Extended harmonic mappings and Euler-Lagrange equations

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Euler-Lagrange equations

Let $\phi:(M,g)\to(N,h)$ be a smooth mapping, where (M,g) and (N,h) are Riemannian manifolds of dimension 2 and 3 with Riemannian metric g and h, respectively. Then we consider the following Lagrangian (density):

$$L_\phi = \frac{1}{2}\sum_{i,j=1}^{2}\sum_{\alpha,\beta=1}^{3} g^{ij}\,\partial_i\phi^\alpha\partial_j\phi^\beta h_{\alpha\beta}(\phi) - G(\phi), \qquad (1)$$

where $\phi^\alpha := y^\alpha\circ\phi$, $\alpha=1,2,3$. $(x^1,x^2),(y^1,y^2,y^3)$ are local coordinates systems on M,N. $\partial_1\phi^\alpha, \partial_2\phi^\alpha$ denote the partial derivatives $\frac{\partial}{\partial x^1}\phi^\alpha, \frac{\partial}{\partial x^2}\phi^\alpha$, respectively.

$$h(\phi) = \sum_{\alpha,\beta=1}^{3} h_{\alpha\beta}(\phi)(dy^\alpha)_\phi\otimes(dy^\beta)_\phi, \quad G(\phi)=G\circ\phi \ \ (G\in C^\infty(N)).$$

The generalized momenta p_α^i can be defined by

$$p_\alpha^i := \frac{\partial L_\phi}{\partial(\partial_i\phi^\alpha)}, \quad i=1,2,\ \alpha=1,2,3,$$

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where $\partial_i\phi^\alpha$ can be regarded as the components of tensor field [N]:

$$(d\phi)((t,x)) = \sum_{i=1}^{2}\sum_{\alpha=1}^{3} \partial_i\phi^\alpha (dx^i)_{(t,x)} \otimes \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi((t,x))}, \quad x^1 = t,\ x^2 = x.$$

Then we have the transformation formulas of $\partial_i\phi^\alpha$ and p^i_α :

$$\tilde{\partial}_j\tilde{\phi}^\alpha = \sum_{i=1}^{2}\sum_{\beta=1}^{3} \frac{\partial x^i}{\partial \tilde{x}^j}\left(\frac{\partial \tilde{y}^\alpha}{\partial y^\beta}(\phi)\right)\partial_i\phi^\beta, \qquad \tilde{p}^i_\alpha = \sum_{j=1}^{2}\sum_{\beta=1}^{3} \frac{\partial \tilde{x}^i}{\partial x^j}\left(\frac{\partial y^\beta}{\partial \tilde{y}^\alpha}(\phi)\right)p^j_\beta,$$

under the transformation of coordinates : $(x^1, x^2) \to (\tilde{x}^1, \tilde{x}^2)$ and $(y^1, y^2, y^3) \to (\tilde{y}^1, \tilde{y}^2, \tilde{y}^3)$. Hence p^i_α can be regarded as the components of tensor field:

$$p((t,x)) = \sum_{i=1}^{2}\sum_{\alpha=1}^{3} p^i_\alpha \left(\frac{\partial}{\partial x^i}\right)_{(t,x)} \otimes (dy^\alpha)_{\phi((t,x))}$$

Proposition 1. Assume that (M, g) is (R^2, g_0), where g_0 is the standard metric on R^2. Then, under the Lagrangian (1) of ϕ , the following (a) and (b)

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are equivalent:

(a) (Euler-Lagrange equations)

$$\sum_{i=1}^{2}\partial_i\, p^i_\alpha - \frac{\partial L_\phi}{\partial \phi^\alpha} = 0, \quad \alpha = 1,2,3. \qquad (2)$$

(b)

$$\tau_\phi = -\, grad_h G(\phi), \qquad (3)$$

where τ_ϕ stands for the tension field of ϕ and

$$grad_h G(\phi) = \sum_{\alpha,\beta=1}^{3} h^{\alpha\beta}(\phi)\Big(\Big(\frac{\partial}{\partial y^\alpha}\Big)_\phi G(\phi)\Big)\Big(\frac{\partial}{\partial y^\beta}\Big)_\phi.$$

Proof. $\sum_{i=1}^{2}\partial_i\, p^i_\gamma = \sum_{i=1}^{2}\sum_{\alpha=1}^{3}\partial_i(\partial_i\phi^\alpha h_{\alpha\gamma}(\phi))$

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$$=\sum_{i=1}^{2}\sum_{\alpha=1}^{3}\partial_i^2\phi^\alpha h_{\alpha\gamma}(\phi)+\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\partial_i\phi^\alpha\frac{\partial h_{\alpha\gamma}(\phi)}{\partial\phi^\beta}\partial_i\phi^\beta,$$

and

$$\frac{\partial L_\phi}{\partial\phi^\gamma}=\frac{1}{2}\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\partial_i\phi^\alpha\partial_i\phi^\beta\frac{\partial h_{\alpha\beta}(\phi)}{\partial\phi^\gamma}-\frac{\partial G(\phi)}{\partial\phi^\gamma}.$$

Then we have

$$\sum_{i=1}^{2}\partial_i\, p_\gamma^i-\frac{\partial L_\phi}{\partial\phi^\gamma}$$

$$=\sum_{i=1}^{2}\sum_{\alpha=1}^{3}\partial_i^2\phi^\alpha h_{\alpha\gamma}(\phi)+\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\left(\frac{\partial h_{\alpha\gamma}(\phi)}{\partial\phi^\beta}-\frac{1}{2}\frac{\partial h_{\alpha\beta}(\phi)}{\partial\phi^\gamma}\right)\partial_i\phi^\alpha\partial_i\phi^\beta+\frac{\partial G(\phi)}{\partial\phi^\gamma}.$$

On the other hand, we have

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$$\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\Gamma^{\mu}_{\alpha\beta}(\phi)\partial_i\phi^{\alpha}\partial_i\phi^{\beta}=\sum_{i=1}^{2}\sum_{\alpha,\beta,\gamma=1}^{3}h^{\mu\gamma}(\phi)\left(\frac{\partial h_{\alpha\gamma}(\phi)}{\partial\phi^{\beta}}-\frac{1}{2}\frac{\partial h_{\alpha\beta}(\phi)}{\partial\phi^{\gamma}}\right)\partial_i\phi^{\alpha}\partial_i\phi^{\beta},$$

where $\Gamma^{\mu}_{\alpha\beta}$ denotes the coefficients of Levi-Civita connection of (N,h).

