Explicit Solving of the Natural Partial Differential Equations of Minimal Surfaces

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1. Weingarten Surfaces in $\mathbb{R}^3$

[25] Weingarten J., *Über die Oberflächen, für welche einer der beiden Hauptkrümmungshalbmesser eine Funktion des anderen ist*, J. reine angew. Math., **62** (1863), 160-173.

[26] Weingarten J., *Über eine Eigenschaft der Flächen, bei denen der eine Hauptkrümmungsradius eine Funktion des anderen ist*, J. reine angew. Math., **103** (1888), 184.

A surface S with principal curvatures ν_1 and ν_2 is *a Weingarten surface* (W-surface) if there exists a function ν on S and two functions (Weingarten functions) f, g of one variable, such that

$$\nu_1 = f(\nu), \quad \nu_2 = g(\nu).$$

We proved in

[9] Ganchev G., Mihova V., *On the invariant theory of Weingarten surfaces in Euclidean space*. J. Phys. A: Math. Theor. **43** (2010) 405210-405236.

that any W-surface admits locally special principal parameters - *natural principal parameters*. With respect to these natural principal parameters the functions

$$\sqrt{E}\,\mathrm{exp}\left(\int \frac{f' d\nu}{f-g}\right), \ \sqrt{G}\,\mathrm{exp}\left(\int \frac{g' d\nu}{g-f}\right)$$

are constants, E, G being the coefficients of the first fundamental form on a W-surface.

With respect to natural principal parameters any W-surface S with Weingarten functions f, g is determined uniquely up to motions by the geometric function ν, which satisfies a non-linear partial differential equation - the *natural PDE* of the surface S [9]. This result solves the Lund-Regge reduction problem (Fokas - Gelfand [5], Lund - Regge [14], Sym [18]) for W-surfaces in Euclidean space.

Parallel surfaces - a family of associated surfaces

Let $S: \; z = z(u,v), \; (u,v) \in \mathcal{D}$ be a surface, parameterized by principal parameters and $l(u,v)$ be the unit normal vector field of S. The parallel surfaces of S are given by

$$\bar{S}(a): \; \bar{z}(u,v) = z(u,v) + a\, l(u,v), \quad a = \text{const} \neq 0, \quad (u,v) \in \mathcal{D}.$$

[10] Ganchev G., Mihova V., *Natural PDE's of Weingarten surfaces with linear relation between their curvatures in Euclidean Space.* (to appear)

In Proposition 3.1 in [10] we prove that

The natural principal parameters of a given W-surface S are natural principal parameters for all surfaces $\{\bar{S}(a)\}$, which are parallel to S.

Theorem 3.2 [10] states that

The natural PDE of a given W-surface S is the natural PDE of any surface $\bar{S}(a)$, $a = \text{const} \neq 0$, which is parallel to S.

2. On the Natural PDE's of Minimal Surfaces in $\mathbb{R}^3$

Canonical Weierstrass principal representation of minimal surfaces *Let* $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y),\ (x, y) \in \mathcal{D} \subset \mathbb{C}$ *be a minimal surface, free of flat points, parameterized by canonical principal parameters. If the normal curvature function* $\nu(x, y)$ *of* $\mathcal{M}$ *is positive, then it has locally a representation of the type*

$$\mathbf{z}: \quad \begin{aligned} z_1 &= \mathfrak{Re}\left(\frac{1}{2}\int_{\mathbf{z_0}}^{\mathbf{z}} \frac{w^2 - 1}{w'}\, dz\right), \\ z_2 &= \mathfrak{Re}\left(-\frac{i}{2}\int_{\mathbf{z_0}}^{\mathbf{z}} \frac{w^2 + 1}{w'}\, dz\right), \\ z_3 &= \mathfrak{Re}\left(-\int_{\mathbf{z_0}}^{\mathbf{z}} \frac{w}{w'}\, dz\right), \end{aligned}$$

where $w = w(z),\ z = x + iy$ *is a holomorphic function in* $\mathbb{C}$ *satisfying the condition* $w' \neq 0$*.*

We show that any minimal surface free of flat points can be endowed locally by canonical asymptotic parameters. We prove a theorem (Canonical Weierstrass asymptotic representation), which is the "asymptotic" analogue to the above statement.

