Concepts

Let M be a smooth surface in n-dimensional Euclidean space $\mathbb{E}^n$ given with the surface patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$. In the chart (u,v) the **coefficients of the first fundamental form** of M are given by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle,$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of the tangent plane T_pM in $\mathbb{E}^n$, that is the normal space of M at p.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent and normal to M respectively. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections on M and $\mathbb{E}^n$, respectively. Given any vector fields X_i and X_j tangent to M consider the **second fundamental map** $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j; \ 1 \leq i, j \leq 2. \tag{2.1}$$

This map is well-defined, symmetric and bilinear.

For any normal vector field N_α $1 \leq \alpha \leq n-2$ of M, recall the **shape operator** $A : \chi^{\perp}(M) \times \chi(M) \rightarrow \chi(M)$;

$$A_{N_\alpha} X_i = -\widetilde{\nabla}_{N_\alpha} X_i + D_{X_i} N_\alpha; \quad 1 \leq i \leq 2.$$

where D denotes the normal connection of M in $\mathbb{E}^n$ (Chen, 1973).This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_\alpha} X_i, X_j \rangle = \langle h(X_i, X_j), N_\alpha \rangle, \ 1 \leq i, j \leq 2. \qquad (2.2)$$

The equation (2.1) is called **Gaussian formula**, and

$$h(X_i, X_j) = \sum_{\alpha=1}^{n-2} h_{ij}^{\alpha} N_\alpha, \qquad 1 \leq i, j \leq 2 \qquad (2.3)$$

where h_{ij}^{α} are the **coefficients of the second fundamental form** h (Chen, 1973). If $h = 0$ then M is called **totally geodesic**. M is **totally umbilical** if all shape operators are proportional to the identity map.

If we define a covariant differentiation $\overline{\nabla} h$ of the second fundamental form h on the direct sum of the tangent bundle and normal bundle $TM \oplus T^{\perp}M$ of M by

$$(\overline{\nabla}_{X_i} h)(X_j, X_k) = D_{X_i} h(X_j, X_k) - h(\nabla_{X_i} X_j, X_k) - h(X_j, \nabla_{X_i} X_k) \tag{2.4}$$

for any vector fields X_i,X_j, X_k tangent to M. Then we have the **Codazzi equation**

$$(\overline{\nabla}_{X_i} h)(X_j, X_k) = (\overline{\nabla}_{X_j} h)(X_i, X_k) \tag{2.5}$$

where $\overline{\nabla}$ is called the Vander Waerden-Bortoletti connection of M (Chen, 1973).

We denote R and $\overline{R}$ the curvature tensors associated with ∇ and D respectively;

$$R(X_i, X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{[X_i,X_j]}X_k, \quad (2.6)$$

$$R^{\perp}(X_i, X_j)N_\alpha = h(X_i, A_{N_\alpha}X_j) - h(X_j, A_{N_\alpha}X_i). \quad (2.7)$$

The **equation of Gauss and Ricci** are given respectively by

$$\langle R(X_i, X_j)X_k, X_l\rangle = \langle h(X_i, X_l), h(X_j, X_k)\rangle - \langle h(X_i, X_k), h(X_j, X_l)\rangle, \quad (2.8)$$

$$\left\langle R^{\perp}(X_i, X_j)N_\alpha, N_\beta\right\rangle = \left\langle [A_{N_\alpha}, A_{N_\beta}]X_i, X_j\right\rangle \quad (2.9)$$

for the vector fields X_i, X_j, X_k tangent to M and N_α, N_β normal to M (Chen, 1973).

Let us $X_i \wedge X_j$ denote the endomorphism $X_k \longrightarrow \langle X_j, X_k \rangle X_i - \langle X_i, X_k \rangle X_j$. Then the curvature tensor R of M is given by the equation

$$R(X_i, X_j)X_k = \sum_{\alpha=1}^{n-2} \left(A_{N_\alpha} X_i \wedge A_{N_\alpha} X_j \right) X_k .$$

It is easy to show that

$$R(X_i, X_j)X_k = K \left(X_i \wedge X_j \right) X_k . \tag{2.10}$$

where K is the **Gaussian curvature** of M defined by

$$K = \langle h(X_1, X_1), h(X_2, X_2) \rangle - \| h(X_1, X_2) \|^2 \tag{2.11}$$

(see, Guadalupe and Rodriguez, 1983).