Then we have

$$\sum_{\gamma=1}^{3}\left(\sum_{i=1}^{2}\partial_i p^i_{\gamma}-\frac{\partial L_{\phi}}{\partial\phi^{\gamma}}\right)h^{\mu\gamma}(\phi)=\sum_{i=1}^{2}\partial_i^2\phi^{\mu}+\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\Gamma^{\mu}_{\alpha\beta}(\phi)\partial_i\phi^{\alpha}\partial_i\phi^{\beta}+\sum_{\gamma=1}^{3}h^{\gamma\mu}(\phi)\frac{\partial G(\phi)}{\partial\phi^{\gamma}}.$$

Since

$$\tau_{\phi}=\sum_{\mu=1}^{3}\left(\sum_{i=1}^{2}\partial_i^2\phi^{\mu}+\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\Gamma^{\mu}_{\alpha\beta}(\phi)\partial_i\phi^{\alpha}\partial_i\phi^{\beta}\right)\left(\frac{\partial}{\partial y^{\mu}}\right)_{\phi},$$

and

$$grad_h G(\phi)=\sum_{\gamma,\mu=1}^{3}h^{\gamma\mu}(\phi)\frac{\partial G(\phi)}{\partial\phi^{\gamma}}\left(\frac{\partial}{\partial y^{\mu}}\right)_{\phi},$$

we obtain

Extended harmonic mapping

$$\tau_{\phi} + grad_{h} G(\phi) = \sum_{\mu,\gamma=1}^{3} (\sum_{i=1}^{2} \partial_{i}\, p_{\gamma}^{i} - \frac{\partial L_{\phi}}{\partial \phi^{\gamma}}) h^{\mu\gamma}(\phi) (\frac{\partial}{\partial y^{\mu}})_{\phi}.$$

This formula proves Prop. 1.

In this paper, if the tension field of ϕ is given by the formula (b) in Prop.1 for some $G \in C^{\infty}(N)$, ϕ is called an extended harmonic mapping and $G(\phi)$ is called the potential function associated with ϕ.

When we give an extended harmonic mapping ϕ such that the associated potential function is $G(\phi)$, we always consider the Lagrangian (1) and the corresponding Euler-Lagrange equations (a), throughout the paper.

ϕ is called an extended harmonic immersion if ϕ is an extended harmonic mapping and an immersion.

Extended harmonic mapping

Let $\phi:(R^2,g_0)\to(H^3(-1),h)$ be an extended harmonic mapping with the associated potential function $G(\phi)$, where g_0 and h stand for the standard metric on R^2 and the Riemannian metric on the hyperbolic 3-space $H^3(-1)$ of constant curvature -1, respectively, and h can be given by

$$\sum_{\alpha,\beta=1}^{3} h_{\alpha\beta}\, d\,y^{\alpha}\otimes d\,y^{\beta}$$

$$=(d\,y^1)\otimes(d\,y^1)+\cosh^2 y^1(d\,y^2)\otimes(d\,y^2)+\cosh^2 y^1\cosh^2 y^2(d\,y^3)\otimes(d\,y^3),$$

where (y^1,y^2,y^3) is a local coordinate system on $H^3(-1)$.

By making use of Euler-Lagrange equations (2), we have

$$(a)\quad \partial_1 p_1^1+\partial_2 p_1^2=\frac{\partial L_\phi}{\partial\phi^1},\qquad (b)\quad \partial_1 p_2^1+\partial_2 p_2^2=\frac{\partial L_\phi}{\partial\phi^2},$$

$$(c)\quad \partial_1 p_3^1+\partial_2 p_3^2=\frac{\partial L_\phi}{\partial\phi^3}.$$

Associated potential function

The formula (a) implies

$$(d) \quad \frac{\partial G(\phi)}{\partial \theta} = -\Delta\theta + \frac{1}{2} < \nabla\varphi, \nabla\varphi > \sinh 2\theta + \frac{1}{2} < \nabla\psi, \nabla\psi > \sinh 2\theta \cosh^2 \varphi,$$

and also, from the formula (b) and (c), we have

$$(e) \quad \frac{\partial G(\phi)}{\partial \varphi} = -\cosh^2 \theta\, \Delta\varphi + \frac{1}{2} < \nabla\psi, \nabla\psi > \cosh^2 \theta \sinh 2\varphi - < \nabla\theta, \nabla\varphi > \sinh 2\theta,$$

$$(f) \quad \frac{\partial G(\phi)}{\partial \psi} = -\cosh^2 \theta \cosh^2 \varphi\, \Delta\psi - < \nabla\theta, \nabla\psi > \sinh 2\theta \cosh^2 \varphi$$

$$- < \nabla\varphi, \nabla\psi > \sinh 2\varphi \cosh^2 \theta,$$

where $\theta = \phi^1 = y^1 \circ \phi, \quad \varphi = \phi^2 = y^2 \circ \phi, \quad \psi = \phi^3 = y^3 \circ \phi,$

and (d),(e) and (f) are obtained by using the following Lagrangian:

$$L_\phi = \frac{1}{2}\{\theta_t{}^2 + \theta_x{}^2 + (\varphi_t{}^2 + \varphi_x{}^2)\cosh^2\theta + (\psi_t{}^2 + \psi_x{}^2)\cosh^2\theta \cosh^2\varphi\} - G(\phi). \quad (4)$$

Determination of tension field

The tension field τ_ϕ of ϕ is given by

$$\tau_\phi = -h^{11}(\phi)\frac{\partial G(\phi)}{\partial \theta}\frac{\partial}{\partial \theta} - h^{22}(\phi)\frac{\partial G(\phi)}{\partial \varphi}\frac{\partial}{\partial \varphi} - h^{33}(\phi)\frac{\partial G(\phi)}{\partial \psi}\frac{\partial}{\partial \psi},$$

where the right hand side formula does not depend on the way to choose a local coordinate system on $H^3(-1)$, and

$$h^{11}(\phi)=1, \quad h^{22}(\phi)=\frac{1}{\cosh^2\theta}, \quad h^{33}(\phi)=\frac{1}{\cosh^2\theta\cosh^2\varphi},$$

and also $\frac{\partial G(\phi)}{\partial \theta}, \frac{\partial G(\phi)}{\partial \varphi}$ and $\frac{\partial G(\phi)}{\partial \psi}$ are given by the formulas (d),(e) and (f).