Theorem B. (Explicit solving of the natural PDE of minimal surfaces) *Any solution $\nu > 0$ of the natural partial differential equation of minimal surfaces*

$$\Delta \ln \nu + 2\nu = 0 \tag{1}$$

locally is given by the formula

$$\nu = \frac{4(u_x^2 + u_y^2)}{(u^2 + v^2 + 1)^2}, \quad u_x^2 + u_y^2 > 0, \tag{2}$$

where $w = u(x, y) + iv(x, y)$ is a holomorphic function in $\mathbb{C}$.

Conversely, any function $\nu(x, y)$ of the type (2) is a solution to (1).

Question: When two holomorphic functions generate one and the same solution to (1)?

O. Kassabov [23] obtained the following result:

Theorem. *Let* w *and* $\hat{w}$ *be two holomorphic functions generating one and the same solution* ν *of the natural equation* (1)*. Then*

$$\hat{w} = \frac{-\bar{b} + \bar{a}\, w}{a + b\, w}, \quad a = \text{const},\ b = \text{const},\ |a|^2 + |b|^2 = 1. \tag{3}$$

Conversely, any two holomorphic functions related by (3) *generate one and the same solution to* (1)*.*

Now, let $\mathcal{M}$ be a minimal surface, free of flat points parameterized by canonical principal parameters and $\mathcal{F}_1$ be the family of parametric lines $y =$ const. It follows from the Frenet type equations that the lines of curvature from this family are regular curves. These lines of curvature are plane curves if and only if the vector fields X, $\nabla_X X$ and $\nabla_X \nabla_X X$ are coplanar at any point, i.e.

$$(4) \qquad \lambda_{xy} - \lambda_x \lambda_y = 0 \qquad \Longleftrightarrow \qquad (e^{-\lambda})_{xy} = 0 \qquad \Longleftrightarrow \qquad \left(\frac{1}{\sqrt{\nu}}\right)_{xy} = 0.$$

The well known fact that if the one family of lines of curvature are plane curves, then the other family of lines of curvature are again plane curves now follows immediately from (4).

Thus, the minimal surfaces whose lines of curvature are plane curves are generated by the solutions $\nu(x, y) > 0$ of (1) satisfying the additional condition $\left(\frac{1}{\sqrt{\nu}}\right)_{xy} = 0.$

Let $\mathcal{M}$ be a minimal surface, parameterized by canonical asymptotic parameters. Then, the asymptotic lines from the family $\mathcal{F}_1 : \ y = \text{const}$ are generalized helices if and only if

$$\left(\frac{(\sqrt{\alpha})_y}{\alpha} \right)_x = 0 \qquad \Longleftrightarrow \qquad \left(\frac{1}{\sqrt{\alpha}} \right)_{xy} = 0.$$

Here, again it follows immediately that if the one family of asymptotic lines are generalized helices, then the other family of asymptotic lines are also generalized helices.

The principal lines of a minimal surface $\mathcal{M}$ are plane curves if and only if the asymptotic lines of the conjugate minimal surface $\overline{\mathcal{M}}$ are generalized helices.