The **normal curvature** K_N of M is defined by (see, Decruyenaere et. al, 1993)

$$K_N = \left\{ \sum_{1=\alpha<\beta}^{n-2} \left\langle R^{\perp}(X_1, X_2) N_\alpha, N_\beta \right\rangle^2 \right\}^{1/2}. \qquad (2.12)$$

We observe that the normal connection D of M is flat if and only if $K_N = 0$, and by a result of Cartan, this equivalent to the diagonalisability of all shape operators A_{N_α} of M, M is of flat normal connection in $\mathbb{E}^n$.
Further, the **mean curvature vector** $\overrightarrow{H}$ of M is defined by

$$\overrightarrow{H} = \frac{1}{2} \sum_{\alpha=1}^{n-2} tr(A_{N_\alpha}) N_\alpha. \qquad (2.13)$$

3. Semiparallel Surfaces

Let M a smooth surface in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\overline{\nabla}$ be the connection of Vander Waerden-Bortoletti of M. Denote the tensors $\overline{\nabla}$ by $\overline{R}$. Then the product tensor $\overline{R} \cdot h$ of the curvature tensor $\overline{R}$ with the second fundamental form h is defined by

$$\begin{aligned}(\overline{R}(X_i, X_j) \cdot h)(X_k, X_l) &= \overline{\nabla}_{X_i}(\overline{\nabla}_{X_j} h(X_k, X_l)) - \overline{\nabla}_{X_j}(\overline{\nabla}_{X_i} h(X_k, X_l)) \\ &\quad - \overline{\nabla}_{[X_i, X_j]} h(X_k, X_l)\end{aligned}$$

for all X_i, X_j, X_k, X_l tangent to M.

3. Semiparallel Surfaces

The surface M is said to be **semi-parallel** if $\overline{R} \cdot h = 0$, i.e. $\overline{R}(X_i, X_j) \cdot h = 0$ ((Deprez, 1985), (Lumiste, 1988), (Deszcz, 1992), (Özgür et. all, 2002)). It is easy to see that

$$\begin{aligned}(\overline{R}(X_i, X_j) \cdot h)(X_k, X_l) &= R^{\perp}(X_i, X_j)h(X_k, X_l) \\ &\quad - h(R(X_i, X_j)X_k, X_l) - h(X_k, R(X_i, X_j)X_l).\end{aligned} \tag{3.1}$$

First, we sketched the proof of the following result.

Lemma (Deprez, 1985)

Let $M \subset \mathbb{E}^n$ a smooth surface given with the patch $X(u,v)$. Then the following equalities are hold;

$$\begin{aligned}
(\overline{R}(X_1,X_2)\cdot h)(X_1,X_1) &= \left(\sum_{\alpha=1}^{n-2} h_{11}^{\alpha}(h_{22}^{\alpha}-h_{11}^{\alpha})+2K\right)h(X_1,X_2) \\
&\quad + \sum_{\alpha=1}^{n-2} h_{11}^{\alpha}h_{12}^{\alpha}(h(X_1,X_1)-h(X_2,X_2)), \\
(\overline{R}(X_1,X_2)\cdot h)(X_1,X_2) &= \left(\sum_{\alpha=1}^{n-2} h_{12}^{\alpha}(h_{22}^{\alpha}-h_{11}^{\alpha})\right)h(X_1,X_2) \qquad (3.2) \\
&\quad +\left(\sum_{\alpha=1}^{n-2} h_{12}^{\alpha}h_{12}^{\alpha}-K\right)(h(X_1,X_1)-h(X_2,X_2)),
\end{aligned}$$

Lemma (Cont.)

$$\begin{aligned}(\overline{R}(X_1, X_2) \cdot h)(X_2, X_2) &= \left(\sum_{\alpha=1}^{n-2} h_{22}^{\alpha}(h_{22}^{\alpha} - h_{11}^{\alpha}) - 2K\right) h(X_1, X_2) \\ &+ \sum_{\alpha=1}^{n-2} h_{22}^{\alpha} h_{12}^{\alpha}(h(X_1, X_1) - h(X_2, X_2)).\end{aligned}$$

Proof.