Assume that θ, φ, ψ are cyclic coordinates, i.e., $\frac{\partial L_\phi}{\partial \theta} = \frac{\partial L_\phi}{\partial \varphi} = \frac{\partial L_\phi}{\partial \psi} = 0.$

Then we have

$$\tau_\phi = -(<\nabla\varphi,\nabla\varphi> + <\nabla\psi,\nabla\psi>\cosh^2\varphi)\sinh\theta\cosh\theta\frac{\partial}{\partial\theta}$$

$$-<\nabla\psi,\nabla\psi>\sinh\varphi\cosh\varphi\frac{\partial}{\partial\varphi}. \qquad (5)$$

Conservative formula for generalized momenta

We consider the following conservative formula for generalized momenta:

$$dp\,((t,x)) = \sum_{i,j=1}^{2}\sum_{\alpha=1}^{3} \partial_i\, p_\alpha^j\, (dx^i)_{(t,x)} \otimes \left(\frac{\partial}{\partial x^j}\right)_{(t,x)} \otimes (dy^\alpha)_{\phi((t,x))} = 0, \qquad (6)$$

Let $\phi : (R^2, g_0) \to (H^3(-1), h)$ be an extended harmonic mapping with associated potential function $G(\phi) = G \circ \phi, \quad G \in C^\infty(H^3(-1))$.

Throughout the paper, we use the following notation:

$$\theta = \phi^1 = y^1 \circ \phi, \quad \varphi = \phi^2 = y^2 \circ \phi, \quad \psi = \phi^3 = y^3 \circ \phi.$$

The formula (6) implies that θ, φ, ψ are the cyclic coordinates with respect to Euler-Lagrange equations (2). Then we holds the following formula:

$$\sum_{i=1}^{2} \partial_i\, p_\alpha^i = 0, \quad \alpha = 1,2,3.$$

This formula may be called the strong Euler-Lagrange equations. Under the strong Euler-Lagrange equations, the tension field of ϕ is given by (5).

Conservative formula for generalized momenta

The conservative formula (6) for generalized momenta implies that

(i) $\partial_1 p_1^1 = \partial_1 p_1^2 = \partial_2 p_1^1 = \partial_2 p_1^2 = 0,$

(ii) $\partial_1 p_2^1 = \partial_2 p_2^1 = 0,$

(iii) $\partial_1 p_2^2 = \partial_2 p_2^2 = 0,$

(iv) $\partial_1 p_3^1 = \partial_2 p_3^1 = 0,$

(v) $\partial_1 p_3^2 = \partial_2 p_3^2 = 0.$

Conservative formula for generalized momenta

The formula (i) implies that

$$\theta_{tt} = \theta_{tx} = \theta_{xt} = \theta_{xx} = 0,$$

from which, we can choose θ as $\theta(t,x) = t + x + c.$ (c: constant) (7)

The formula (ii) and (iii) imply that

$$\partial_t(\varphi_t \cosh^2\theta) = \partial_x(\varphi_t \cosh^2\theta) = 0,$$

$$\partial_t(\varphi_x \cosh^2\theta) = \partial_x(\varphi_x \cosh^2\theta) = 0,$$

from which, we can choose φ as $\varphi_t = \varphi_x = \dfrac{1}{\cosh^2\theta}.$

Similarly, by using (iv) and (v),

We can choose ψ such as

$$\psi_t = \psi_x = \frac{1}{\cosh^2\theta(t,x)\cosh^2\varphi(t,x)}.$$

Conservative formula for generalized momenta

Hence, we have

$$\varphi(t,x) = \int \frac{dt}{\cosh^2 \theta(t,x)} + \int \frac{dx}{\cosh^2 \theta(t,x)}, \qquad (8)$$

from the formula (iv),(v), we can choose ψ as follows:

$$\psi(t,x) = \int \frac{dt}{\cosh^2 \theta(t,x) \cosh^2 \varphi(t,x)} + \int \frac{dx}{\cosh^2 \theta(t,x) \cosh^2 \varphi(t,x)}. \qquad (9)$$

Determination of associated potential function

The formulas (2) and (6) imply that

$$\frac{\partial L_\phi}{\partial \theta} = 0, \quad \frac{\partial L_\phi}{\partial \varphi} = 0 \ \ and \ \ \frac{\partial L_\phi}{\partial \psi} = 0, \qquad (10)$$

that is, $\theta, \varphi,$ and ψ are the cyclic coordinates.

Since $\theta, \varphi,$ and ψ is the cyclic coordinates and (4), we have

$$\frac{\partial G(\phi)}{\partial \theta} = \frac{1}{2}(|\nabla \varphi|^2 + |\nabla \psi|^2 \cosh^2 \varphi) \sinh 2\theta, \qquad (11)$$

$$\frac{\partial G(\phi)}{\partial \varphi} = \frac{1}{2} |\nabla \psi|^2 \cosh^2 \theta \sinh 2\varphi, \qquad (12)$$

$$\frac{\partial G(\phi)}{\partial \psi} = 0, \qquad (13)$$

where $\nabla \varphi, \nabla \psi$ stand for the gradients of φ, ψ on (R^2, g_0), respectively.

Determination of tension field

The formulas (8) and (9) imply that

$$|\nabla\varphi(t,x)| = \frac{\sqrt{2}}{\cosh^2\theta(t,x)}, \qquad |\nabla\psi(t,x)| = \frac{\sqrt{2}}{\cosh^2\theta(t,x)\cosh^2\varphi(t,x)}, \tag{14}$$

from the formulas (11),(12),(13) and (14), we obtain

$$G(\phi) = \int \frac{2(\cosh^2\varphi(t,x)+1)}{\cosh^3\theta(t,x)\cosh^2\varphi(t,x)}\sinh\theta(t,x)\,d\theta$$
$$+ \int \frac{2\sinh\varphi(t,x)}{\cosh^2\theta(t,x)\cosh^3\varphi(t,x)}\,d\varphi.$$

Using (11),(12) and (13), we have the tension field τ_ϕ of ϕ :

$$\tau_\phi = -\frac{1}{2}(<\nabla\varphi,\nabla\varphi> + <\nabla\psi,\nabla\psi>\cosh^2\varphi)\sinh 2\theta\,\frac{\partial}{\partial\theta}$$
$$-\frac{1}{2}\sinh 2\varphi <\nabla\psi,\nabla\psi> \frac{\partial}{\partial\varphi}.$$

Extended harmonic mapping

Thus, by making use of conservative formula (6) of generalized momenta, we can construct an example of extended harmonic mapping, which is not an immersion.