Taking into account Theorem A, we formulate the classical result for the minimal surfaces with plane curvature lines in the following form:

Examples 2.1. [13] Up to similarities, the minimal surfaces whose principal lines are plane curves, are as follows:

(i) *The Enneper minimal surface*:

generating holomorphic function: $w = z$;

canonical principal representation:

$$
\mathcal{M}: \quad \begin{aligned} z_1 &= \frac{1}{6}(x^3 - 3xy^2) - \frac{x}{2}, \\ z_2 &= \frac{1}{6}(3x^2y - y^3) + \frac{y}{2}, \\ z_3 &= -\frac{1}{2}(x^2 - y^2). \end{aligned} \tag{2.1}
$$

The normal curvature function ν of S is the following:

$$\nu(x,y) = \frac{4}{[(x^2+y^2)+1]^2}\,.$$

(ii) *The Bonnet minimal surfaces*:

generating holomorphic function: $w = ie^k \tan\dfrac{z}{2},\ k = \text{const} \in \mathbb{R}$;

canonical principal representation:

$$(2.2) \quad \tilde{S}(k): \qquad \begin{aligned} z_1 &= b\cos x \sinh y - ay, \\ z_2 &= -a\sin x \cosh y + bx, \\ z_3 &= \cos x \cosh y, \end{aligned} \qquad a = \cosh k,\ b = \sinh k.$$

In the case $k = 0$ the surface $\tilde{S}(0)$ is the usual catenoid.

The normal curvature function ν of $\tilde{S}(k)$ is as follows:

$$\nu(x,y) = \frac{1}{(\,a\cosh y - b\cos x\,)^2}\,.$$

The minimal surfaces whose asymptotic lines are generalized helices are as follows.

Examples 2.2. All minimal surfaces whose asymptotic lines are generalized helices, up to similarity are the following:

(i) *The Enneper minimal surface* S .

The Enneper surface S (2.1) is congruent to its associated minimal surfaces. Hence, S is the unique minimal surface whose principal lines are plane curves and simultaneously its asymptotic lines are generalized helices.

(ii) *The conjugate surfaces to the Bonnet minimal surfaces*:

generating holomorphic function: $w = ie^k \tan \frac{e^{i\frac{\pi}{4}} z}{2},\ k = \text{const} \in \mathbb{R}$;

canonical principal representation:

$$\tilde{S}(k): \qquad \begin{aligned} z_1 &= a\,\frac{x-y}{\sqrt{2}} - b\sin\frac{x-y}{\sqrt{2}}\cosh\frac{x+y}{\sqrt{2}}, \\ z_2 &= b\,\frac{x+y}{\sqrt{2}} - a\cos\frac{x-y}{\sqrt{2}}\sinh\frac{x+y}{\sqrt{2}}, \\ z_3 &= -\sin\frac{x-y}{\sqrt{2}}\sinh\frac{x+y}{\sqrt{2}}, \end{aligned} \qquad a = \cosh k,\ b = \sinh k.$$

In the case $k = 0$ the surface $\tilde{S}(0)$ is the usual helicoid.

The normal curvature function ν of $\tilde{S}(k)$ is as follows:

$$\nu(x,y) = \frac{1}{\left(\,a\cosh\frac{x+y}{\sqrt{2}} - b\cos\frac{x-y}{\sqrt{2}}\,\right)^2}.$$

3. The Natural PDE's of Space-like Minimal Surfaces in $\mathbb{R}^3_1$

Theorem. (Explicit solving of the natural PDE of minimal space-like surfaces) *Any solution $\nu > 0$ of the natural partial differential equation of minimal space-like surfaces*

$$\Delta \ln \nu - 2\nu = 0 \tag{3.1}$$

locally is given by the formula

$$\nu = \frac{4(u_x^2 + u_y^2)}{(u^2 + v^2 - 1)^2}, \quad u_x^2 + u_y^2 > 0, \tag{3.2}$$

where $w = u(x,y) + iv(x,y)$ is a holomorphic function in $\mathbb{C}$.

Conversely, any function $\nu(x,y)$ of the type (3.2) is a solution to (3.1).