Substituting (2.3) and (2.2) into (2.7) we get

$$\begin{aligned} R^{\perp}(X_1, X_2)N_{\alpha} &= h^{\alpha}_{12}(h(X_1, X_1) - h(X_2, X_2)) \\ &+ (h^{\alpha}_{22} - h^{\alpha}_{11})h(X_1, X_2). \end{aligned} \tag{3.3}$$

Further, by the use of (2.10) we get

$$\begin{aligned} R(X_1, X_2)X_1 &= -KX_2 \\ R(X_1, X_2)X_2 &= KX_1. \end{aligned} \tag{3.4}$$

So, substituting (3.3) and (3.4) into (3.1) we get the result. □

Semi-parallel surfaces in $\mathbb{E}^n$ are classified by J. Deprez (Deprez, 1985):

Theorem

Let M a surface in n-dimensional Euclidean space $\mathbb{E}^n$. Then M is semi-parallel if and only if locally;
i) M is equivalent to a 2-sphere, or
ii) M has trivial normal connection, or
iii) M is an isotropic surface in $\mathbb{E}^5 \subset \mathbb{E}^n$ satisfying $\|H\|^2 = 3K$.

4.1. Semiparallel tensor product surfaces in E4

In the following section, we will consider the tensor product immersions, actually surfaces in $\mathbb{E}^4$, which are obtained from two Euclidean plane curves. We recall definitions and results of (Decruyenaere et. all, 1993).

Let $c_1 : \mathbb{R} \to \mathbb{E}^2$ and $c_2 : \mathbb{R} \to \mathbb{E}^2$ be two Euclidean curves. Put $c_1(t) = (\gamma(t), \delta(t))$ and $c_2(s) = (\alpha(s), \beta(s))$. Then their **tensor product surface** is given by patch

$$f = c_1 \otimes c_2 : \mathbb{R}^2 \to \mathbb{E}^4$$

$$f(t,s) = (\alpha(s)\gamma(t), \beta(s)\gamma(t), \alpha(s)\delta(t), \beta(s)\delta(t)). \qquad (4.1.1)$$

(see (Mihai et. all, 1994-1995), (Decruyenaere et. all, 1994), (Arslan and Murathan, 1994)).

If we take c_1 as an unit plane circle centered at 0 and $c_2(s) = (\alpha(s), \beta(s))$ is an Euclidean plane curve. Then the surface patch becomes

$$M: \quad f(t,s) = (\alpha(s)\cos t, \beta(s)\cos t, \alpha(s)\sin t, \beta(s)\sin t). \qquad (4.1.2)$$

An orthonormal tangent basis and normal space of M is given by

$$\begin{aligned}
X_1 &= \frac{1}{\|c_2\|}\frac{\partial f}{\partial t}, X_2 = \frac{1}{\|c_2'\|}\frac{\partial f}{\partial s} \\
N_1 &= \frac{1}{\|c_2'\|}(-\beta'(s)\cos t, \beta'(s)\cot s, \alpha'(s)\sin t, -\alpha'(s)\ \sin t), \\
N_2 &= \frac{1}{\|c_2\|}(-\beta(s)\sin t, \beta(s)\sin t, \alpha(s)\cos t, -\alpha(s)\cos t).
\end{aligned}$$

By covariant differentiation with respect to X_1 and X_2 a straightforward calculation gives

$$\begin{aligned} \tilde{\nabla}_{X_1} X_1 &= -a(s)X_2 + b(s)N_1, \\ \tilde{\nabla}_{X_2} X_2 &= c(s)N_1, \\ \tilde{\nabla}_{X_2} X_1 &= b(s)N_2, \\ \tilde{\nabla}_{X_1} X_2 &= a(s)X_1 - b(s)N_2, \end{aligned} \tag{4.1.3}$$

and

$$\begin{aligned} \tilde{\nabla}_{X_1} N_1 &= -b(s)X_1 - a(s)N_2, \\ \tilde{\nabla}_{X_1} N_2 &= b(s)X_2 + a(s)N_1, \\ \tilde{\nabla}_{X_2} N_1 &= -c(s)X_2, \\ \tilde{\nabla}_{X_2} N_2 &= -b(s)X_1, \end{aligned} \tag{4.1.4}$$

where

$$\begin{aligned} a(s) &= \frac{\alpha(s)\alpha'(s)+\beta(s)\beta'(s)}{\|c_2(s)\|^2\|c_2'\|}, \\ b(s) &= \frac{\alpha(s)\beta'(s)-\beta(s)\alpha'(s)}{\|c_2(s)\|^2\|c_2'\|}, \\ c(s) &= \frac{\alpha'(s)\beta''(s)-\alpha''(s)\beta'(s)}{\|c_2'\|^3}. \end{aligned} \tag{4.1.5}$$

are the differentiable functions.