Extended harmonic CMC-H immersion

Let $\phi:(R^2, g_0) \to (H^3(-1), h)$ be an extended harmonic CMC-H immersion with associated potential function $G(\phi) = G \circ \phi$.

Then, since $\tau(\phi) = -\,grad_h G(\phi)$, we have $\| grad_h G(\phi) \|_h = 2H$ and the mean curvature vector field of ϕ is given by $\frac{1}{2}\tau_\phi$. .

H stands for the constant mean curvature of ϕ.

Under the assumption of the cyclic coordinates, (11),(12) and (13) hold, then

$$grad_h G(\phi) = h^{11}(\phi)\frac{\partial G(\phi)}{\partial \theta}\frac{\partial}{\partial \theta} + h^{22}(\phi)\frac{\partial G(\phi)}{\partial \varphi}\frac{\partial}{\partial \varphi} + h^{33}(\phi)\frac{\partial G(\phi)}{\partial \psi}\frac{\partial}{\partial \psi}$$

$$= \frac{1}{2}(|\nabla \varphi|^2 + |\nabla \psi|^2 \cosh^2 \varphi)\sinh 2\theta \frac{\partial}{\partial \theta} + \frac{1}{2}|\nabla \psi|^2 \sinh 2\varphi \frac{\partial}{\partial \varphi},$$

where

$$h_{11}(\phi) = 1, \quad h_{22}(\phi) = \cosh^2 \theta, \quad h_{33}(\phi) = \cosh^2 \theta \cosh^2 \varphi,$$

$$h_{12}(\phi) = h_{21}(\phi) = h_{13}(\phi) = h_{31}(\phi) = h_{23}(\phi) = h_{32}(\phi) = 0.$$

Extended harmonic CMC-H immersion

Then, since $\| grad_h G(\phi) \|_h^2 = 4H^2$, we have

$$(|\nabla\varphi|^2 + |\nabla\psi|^2 \cosh^2\varphi)^2 \sinh^2 2\theta + |\nabla\psi|^4 \sinh^2 2\varphi \cosh^2\theta = 16H^2.$$

Hence, we can take the parameter function $\rho = \rho(t, x)$ such that

$$(|\nabla\varphi|^2 + |\nabla\psi|^2 \cosh^2\varphi)\sinh 2\theta = 4H\cos\rho,$$

$$|\nabla\psi|^2 \sinh 2\varphi \cosh\theta = 4H\sin\rho.$$

Then, under the assumption of cyclic coordinates, we can choose the associated potential function with respect to ϕ as follows:

$$G(\phi) = 2H(\int \cos\rho \, d\theta + \int \cosh\theta \sin\rho \, d\varphi).$$

Consequently, the potential function $G(\phi)$ contains the constant mean curvature H itself.

Hamiltonians and conservation laws

Let $\phi:(R^2,g_0)\to(N,h)$ ($\dim N=3$) be an extended harmonic mapping with the associated potential function $G(\phi)$.

We use the notations: $\phi_t^\alpha:=\partial_t\phi^\alpha,\ \phi_x^\alpha:=\partial_x\phi^\alpha,\ \alpha=1,2,3.$

Then we define the Hamiltonian densities $H_\phi^{(t)}$ and $H_\phi^{(x)}$ with respect to ϕ :

$$H_\phi^{(t)}:=\sum_{\alpha=1}^{3}\phi_t^\alpha p_\alpha^1 - L_\phi(\phi,d\phi),\quad H_\phi^{(x)}:=\sum_{\alpha=1}^{3}\phi_x^\alpha p_\alpha^2 - L_\phi(\phi,d\phi),$$

where the Lagrangian L can be regarded as a smooth function on the 1-jet bundle $J^1((R^2,g_0),(N,h))$.

$$\frac{\partial}{\partial t}H_\phi^{(t)}=\sum_{\alpha=1}^{3}\phi_{tt}^\alpha\ p_\alpha^1+\sum_{\alpha=1}^{3}\phi_t^\alpha\ \partial_1 p_\alpha^1-\sum_{\alpha=1}^{3}\frac{\partial L_\phi}{\partial\phi^\alpha}\phi_t^\alpha-\sum_{\alpha=1}^{3}\frac{\partial L_\phi}{\partial\phi_t^\alpha}\phi_{tt}^\alpha-\sum_{\alpha=1}^{3}\frac{\partial L_\phi}{\partial\phi_x^\alpha}\phi_{tx}^\alpha,$$

Hamiltonians and conservation laws

by using Euler-Lagrange equations (2), we have

$$
\begin{aligned}
\frac{\partial}{\partial t} H_{\phi}^{(t)} &= \sum_{\alpha=1}^{3} \left(\frac{\partial L_{\phi}}{\partial \phi^{\alpha}} - \partial_2 p_{\alpha}^{2}\right)\phi_t^{\alpha} - \sum_{\alpha=1}^{3} \frac{\partial L_{\phi}}{\partial \phi^{\alpha}}\ \phi_t^{\alpha} - \sum_{\alpha=1}^{3} \frac{\partial L_{\phi}}{\partial \phi_x^{\alpha}}\ \phi_{t\,x}^{\alpha} \\
&= -\sum_{\alpha=1}^{3} \frac{\partial}{\partial x}\left(\phi_t^{\alpha}\ \frac{\partial L_{\phi}}{\partial \phi_x^{\alpha}}\right) \\
&= -\frac{\partial}{\partial x} h(\phi)\left(\phi_*\left(\frac{\partial}{\partial t}\right), \phi_*\left(\frac{\partial}{\partial x}\right)\right), \qquad (15)
\end{aligned}
$$

by using the formula : $p_{\alpha}^{i} = \sum_{\beta=1}^{3} \partial_i \phi^{\beta} h_{\alpha\beta}(\phi), \quad i = 1,2; \ \alpha = 1,2,3,$

where $\phi_*\left(\frac{\partial}{\partial t}\right) = \theta_t \frac{\partial}{\partial \theta} + \varphi_t \frac{\partial}{\partial \varphi} + \psi_t \frac{\partial}{\partial \psi}, \quad \phi_*\left(\frac{\partial}{\partial x}\right) = \theta_x \frac{\partial}{\partial \theta} + \varphi_x \frac{\partial}{\partial \varphi} + \psi_x \frac{\partial}{\partial \psi}.$

Hamiltonians and conservation laws

Similarly, we have $\frac{\partial}{\partial x} H_{\phi}^{(x)} = -\frac{\partial}{\partial t} h(\phi)(\phi_*(\frac{\partial}{\partial x}), \phi_*(\frac{\partial}{\partial t}))$.