4. The Natural PDE's of Time-like Minimal Surfaces in $\mathbb{R}^3_1$

Theorem. (Explicit solving of the natural PDE of minimal time-like surfaces) *Any solution $\nu = n(x, y) > 0$ of the natural partial differential equation of minimal time-like surfaces*

$$(\ln \nu)_{xx} - (\ln \nu)_{yy} = 2\nu \tag{4.1}$$

locally is given by the formula

$$\nu = \frac{4(u_x^2 - u_y^2)}{(u^2 - v^2 - 1)^2}, \quad u^2 - v^2 < 1, \quad u_x^2 - u_y^2 > 0, \tag{4.2}$$

where $w = u(x, y) + jv(x, y)$ is a holomorphic function in the Minkowski plane $\mathbb{R}^2_1$ considered as the algebra of double numbers.

Conversely, any function $\nu(x, y)$ of the type (4.2) is a solution to (4.1).

5. The System of natural PDE's of a minimal surface in $\mathbb{R}^4$

By *minimal non-superconformal surface* in the four-dimensional Euclidean space $\mathbb{R}^4$ we mean a surface $(\mathcal{M}, \mathrm{z})$, $\mathrm{z} : \mathcal{M} \to \mathbb{R}^4$ with zero mean curvature, whose ellipse of curvature is not a circle. Studying minimal surfaces in $\mathbb{R}^4$, T. Itoh [12] proved that any minimal non-superconformal surface admits locally special isothermal parameters (u, v). These parameters are characterized by the following conditions:

$$\begin{array}{l} \mathrm{z}_u^2 = \mathrm{z}_v^2 \\ \mathrm{z}_u \cdot \mathrm{z}_v = 0 \\ \sigma^2(\mathrm{z}_u, \mathrm{z}_u) - \sigma^2(\mathrm{z}_u, \mathrm{z}_v) = 1 \\ \sigma(\mathrm{z}_u, \mathrm{z}_u) \cdot \sigma(\mathrm{z}_u, \mathrm{z}_v) = 0 \ , \end{array}$$

where $\sigma(X, Y)$ denotes the second fundamental form of $\mathcal{M}$.

It can be proved that these parameters are uniquely determined up to renumbering, signs and additive constants. Thus if $(\tilde{u}, \tilde{v})$ is another pair of such parameters we have:

$$\begin{array}{l} \tilde{u} = \varepsilon u + a \\ \tilde{v} = \delta v + b \end{array} \quad \text{or} \quad \begin{array}{l} \tilde{u} = \varepsilon v + a \\ \tilde{v} = \delta u + b \end{array}, \quad \text{where} \quad \begin{array}{l} \varepsilon, \delta = \pm 1 \\ a, b = \text{const} \end{array}.$$

Further we shall call these parameters *canonical parameters* of the minimal non-superconformal surface $\mathcal{M}$ in $\mathbb{R}^4$.

On the base of the existence of canonical parameters, R. de Azevero Tribuzy and I. Guadalupe [24] proved the following:

Theorem. The Gauss curvature K and the curvature of the normal connection $\varkappa$ (the normal curvature) of a minimal non-superconformal surface in $\mathbb{R}^4$, parameterized by canonical parameters, satisfy the following system of partial differential equations:

$$(1) \qquad \begin{array}{rcl} (K^2 - \varkappa^2)^{\frac{1}{4}} \Delta \ln |\varkappa - K| & = & 2(2K - \varkappa) \\ (K^2 - \varkappa^2)^{\frac{1}{4}} \Delta \ln |\varkappa + K| & = & 2(2K + \varkappa)\ . \end{array}$$

Conversely, any solution $(K, \varkappa)$ to system (1) determines locally uniquely (up to a motion in $\mathbb{R}^4$) a minimal non-superconformal surface with Gauss curvature K and normal curvature $\varkappa$.

We call (1): *the system of natural PDE's* of minimal surfaces in $\mathbb{R}^4$ and our aim is to solve locally explicitly this system.