By the use of (4.1.4) with (2.1) we get the following result.

Remark

We have suppose that c_2 is not a straight line passing through the origin. In other case M is a plane (Guadalupe and Rodriguez, 1983).

Lemma

Let $f = c_1 \otimes c_2$ be tensor product immersion of a plane circle c_1 centered at 0 with any Euclidean planar curve $c_2(s) = (\alpha(s), \beta(s))$ then the shape operator matrices are

$$A_{N_1} = \begin{bmatrix} b(s) & 0 \\ 0 & c(s) \end{bmatrix}, \; A_{N_2} = \begin{bmatrix} 0 & -b(s) \\ -b(s) & 0 \end{bmatrix}. \quad (4.1.6)$$

Thus by the use of (2.7) together with (2.11) and (2.12) we get the following result.

Proposition

Let M a tensor product surface given with the surface patch (4.1.2). Then the Gaussian curvature K coincides with the normal curvature K_N of M. That is ;

$$K = K_N = b(s)\left(c(s) - b(s)\right). \tag{4.1.7}$$

By the use of (4.1.5) with (4.1.7) we get the following result.

Corollary

Let M a tensor product surface given with the surface patch (4.1.2). If M has vanishing Gaussian curvature then c_2 is a logarithmic spiral given with the parametrization

$$\alpha(s) = e^{\lambda s} \cos s, \beta(s) = e^{\lambda s} \sin s$$

Theorem (Bulca and Arslan, 2014b)

Let M a tensor product surface in $\mathbb{E}^4$ given with the surface patch (4.1.2). If M is semi-parallel then it has flat normal connection in $\mathbb{E}^4$.

Proof. Let M be a tensor product surface in $\mathbb{E}^4$ given with the patch (4.1.2). Then by the use of (4.1.3) with (4.1.6) we get

$$\begin{aligned} h_{11}^1 &= b(s), h_{12}^1 = h_{21}^1 = 0, h_{22}^1 = c(s), \\ h_{11}^2 &= 0, h_{12}^2 = h_{21}^2 = -b(s), h_{22}^2 = 0. \end{aligned} \tag{4.1.8}$$

and

$$\begin{aligned} h(X_1, X_2) &= -b(s)N_2 \\ h(X_1, X_1) - h(X_2, X_2) &= (b(s) - c(s))\, N_1. \end{aligned} \tag{4.1.9}$$

Proof. [Cont] Further, substituting (4.1.8) and (4.1.9) into (3.2) and after some computation one can get

$$
\begin{aligned}
(\overline{R}(X_1,X_2)\cdot h)(X_1,X_1) &= -b(s)\left(b(s)\left(c(s)-b(s)\right)+2K\right)N_2 \\
(\overline{R}(X_1,X_2)\cdot h)(X_1,X_2) &= \left(b^2(s)-K\right)\left(b(s)-c(s)\right)N_1 \\
(\overline{R}(X_1,X_2)\cdot h)(X_2,X_2) &= -b(s)\left(c(s)\left(c(s)-b(s)\right)-2K\right)N_2
\end{aligned}
$$

Suppose that, M is semi-parallel then by definition $(\overline{R}(X_1,X_2)\cdot h)(X_i,X_j)=0$, $(1\leq i,j\leq 2)$. So, we get

$$
\begin{aligned}
b(s)\left(b(s)\left(c(s)-b(s)\right)+2K\right) &= 0, \\
\left(b^2(s)-K\right)\left(b(s)-c(s)\right) &= 0, \\
b(s)\left(c(s)\left(c(s)-b(s)\right)-2K\right) &= 0.
\end{aligned}
\tag{4.1.10}
$$

Proof [Cont.] So, substituting $K = b(s)\left(c(s) - b(s)\right)$ into previous equation we obtain

$$\begin{aligned} b^2(s)\left(c(s) - b(s)\right) &= 0, \\ b(s)\left(b(s) - c(s)\right)\left(2b(s) - c(s)\right) &= 0, \\ b(s)\left(c(s) - b(s)\right)\left(2b(s) - c(s)\right) &= 0, \end{aligned} \qquad (4.1.11)$$