Thus we have

Theorem 2. Let $\phi : (R^2, g_0) \rightarrow (N, h)$ be an extended harmonic mapping with associated potential function $G(\phi)$ and assume that $h(\phi)(\phi_*(\frac{\partial}{\partial t}), \phi_*(\frac{\partial}{\partial x}))$ is constant as a smooth function on R^2.

Then

$$\frac{\partial}{\partial t} H_{\phi}^{(t)} = \frac{\partial}{\partial x} H_{\phi}^{(x)} = 0.$$

Theorem 3. Let an extended harmonic mapping $\phi : (R^2, g_0) \rightarrow (N, h)$ with associated potential function $G(\phi)$ be conformal as a smooth mapping between Riemannian manifolds. Then

(a) $\frac{\partial}{\partial t} H_{\phi}^{(t)} = \frac{\partial}{\partial x} H_{\phi}^{(x)} = 0,$

(b) $H_{\phi}^{(t)} = H_{\phi}^{(x)} = G(\phi).$

Hamiltonians and conservation laws

Proof of Theorem 3. We have

$$H_\phi^{(t)} = \frac{1}{2}(h(\phi)(\phi_*(\frac{\partial}{\partial t}),\phi_*(\frac{\partial}{\partial t})) - h(\phi)(\phi_*(\frac{\partial}{\partial x}),\phi_*(\frac{\partial}{\partial x}))) + G(\phi), \qquad (16)$$

$$H_\phi^{(x)} = \frac{1}{2}(h(\phi)(\phi_*(\frac{\partial}{\partial x}),\phi_*(\frac{\partial}{\partial x})) - h(\phi)(\phi_*(\frac{\partial}{\partial t}),\phi_*(\frac{\partial}{\partial t}))) + G(\phi). \qquad (17)$$

Since ϕ is conformal, there exists a positive smooth function σ on R^2 such that

$$h(\phi)(\phi_*(\frac{\partial}{\partial x^i}),\phi_*(\frac{\partial}{\partial x^j})) = \phi^* h\,(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) = \sigma\, g_0(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}),\ \ i, j = 1, 2.$$

Then we have the following as smooth functions on R^2.

$$h(\phi)(\phi_*(\frac{\partial}{\partial t}),\phi_*(\frac{\partial}{\partial x})) = 0, \quad h(\phi)(\phi_*(\frac{\partial}{\partial t}),\phi_*(\frac{\partial}{\partial t})) = h(\phi)(\phi_*(\frac{\partial}{\partial x}),\phi_*(\frac{\partial}{\partial x})) = \sigma. \qquad (18)$$

The first formula of (18) implies (a) by using Theorem 2.

Furthermore, from (16),(17) and the second formula of (18), we obtain (b).

Q.E.D.

Canonical energy momentum tensor and conservation laws

We introduce the canonical energy momentum tensor $T(\phi)^i_j$ of ϕ :

$$T(\phi)^i_j = \sum_{\alpha=1}^{3} (\partial_j \phi^\alpha) p^i_\alpha - \delta^i_j L_\phi, \qquad i, j = 1,2.$$

Then, from the formula (2), the direct computation implies

Proposition 4. Let $\phi : (R^2, g_0) \to (N, h)$ be an extended harmonic mapping with associated potential function $G(\phi)$. Then we have the conservation laws :

$$\sum_{i=1}^{2} \partial_i T(\phi)^i_j = 0, \quad j = 1,2.$$

Equivalently,

$$\frac{\partial}{\partial t} H^{(t)}_\phi + \frac{\partial}{\partial x} T(\phi)^2_1 = 0, \quad \frac{\partial}{\partial t} T(\phi)^1_2 + \frac{\partial}{\partial x} H^{(x)}_\phi = 0.$$

Conservation laws and harmonic map

By using Theorem 3, we have

Theorem 5. Let (N,h) be a Riemannian manifold of dimension 3.
Let $\phi:(R^2,g_0)\to(N,h)$ be an extended harmonic mapping with associated potential function $G(\phi)=G\circ\phi$ and assume that ϕ is conformal as a mapping between Riemannian manifolds.
Furthermore, assume that the gradient vector fields $\nabla\phi^1,\nabla\phi^2$ and $\nabla\phi^3$ on (R^2,g_0) are linearly independent at each point , where this linearly independency does not depend on the way to choose a local coordinate system on N. Then, ϕ is a harmonic mapping.

APPENDIX 1. First variation formula

Let $\phi:(R^2, g_0)\to(N,h)$ be a smooth mapping, where (N,h) is an n-dimensional Riemannian manifold ($n\geq 3$). We take a smooth1-parameter variation

$$\Phi:(-\varepsilon,\varepsilon)\times R^2\to N,$$

of ϕ such that $\phi_t(p):=\Phi(t,p),\quad \phi_0:=\phi,\ t\in(-\varepsilon,\varepsilon),\ p\in R^2.$

