Submanifolds and Gauss map related to some differential operators

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Abstract & Keyword

- **Abstract** Gauss map is one of interesting smooth maps on a submanifolds of Euclidean and pseudo-Euclidean space, which describes how the immersion behaves in the ambient space. In this talk, we introduce how the Gauss map plays a role to classify or characterize ruled submanifolds in Euclidean space or Minkowski space related to the Laplace operator and the Cheng-Yau operator.

Gauss map of a submanifold in Euclidean space

- M : an n-dimensional submanifold of Euclidean space $\mathbb{E}^m$.
- $G : M \to G(n, m)$, $G(p) = (e_1 \wedge e_2 \wedge \ldots \wedge e_n)(p)$, $G(n, m)$: the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$ and $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ an adapted local orthonormal frame field in $\mathbb{E}^m$.
- $G(n, m)$: a unit hypersphere in $\mathbf{E}^N$, $N = {}_mC_n$.

In particular, if M is a hypersurface of $\mathbb{E}^m$, the Gauss map G is obviously identified with a unit normal vector field on M.

Gauss map of a hypersurface in Euclidean space

- M : an n-dimensional oriented hypersurface of Euclidean space $\mathbb{E}^{n+1}$.
  $\Delta G = \|A_G\|^2 G + n\nabla H$. (U. Dursun, Taiwanese J. Math. (2007))

L_k-operator

- M : an n-dimensional oriented hypersurface of Euclidean space $\mathbb{E}^{n+1}$.
- A: the shape operator of M and $\kappa_1, \kappa_2, \ldots, \kappa_n$ the principal curvatures
- $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ the elementary symmetric function defined by

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$

- H_k: the k-th mean curvature defined by

$$\binom{n}{k} H_k = s_k$$

$(0 \leqslant k \leqslant n)$, where $s_0 = 1$.

- If $k = 1$, $H_1 = 1/n \sum_{i=1}^{n} \kappa_i = H$
- When k is even, H_k is intrinsic.

L_k-operator

- $P_k : \chi(M) \to \chi(m)$, $p_0 = I$,
  $P_k = s_k I - S \circ P_{k-1} = \binom{n}{k} H_k I - S \circ P_{k-1}$, $k = 1, 2, ..., n$.
- $P_k = \sum_{j=0}^{k} (-1)^j s_{k-j} S^j = \sum_{j=0}^{k} (-1)^j \binom{n}{k} H_{k-j} S^j$.
- $L_k : C^\infty(M) \to C^\infty(M)$ defined by

$$L_k(f) = \mathrm{trace}(P_k \circ \nabla^2 f).$$

- $L_1 = \Box$: Cheng-Yau operator
- $\Box G = -\nabla K - nHKG$.

Ruled submanifold

Definition

A submanifold M of the Minkowski space $\mathbb{L}^m$ is ruled if M is foliated by codimension one totally geodesic submanifolds of $\mathbb{L}^m$ over a curve, i.e.,

$$x = x(s, t_1, t_2, \ldots t_r) = \alpha(s) + \sum_{i=1}^{r} t_i \mathbf{e}_i(s), \quad s \in I, \quad t_i \in I_i$$

where I_i are some open intervals for $i = 1, 2, \ldots, r$ and $\mathrm{Span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r\} = E(s, r)$ which is the linear span of linearly independent vector fields $\mathbf{e}_1(s), \mathbf{e}_2(s), \ldots, \mathbf{e}_r(s)$ along the curve α, where $E(s, r)$ is either non-degenerate or degenerate for each s along α. We call $E(s, r)$ the ruling and α the base curve of the ruled submanifold M.

Ruled submanifold

- If the base curve α is null, then $E(s, r)$ is degenerate for each $s \in I$ so that the defined ruled submanifold M is non-degenerate. Such a ruled submanifold M is called a ruled submanifold of *the null scroll type* or simply a NS-ruled submanifold.