So, two possible cases occur; either $b(s) = 0$ or $b(s) = c(s)$. For the first case c_2 is a straight line passing through the origin and the surface M becomes a plane. So we don't consider this case. Hence, $b(s) = c(s)$ which means that $R^{\perp} = 0$ by (3.3) and (4.1.8). This is equivalent to say that M has vanishing normal curvature K_N. So, M has flat normal connection in $\mathbb{E}^4$. □

4.2. Semiparallel Vranceanu surfaces in E4

Rotation surfaces were studied in (Vranceanu, 1977) by Vranceanu as surfaces in $\mathbb{E}^4$ which are defined by the following parametrization;

$$\begin{aligned} X(u,v) &= (r(v)\cos v \cos u, r(v)\cos v \sin u, \\ &\quad r(v)\sin v \cos u, r(v)\sin v \sin u) \end{aligned} \tag{4.2.1}$$

where $r(v)$ is a real valued non-zero function.

We choose a moving frame $\{X_1, X_2, N_1, N_2\}$ such that X_1, X_2 are tangent to M and N_1, N_2 are normal to M as given the following (see (Yoon, 2001)):

$$\begin{aligned}
X_1 &= \frac{\partial}{r(v)\partial u} = (-\cos v \sin u, \cos v \cos u, -\sin v \sin u, \sin v \cos u), \\
X_2 &= \frac{\partial}{A\partial v} = \frac{1}{A}(B(v)\cos u, B(v)\sin u, C(v)\cos u, C(v)\sin u), \\
N_1 &= \frac{1}{A}(-C(v)\cos u, -C(v)\sin u, B(v)\cos u, B(v)\sin u), \\
N_2 &= (-\sin v \sin u, \sin v \cos u, \cos v \sin u, -\cos v \cos u)
\end{aligned}$$

where

$$
\begin{aligned}
A(v) &= \sqrt{r^2(v) + (r')^2(v)}, \\
B(v) &= r'(v)\cos v - r(v)\sin v, \\
C(v) &= r'(v)\sin v + r(v)\cos v.
\end{aligned}
$$

Furthermore, by covariant differentiation with respect to X_1 and X_2 a straightforward calculation gives:

$$\begin{aligned}
\widetilde{\nabla}_{X_1} X_1 &= -a(v)k(v)X_2 + a(v)N_1, \\
\widetilde{\nabla}_{X_2} X_2 &= b(v)N_1, \\
\widetilde{\nabla}_{X_2} X_1 &= -a(v)N_2,
\end{aligned} \tag{4.2.2}$$

where

$$\begin{aligned} k(v) &= \frac{r'(v)}{r(v)}, \\ a(v) &= \frac{1}{\sqrt{r^2(v)+(r')^2(v)}}, \\ b(v) &= \frac{2(r'(v))^2 - r(v)r''(v) + r^2(v)}{(r^2(v)+(r')^2(v))^{3/2}} \end{aligned} \tag{4.2.3}$$

are differentiable functions.

Thus by the use of (2.7) together with (2.11) and (2.12) we get the following result.

Proposition

Let M a Vranceanu surface given with the surface patch (4.2.1). Then the Gaussian curvature K of M is ;

$$K = K_N = a(v)b(v) - a^2(v). \tag{4.2.4}$$

Corollary (Bulca and Arslan, 2014a)

Let M a Vranceanu surface given with the surface patch (4.2.1). If M is semi-parallel then M is a flat surface satisfying $r(v) = c_1 e^{c_2 v}$.

Proof.

Suppose the Vranceanu surface M is semi-parallel then by the use of (3.2) with (4.2.2) we get

$$\begin{aligned}(\overline{R}(X_1,X_2)\cdot h)(X_1,X_1) &= \left(3a^2(v)\left(a(v)\text{-}b(v)\right)\right)N_2\\ (\overline{R}(X_1,X_2)\cdot h)(X_1,X_2) &= \left(a(v)\left(a(v)\text{-}b(v)\right)\left(2a(v)\text{-}b(v)\right)\right)N_1\\ (\overline{R}(X_1,X_2)\cdot h)(X_2,X_2) &= a(v)\left(3a(v)b(v)\text{-}2a(v)^2\text{-}b(v)^2\right)N_2.\end{aligned}$$