The variation vector field W is denoted by

$$W(p)=\sum_{\alpha=1}^{n}W^{\alpha}(p)\left(\frac{\partial}{\partial y^{\alpha}}\right)_{\phi(p)}=\sum_{\alpha=1}^{n}\frac{d}{dt}\Big|_{t=0}\phi_t^{\alpha}(p)\left(\frac{\partial}{\partial y^{\alpha}}\right)_{\phi(p)},\ p\in R^2,$$

where $\{y^1,\ldots,y^n\}$ is a local coordinate system of $N.$ Let D be a bounded domain in R^2 such that its boundary is a smooth Jordan curve and assume boundary condition: $W|_{\partial D}=0.$ Assume that the Lagrangian L_ϕ of ϕ is

$$L_\phi=\frac{1}{2}\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{n}\partial_i\phi^{\alpha}\partial_i\phi^{\beta}h_{\alpha\beta}(\phi)-G(\phi),\quad G(\phi)=G\circ\phi,\ \ G\in C^{\infty}(N).$$

First variation formula

We consider the action integral :

$$E_\phi = \iint_D L_\phi(\phi(p),(d\phi)(p))\, dx^1 dx^2, \quad p=(x^1,x^2).$$

Then, by using Proposition 1, we have **the first variation formula** as follows:

$$\frac{d}{dt}\Big|_{t=0} E_{\phi_t} = \sum_{\gamma=1}^{n} \iint_D \frac{\partial L_\phi}{\partial \phi^\gamma} W^\gamma\, dx^1 dx^2 + \sum_{j=1}^{2}\sum_{\gamma=1}^{n} \iint_D \frac{\partial L_\phi}{\partial(\partial_j \phi^\gamma)} \partial_j W^\gamma\, dx^1 dx^2$$

$$= -\sum_{\gamma=1}^{n} \iint_D \Big\{ \sum_{j=1}^{2} \partial_j \Big(\frac{\partial L_\phi}{\partial(\partial_j \phi^\gamma)}\Big) - \frac{\partial L_\phi}{\partial \phi^\gamma} \Big\} W^\gamma\, dx^1 dx^2$$

$$= -\iint_D h_\phi\Big(\sum_{\beta,\gamma=1}^{n} h^{\beta\gamma} \Big(\sum_{j=1}^{2} \partial_j\, p_\gamma^j - \frac{\partial L_\phi}{\partial \phi^\gamma}\Big)\Big(\frac{\partial}{\partial y^\beta}\Big)_\phi, W\Big)\, dx^1 dx^2,$$

$$\frac{d}{dt}\Big|_{t=0} E_{\phi_t} = -\iint_D h_\phi(\ \tau_\phi + grad_h G(\phi), W\ )\, dx^1 dx^2.$$

First variation formula

Proposition. The following (a) and (b) are equivalent:

(a) $\frac{d}{dt}|_{t=0}\, E\phi_t = 0.$

(b) $\tau_\phi = -grad_h G(\phi),$

where we call ϕ the extended harmonic mapping with the potential function $G \in C^\infty(N).$

Remark. If ϕ is an extended harmonic mapping, whose associated potential function G is a constant function on $N,$ then we have

$$\frac{d}{dt}|_{t=0}\, E_{\phi_t} = \frac{d}{dt}|_{t=0}\, E^0_{\phi_t},$$

where

$$E^0_{\phi_t} = \frac{1}{2}\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{n} \iint_D \partial_i\, {\phi_t}^\alpha\, \partial_i\, {\phi_t}^\beta\, h_{\alpha\beta}(\phi_t)\, dx^1 dx^2.$$

APPENDIX 2. Second variation formula

Next we consider a smooth 2-parameter variation Ψ of extended harmonic mapping $\phi:(R^2, g_0) \to (N, h)$ such that

$$\Psi:(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times R^2 \to N, \quad \phi_{s,t}(p) := \Psi(s,t,p),\ \phi_{0,0} := \phi,\ p \in R^2,$$

$$s,t \in (-\varepsilon,\varepsilon).$$

The variation vector fields $V = \Psi_*(\frac{\partial}{\partial s})|_{s=t=0},\ W = \Psi_*(\frac{\partial}{\partial t})|_{s=t=0}$

with the boundary condition: $V_t|_{\partial D} = W_s|_{\partial D} = 0,$

where

$$V_t := \Psi_*(\frac{\partial}{\partial s})|_{s=0},\ W_s := \Psi_*(\frac{\partial}{\partial t})|_{t=0}, \quad V_0 = V, \quad W_0 = W.$$

Then we have

$$\frac{\partial}{\partial t}|_{t=0} E_\Psi = \sum_{\gamma=1}^{n} \iint_D \Big(\frac{\partial L_{\phi_{s,t}}}{\partial \phi_{s,t}^\gamma}|_{t=0}\, \partial_t|_{t=0}\, \phi_{s,t}^\gamma + \sum_{j=1}^{2} \frac{\partial L_{\phi_{s,t}}}{\partial(\partial_j \phi_{s,t}^\gamma)}|_{t=0}\, \partial_j(\partial_t|_{t=0}\, \phi_{s,t}^\gamma)\Big) dx^1 dx^2.$$

Second variation formula

By using the boundary condition,

$$\frac{\partial}{\partial t}\Big|_{t=0} E_\Psi = \sum_{\gamma=1}^{n} \iint_D \left(\frac{\partial L_{\phi_{s,0}}}{\partial \phi_{s,0}^\gamma} - \sum_{j=1}^{2} \partial_j \left(\frac{\partial L_{\phi_{s,0}}}{\partial(\partial_j \phi_{s,0}^\gamma)} \right) \right) W_s^\gamma \, dx^1 dx^2,$$

where

$$W_s = \sum_{\gamma=1}^{n} \left(\frac{\partial}{\partial t} \phi_{s,t}^\gamma \left(\frac{\partial}{\partial y^\gamma} \right)_{\phi_{s,t}} \right)\Big|_{t=0} = \sum_{\gamma=1}^{n} W_s^\gamma \left(\frac{\partial}{\partial y^\gamma} \right)_{\phi_{s,t}} \Big|_{t=0} .$$

By using Proposition 1, we have

$$\frac{\partial}{\partial t}\Big|_{t=0} E_\Psi = -\iint_D h_{\phi_{s,0}} \left(\tau_{\phi_{s,0}} + grad_{h_{\phi_{s,0}}} G(\phi_{s,0}),\ W_s \right) dx^1 dx^2$$

$$= -\iint_D h_{\phi_{s,0}} \left(\sum_{i=1}^{2} \hat{\nabla}_{e_i} \Psi_*(e_i)\Big|_{t=0}, W_s \right) dx^1 dx^2 - \iint_D h_{\phi_{s,0}} \left(grad_{h_{\phi_{s,0}}} G(\phi_{s,0}),\ W_s \right) dx^1 dx^2,$$

where $\hat{\nabla}$ denotes the induced connection of the induced vector bundle $\Psi^{-1}TN$ and $e_1 = \frac{\partial}{\partial x^1}, e_2 = \frac{\partial}{\partial x^2}$ stand for the standard basis of R^2.