Remark. Introducing natural parameters on any minimal non-superconformal surface in $\mathbb{R}^4$ reduces the number of the invariants determining the surface to two: K and $\varkappa$. Further, these two invariants satisfy the system of two natural PDE's and determine the minimal non-superconformal surface uniquely up to a motion. It is clear that the number of the invariants and the number of the PDE's can not be reduced further. Therefore this solves the reduction problem of Lund-Regge for minimal non-superconformal surfaces in $\mathbb{R}^4$.

In [8] G. Ganchev and K. Kanchev proved the following

Theorem 1.(Explicit solving of the system of natural PDE's of minimal surfaces in $\mathbb{R}^4$) *Let K and $\varkappa$ be a solution to the system of natural PDE's* (1). *Then we have locally:*

$$\begin{aligned} K &= \frac{-8|g_1' g_2'|}{(|g_1|^2+1)(|g_2|^2+1)} \left(\frac{|g_1'|^2}{(|g_1|^2+1)^2} + \frac{|g_2'|^2}{(|g_2|^2+1)^2} \right) \\ \varkappa &= \frac{8|g_1' g_2'|}{(|g_1|^2+1)(|g_2|^2+1)} \left(\frac{|g_1'|^2}{(|g_1|^2+1)^2} - \frac{|g_2'|^2}{(|g_2|^2+1)^2} \right) \end{aligned} \tag{2}$$

for some pair of holomorphic functions (g_1, g_2) *in* $\mathbb{C}$ *with* $g_1' g_2' \neq 0$.

Conversely, for any such pair of holomorphic functions (g_1, g_2), *the functions K and $\varkappa$ given by* (2), *satisfy the system* (1).

Theorem 2. *Let* (g_1, g_2) *and* $(\widehat{g}_1, \widehat{g}_2)$ *be two pairs of holomorphic functions generating one and the same solution* $(K, \varkappa)$ *of the system* (1) *of natural PDE's of minimal surfaces in* $\mathbb{R}^4$. *Then:*

$$\widehat{g}_k = \frac{-\bar{b}_k + \bar{a}_k\, g_k}{a_k + b_k\, g_k}, \tag{3}$$

where $a_k = \text{const}$, $b_k = \text{const}$, $|a_k|^2 + |b_k|^2 = 1$; $(k = 1; 2)$.

Conversely, any two pairs of holomorphic functions related by the above formula, generate one and the same solution of (1).

Remark. The conditions for the constants a_k and b_k in **Theorem 2** mean that $\begin{pmatrix} \bar{a}_k & -\bar{b}_k \\ b_k & a_k \end{pmatrix}$ is a special unitary matrix in the group $\mathbf{SU}(2)$. Therefore we can state **Theorem 2** in the following equivalent form:

Theorem 2a. *Two pairs of holomorphic functions* (g_1, g_2) *and* $(\hat{g}_1, \hat{g}_2)$ *generate one and the same solution* $(K, \varkappa)$ *of the system* (1) *iff the pair* $(\hat{g}_1, \hat{g}_2)$ *is obtained from the pair* (g_1, g_2) *by linear fractional transformations with matrices in* $\mathbf{SU}(2)$.

In

G. Ganchev, V. Milousheva, *Timelike surfaces with zero mean curvature in Minkowski 4-space.* Israel Journal of Mathematics, 196 (2013), 1, 413-433

we gave the systems of natural PDE's of minimal surfaces in $\mathbb{R}^4$ and $\mathbb{R}^4_1$ in the following form

Minimal surfaces in $\mathbb{R}^4$

$$\Delta X = 2\, e^X \cosh Y$$
$$\Delta Y = 2\, e^X \sinh Y$$

Minimal space-like surfaces in $\mathbb{R}^4_1$

$$\Delta X = 2\, e^X \cos Y$$
$$\Delta Y = 2\, e^X \sin Y$$

Minimal time-like surfaces in $\mathbb{R}^4_1$

$$\Delta^h X = 2\, e^X \cos Y$$
$$\Delta^h Y = 2\, e^X \sin Y$$

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Thank you!