Suppose that, M is semi-parallel then by (3.1) $(\overline{R}(X_1,X_2)\cdot h)(X_i,X_j)=0$, $(1\leq i,j\leq 2)$. Which implies that $a(v)-b(v)=0$. So, by (4.2.4) $K=K_N=0$. Further, from (4.2.3) we get the result. □

4.3. Semiparallel Meridian surfaces in E4

In this section, we will consider the meridian surfaces in $\mathbb{E}^4$ which is first defined by Ganchev and Milousheva (Ganchev and Milousheva, 2010). The meridian surfaces are one-parameter systems of meridians of the standard rotational hypersurface in $\mathbb{E}^4$. Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in $\mathbb{E}^4$, and $S^2(1)$ be a 2-dimensional sphere in $\mathbb{E}^3 = span\{e_1, e_2, e_3\}$, centered at the origin O. We consider a smooth curve $C : r = r(v),\ v \in J,\ J \subset \mathbb{R}$ on $S^2(1)$, parameterized by the arc-length $(\|(r')^2(v)\| = 1)$. We denote $t = r'$ and consider the moving frame field $\{t(v), n(v), r(v)\}$ of the curve C on $S^2(1)$.

With respect to this orthonormal frame field the following Frenet formulas hold good:

$$r' = t;$$
$$t' = \kappa\, n - r;$$
$$n' = -\kappa\, t,$$

where κ is the spherical curvature of C.

Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that

$$(f')^2(u) + (g')^2(u) = 1, \ u \in I. \tag{4.3.1}$$

In (Ganchev and Milousheva, 2010) Ganchev and Milousheva constructed a surface M^2 in $\mathbb{E}^4$ in the following way:

$$M^2 : X(u, v) = f(u)\, r(v) + g(u)\, e_4, \quad u \in I, \, v \in J. \tag{4.3.2}$$

The surface M^2 lies on the rotational hypersurface M^3 in $\mathbb{E}^4$ obtained by the rotation of the meridian curve $\alpha : u \longrightarrow (f(u), g(u))$ around the Oe_4-axis in $\mathbb{E}^4$. Since M^2 consists of meridians of M^3, we call M^2 a **meridian surface** (Ganchev and Milousheva, 2010). If we denote by κ_α the curvature of meridian curve α, i.e.,

$$\kappa_\alpha = f'(u)g''(u) - f''(u)g(u) = \frac{-f''(u)}{\sqrt{1 - f'^2(u)}}. \qquad (4.3.3)$$

We consider the following orthonormal moving frame fields, X_1, X_2, N_1, N_2 on the meridian surface M^2 such that X_1, X_2 are tangent to M^2 and N_1, N_2 are normal to M^2. The tangent and normal space of M^2 is spanned by the vector fields:

$$X_1 = \frac{\partial X}{\partial u}, \quad X_2 = \frac{1}{f}\frac{\partial X}{\partial v},$$

$$N_1 = n(v), \quad N_2 = -g'(u)\,r(v) + f'(u)\,e_4.$$

By a direct computation we have the components of the second fundamental forms as;

$$\begin{aligned} h^1_{11} &= h^1_{12} = h^1_{21} = 0, \quad h^1_{22} = \tfrac{\kappa}{f}, \\ h^2_{11} &= \kappa_\alpha \quad h^2_{12} = h^2_{21} = 0, \quad h^2_{22} = \tfrac{g'}{f}. \end{aligned} \tag{4.3.4}$$

Lemma

Let M be meridian surface in $\mathbb{E}^4$ given with the parametrization (4.3.2) then the shape operator matrices are

$$A_{N_1} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\kappa}{f} \end{bmatrix}, A_{N_2} = \begin{bmatrix} \kappa_\alpha & 0 \\ 0 & \frac{g'}{f} \end{bmatrix}$$

and hence $K = \frac{\kappa_\alpha g'}{f}$ and $K_N = 0$, which implies that the meridian surface M^2 is totally umbilical surface in $\mathbb{E}^4$.