Second variation formula

Hence, we have

$$(\#)\quad \frac{\partial}{\partial s}\Big|_{s=0}\Big(\frac{\partial}{\partial t}\Big|_{t=0} E_{\Psi}\Big) = -\iint_D h_{\phi}\Big(\sum_{i=1}^{2}\hat{\nabla}_{\frac{\partial}{\partial s}}\hat{\nabla}_{e_i}\Psi_*(e_i)\Big|_{s=t=0}, W\Big)\, dx^1 dx^2$$

$$-\iint_D h_{\phi}\Big(\sum_{i=1}^{2}\hat{\nabla}_{e_i}\phi_*(e_i), \hat{\nabla}_{\frac{\partial}{\partial s}}\Psi_*\big(\frac{\partial}{\partial t}\big)\Big|_{s=t=0}\Big)\, dx^1 dx^2$$

$$-\iint_D h_{\phi}(\nabla_V{}^N grad_h G(\phi), W)\, dx^1 dx^2 - \iint_D h_{\phi}\Big(grad_h G(\phi), \hat{\nabla}_{\frac{\partial}{\partial s}}\Psi_*\big(\frac{\partial}{\partial t}\big)\Big|_{s=t=0}\Big)\, dx^1 dx^2,$$

In this formula, the sum of the second and the fourth terms vanishes, since ϕ is an extended harmonic mapping. Note that the first $\hat{\nabla}$ in the second term of (#) stands for the induced connection of $\phi^{-1}TN$.

Second variation formula

Note that

$$\hat{\nabla}_{\frac{\partial}{\partial s}} \hat{\nabla}_{e_i} \Psi_*(e_i)\,|_{s=t=0} = R^N(\Psi_* \frac{\partial}{\partial s}, \Psi_* e_i)\Psi_* e_i\,|_{s=t=0} + \hat{\nabla}_{e_i} \hat{\nabla}_{\frac{\partial}{\partial s}} (\Psi_* e_i)\,|_{s=t=0}$$

$$= R^N(V, \phi_* e_i)\phi_* e_i + \hat{\nabla}_{e_i} \hat{\nabla}_{e_i} V,$$

where R^N denotes the Riemannian curvature tensor field of (N,h).

The first term of (#) is

$$\iint_D h_\phi(\overline{\Delta}_\phi V - R_\phi V, W)\, dx^1 dx^2 = \iint_D h_\phi(J_\phi V, W)\, dx^1 dx^2,$$

where $\overline{\Delta}_\phi$ stands for the rough Laplacian with respect to $\phi^{-1}TN$ [U] and R_ϕ is defined by

$$R_\phi X := \sum_{i=1}^{2} R^N(X, \phi_* e_i)\phi_* e_i, \quad X \in \Gamma(\phi^{-1}TN).$$

Second variation formula

Proposition (Second variation formula).

$$\frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} E_\Psi = \iint_D h_\phi (J_\phi V - \nabla^N_V grad_h G(\phi), W)\, dx^1 dx^2$$
$$= \iint_D h_\phi (J_\phi V - H_{G(\phi)} V, W)\, dx^1 dx^2, \qquad H_{G(\phi)} V := \nabla^N_V grad_h G(\phi),$$

where ∇^N denotes the Levi-Civita connection of (N, h).

$J_\phi = \overline{\Delta}_\phi - R_\phi$ is called the Jacobi operator [U] and we call $H_{G(\phi)}$ the Hesse operator. They are the self-adjoint operators with respect to L^2 -metric. This second variation formula may be valid for a smooth mapping $\phi : (M, g) \to (N, h),$ where (M, g) and (N, h) are a compact Riemannian manifold (without boundary) and a Riemannian manifold, respectively [K].

Complex Lagrangian

Let $\phi:(C,g_0)\to(N,h)$ be a holomorphic mapping, where (C,g_0) is 1-dim. complex Euclidean space with standard metric g_0 and (N,h) is an n-dim. complex manifold with Hermitian metric h, respectively.

We consider the following complex Lagrangian of ϕ :

$$L_\phi=\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{n}\partial_i\phi^\alpha\partial_i\overline{\phi}^\beta h_{\alpha\overline{\beta}}(\phi)-G(\phi),$$

where $\phi^\alpha:=\varsigma^\alpha\circ\phi,\ \ \overline{\phi}^\alpha:=\overline{\varsigma}^\alpha\circ\phi,\ (\varsigma^1,\dots,\varsigma^n)$ is a complex local coordinates system on N , and $G\in C_c^\infty(N)$ is a complex valued smooth function on N,

$$g_0:=\mathrm{Re}(d\,z\otimes d\,\overline{z})=\sum_{i=1}^{2}dx^i\otimes dx^i,\qquad \phi_*\left(\frac{\partial}{\partial x^i}\right)=\sum_{\alpha=1}^{n}\left(\partial_i\phi^\alpha\left(\frac{\partial}{\partial\varsigma^\alpha}\right)_\phi+\partial_i\overline{\phi}^\alpha\left(\frac{\partial}{\partial\overline{\varsigma}^\alpha}\right)_\phi\right),$$

$(z=x^1+\sqrt{-1}\,x^2)$

$$\partial_i\phi^\alpha:=\frac{\partial}{\partial x^i}\phi^\alpha=\frac{\partial}{\partial x^i}u^\alpha(x^1,x^2)+\sqrt{-1}\frac{\partial}{\partial x^i}v^\alpha(x^1,x^2),\ \ i=1,2,\ \alpha=1,\dots,n.$$

Complex Lagrangian

We can define the generalized momenta:

$$p_\gamma^i := \frac{\partial L_\phi}{\partial(\partial_i \phi^\gamma)}, \quad \overline{p}_\gamma^i := \frac{\partial L_\phi}{\partial(\partial_i \overline{\phi}^\gamma)} \quad , \quad i = 1,2, \ \gamma = 1,\ldots,n.$$