In (Ganchev and Milousheva, 2009) Ganchev and Milousheva constructed three **main classes of meridian surfaces**:
I. $\varkappa = 0$; i.e. the curve C is a great circle on $S^2(1)$. In this case $N_1 = const.$ and M^2 is a planar surface lying in the constant 3-dimensional space spanned by $\{X_1, X_2, N_2\}$. Particularly, if in addition $\varkappa_\alpha = 0$, i.e. the meridian curve is a part of a straight line, then M^2 is a developable surface in the 3-dimensional space spanned by $\{X_1, X_2, N_2\}$.
II. $\varkappa_\alpha = 0$, i.e. the meridian curve is a part of a straight line. In such case M^2 is a developable ruled surface. If in addition $\varkappa = const.$, i.e. C is a circle on $S^2(1)$, then M^2 is a developable ruled surface in a 3-dimensional space. If $\varkappa \neq const.$,i.e. C is not a circle on $S^2(1)$, then M^2 is a developable ruled surface in $\mathbb{E}^4$.
III. $\varkappa_\alpha \varkappa \neq 0$, i.e. C is not a circle on $S^2(1)$ and α is not a straight line. In this general case the parametric lines of M^2 given by (4.3.2) are orthogonal and asymptotic.

We proved the following Theorem (Bulca and Arslan, 2015)

Theorem

Let M^2 be a meridian surface in $\mathbb{E}^4$ given with the parametrization (4.3.2). Then M^2 is semi-parallel if and only if one of the following holds:

i) *M^2 is a developable ruled surface in $\mathbb{E}^3$ or $\mathbb{E}^4$ which considered in Case II of the classification above,*

ii) *the curve C is a circle on $S^2(1)$ with non-zero constant spherical curvature and the meridian curve is determined by*

$$\begin{aligned} f(u) &= \pm\sqrt{u^2 - 2au + 2b}; \\ g(u) &= -\sqrt{2b - a^2}\ln\left(u - a - \sqrt{u^2 - 2au + 2b}\right), \end{aligned}$$

where $a = const$, $b = const$. In this case M^2 is a planar surface lying in 3-dimensional space spanned by $\{X_1, X_2, N_2\}$.

Proof. Let M^2 be a meridian surface in $\mathbb{E}^4$ given with the parametrization (4.3.2). Then by the use of (2.3) with (4.3.4) we see that

$$\begin{aligned} h(X_1, X_2) &= 0, \\ h(X_1, X_1) - h(X_2, X_2) &= -\frac{\kappa}{f} N_1 + \left(\kappa_\alpha - \frac{g'}{f} \right) N_2. \end{aligned} \tag{4.3.5}$$

Further, substituting (4.3.5) and (4.3.4) into (3.2) and after some computation one can get

$$\begin{aligned} (\overline{R}(X_1, X_2) \cdot h)(X_1, X_1) &= 0, \\ (\overline{R}(X_1, X_2) \cdot h)(X_1, X_2) &= -K \left(-\frac{\kappa}{f} N_1 + \left(\kappa_\alpha - \frac{g'}{f} \right) N_2 \right), \\ (\overline{R}(X_1, X_2) \cdot h)(X_2, X_2) &= 0. \end{aligned}$$

Proof [Cont.] Suppose that, M^2 is semi-parallel then by definition

$$(\overline{R}(X_1, X_2) \cdot h)(X_i, X_j) = 0, \ 1 \leq i, j \leq 2,$$

is satisfied. So, we get

$$K\left(-\frac{\kappa}{f}N_1 + \left(\kappa_\alpha - \frac{g'}{f}\right)N_2\right) = 0.$$

Hence, two possible cases occur; $K = 0$ or $\kappa = 0$ and $\kappa_\alpha - \frac{g'}{f} = 0$.

Proof [Cont.] For the first case $\kappa_\alpha = 0$, i.e. the meridian curve is a part of a straight line. In such case M^2 is a developable ruled surface given in the Case II. For the second case $\kappa = 0$ means that the curve c is a great circle on $S^2(1)$. In this case M^2 lies in the 3-dimensional space spanned by $\{X_1, X_2, N_2\}$. Further, using (4.3.3) the equation $\kappa_\alpha - \frac{g'}{f} = 0$ can be rewritten in the form

$$f(u)f''(u) - (f'(u))^2 + 1 = 0,$$

which has the solution

$$f(u) = \pm\sqrt{u^2 - 2au + 2b}. \qquad (4.3.6)$$

Proof [Cont.] Consequently, by substituting (4.3.6) into (4.3.1) one can get

$$g(u) = -\sqrt{2b - a^2} \ln \left(u - a - \sqrt{u^2 - 2au + 2b} \right).$$

This completes the proof of the theorem.

5.References

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****THANK YOU****