Then we have

$$\sum_{\gamma,\mu=1}^{n} \left(\sum_{i=1}^{2} \partial_i p_\gamma^i - \frac{\partial L_\phi}{\partial \phi^\gamma}\right) h^{\gamma\overline{\mu}} \left(\frac{\partial}{\partial \overline{\varsigma}^\mu}\right)_\phi = \tau_\phi^{(-)} + grad_h^{(-)} G(\phi),$$

$$\sum_{\gamma,\mu=1}^{n} \left(\sum_{i=1}^{2} \partial_i \overline{p}_\gamma^i - \frac{\partial L_\phi}{\partial \overline{\phi}^\gamma}\right) h^{\overline{\gamma}\mu} \left(\frac{\partial}{\partial \varsigma^\mu}\right)_\phi = \tau_\phi^{(+)} + grad_h^{(+)} G(\phi),$$

where

$$grad_h^{(+)} G(\phi) = \sum_{\lambda,\mu=1}^{n} h^{\lambda\overline{\mu}} \frac{\partial G(\phi)}{\partial \overline{\phi}^\mu} \left(\frac{\partial}{\partial \varsigma^\lambda}\right)_\phi, \quad grad_h^{(-)} G(\phi) = \sum_{\lambda,\mu=1}^{n} h^{\lambda\overline{\mu}} \frac{\partial G(\phi)}{\partial \phi^\lambda} \left(\frac{\partial}{\partial \overline{\varsigma}^\mu}\right)_\phi,$$

Complex Lagrangian

and, using the coefficients of torsion-free affine connection of N,

$$\tau_{\phi}^{(+)} := \sum_{i=1}^{2} \sum_{\gamma=1}^{n} (\partial_i^2 \phi^{\gamma} + \sum_{\alpha,\beta=1}^{n} \Gamma_{\alpha\beta}^{\gamma}(\phi)\partial_i \phi^{\alpha} \partial_i \phi^{\beta} + 2 \sum_{\alpha,\beta=1}^{n} \Gamma_{\alpha\overline{\beta}}^{\gamma}(\phi)\partial_i \phi^{\alpha} \partial_i \overline{\phi}^{\beta})(\frac{\partial}{\partial \varsigma^{\gamma}})_{\phi},$$

$$\tau_{\phi}^{(-)} := \sum_{i=1}^{2} \sum_{\gamma=1}^{n} (\partial_i^2 \overline{\phi}^{\gamma} + \sum_{\alpha,\beta=1}^{n} \Gamma_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}}(\phi)\partial_i \overline{\phi}^{\alpha} \partial_i \overline{\phi}^{\beta} + 2 \sum_{\alpha,\beta=1}^{n} \Gamma_{\alpha\overline{\beta}}^{\overline{\gamma}}(\phi)\partial_i \phi^{\alpha} \partial_i \overline{\phi}^{\beta})(\frac{\partial}{\partial \overline{\varsigma}^{\gamma}})_{\phi}.$$

Note that the tension field τ_{ϕ} of ϕ is $\tau_{\phi} = \tau_{\phi}^{(+)} + \tau_{\phi}^{(-)}.$

Then the following (a) and (b) are equivalent:

(a) (Euler-Lagrange equations)

$$\sum_{i=1}^{2} \partial_i p_{\gamma}^{i} - \frac{\partial L_{\phi}}{\partial \phi^{\gamma}} = 0, \quad \sum_{i=1}^{2} \partial_i \overline{p}_{\gamma}^{i} - \frac{\partial L_{\phi}}{\partial \overline{\phi}^{\gamma}} = 0, \quad \gamma = 1, \dots, n.$$

(b) $\tau_{\phi}^{(-)} = -\,grad_h^{(-)} G(\phi), \quad \tau_{\phi}^{(+)} = -\,grad_h^{(+)} G(\phi).$

Complex Hamiltonians

We have the next relationship:

The Euler-Lagrange equations are equivalent to

$$\tau_{\phi} = -\, grad_h G(\phi),$$

where

$$grad_h G(\phi) := grad_h^{(+)} G(\phi) + grad_h^{(-)} G(\phi).$$

We can define the Hamiltonians as follows:

$$H_{\phi}^{(i)} := \sum_{\alpha=1}^{n} \partial_i \phi^{\alpha} \, p_{\alpha}^{i} + \sum_{\alpha=1}^{n} \partial_i \bar{\phi}^{\alpha} \, \bar{p}_{\alpha}^{i} - L_{\phi}, \quad i = 1,2.$$

Then we have

$$\partial_1 H_{\phi}^{(1)} = -\,\partial_2 \, h_{\phi}(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^2})), \;\; \partial_2 H_{\phi}^{(2)} = -\,\partial_1 \, h_{\phi}(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^2})).$$

Conservation laws

From the definitions of Hamiltonians, we see

$$H_\phi^{(1)} = \frac{1}{2}\{h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^1})) - h_\phi(\phi_*(\frac{\partial}{\partial x^2}), \phi_*(\frac{\partial}{\partial x^2}))\} + G(\phi),$$

$$H_\phi^{(2)} = \frac{1}{2}\{h_\phi(\phi_*(\frac{\partial}{\partial x^2}), \phi_*(\frac{\partial}{\partial x^2})) - h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^1}))\} + G(\phi).$$

If ϕ has the conformal property such as

$$h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^2})) = 0, \quad h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^1})) = h_\phi(\phi_*(\frac{\partial}{\partial x^2}), \phi_*(\frac{\partial}{\partial x^2})),$$

then we have the conservation laws:

$$\partial_1 H_\phi^{(1)} = \partial_2 H_\phi^{(2)} = 0,$$

where

$$H_\phi^{(1)} = H_\phi^{(2)} = G(\phi).$$

Extended harmonic mapping

Assume that $\phi:(C,g_0)\to(N,h)$ is an extended harmonic, holomorphic mapping equipped with potential function G with respect to the complex Lagrangian and ϕ has the conformal property.

If the complex 1-forms $d\phi^1,\ldots,d\phi^n,d\overline{\phi}^1,\ldots,d\overline{\phi}^n$ are linearly independent over the complex number field at each point, where this linearly indepen-dency does not depend on the way to choose a complex local coordinate system on $N,$ then the tension field τ_ϕ vanishes.

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Thank you !