B-scroll

Definition

$\alpha(s)$: a null curve in $\mathbb{L}^3$ with $\dot{\alpha} = A(s)$ satisfying

$$\langle A, A\rangle = \langle B, B\rangle = 0, \langle A, B\rangle = -1, \langle C, C\rangle = 1, \langle A, C\rangle = \langle B, C\rangle = 0$$

and $\dot{x} = A$, $\dot{A} = k(s)C, \dot{B} = w_0 C$ (w_0:a nonzero constant),
$\dot{C} = w_0 A + k(s)B$. ($\{A, B, C\}$: Cartan frame along γ.)
$x(s,t) = \alpha(s) + tB(s)$ defines *B-scroll* of $\mathbb{L}^3$.

Remark

B-scroll is flat **iff** $k \equiv 0$.

B-scroll

- In 1979, L.K.Graves, *Math.Ann.*
  $i : \mathbb{L}^2 \longrightarrow \mathbb{L}^3$ isometric immersion,
  $T_0(x) = \mathrm{Ker}(A_x) = \{X \in T_x M | (AX)_x = 0\}$,
  A_x : the shape operator at x (the relative nullity space at x).

Theorem (Graves)

Let $i : \mathbb{L}^2 \to \mathbb{L}^3$ be an isometric immersion and $T_0(x)$ be degenerate. Then, i is a B-scroll immersion. There are various types of B-scrolls which are flat or non-flat.

Theorem (Alias, Ferrandez, Moreno, 2002)

Let M be a null scroll in $\mathbb{L}^3$ satisfying $\Delta G = AG$ for some 3×3–matrix A. Then, M is part of a B-scroll.

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B-scroll

Theorem (Kim - Kim, *2003*)

Let M be a null 2-type timelike surface in a Lorentz space form. Suppose there is a point p of M such that the shape operator A is not diagonalizable at x. Then, M is locally a B-scroll.

- Complex circle of radius κ
  $c + id = \kappa \in \mathbb{C}$ with $c^2 - d^2 = -1$.
  $\mathbb{C}^2 \simeq \mathbb{E}^4_2$ with $(x_1 + ix_3, x_2 + ix_4) \mapsto (x_1, x_2, x_3, x_4)$,
  $ds^2 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$.
  Define $x : \mathbb{C} \to \mathbb{C}^2$ by $z \mapsto \kappa(\cos z, \sin z)$ where $z = u_1 + iu_2$
  $= (u_1, u_2)$, x defines a non-minimal flat timelike surface into $\mathbb{H}^3_1$ (anti-de Sitter space-time).

Generalized B-scroll (Kim, Kim, Yoon)

Let $\alpha(s)$ be a null curve in $\mathbb{L}^m$.
Consider a null frame $\{A(s), B(s), C_1(s), \cdots, C_{m-2}\}$ along α such that

$$\langle A, A\rangle = \langle B, B\rangle = \langle A, C_i\rangle = \langle B, C_i\rangle = 0,$$
$$\langle A, B\rangle = -1, \langle C_i, C_j\rangle = \delta^i_j = 0.$$

$X(s) = \big(A(s)B(s)C_1(s)\cdots C_{m-2}(s)\big)$: a square matrix of degree m displayed as column vectors of $A, \cdots, C_{m-2}$. Then,

$$X^t(s)EX(s) = T$$

where $E = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_{m-1} \end{pmatrix}$, $T = \begin{pmatrix} \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & I_{m-2} \end{pmatrix}$.

Generalized B-scroll (Kim, Kim, Yoon)

Consider a system of O.D.E

$$\dot{X} = XM$$

where

$$M = \begin{pmatrix} 0 & 0 & -a & 0 & & \cdots & 0 \\ 0 & 0 & -k_1(s) & -k_2(s) & & \cdots & -km-2(s) \\ -k_1(s) & -a & \omega_2(s) & 0 & -z_{23} & \cdots & -z_{2,m-2}(s) \\ -k_2(s) & 0 & \omega_3(s) & z_{23} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ -k_{m-2}(s) & 0 & \omega_{m-2}(s) & z_{2,m-2} & & & \mathbf{O}_l \end{pmatrix}$$

$a \in \mathbb{R}, k_1, \cdots, k_{m-2}, \omega_1, \cdots, \omega_{m-2}$, and $z_{ij}(2 \leqslant i \leqslant m-3, 3 \leqslant j \leqslant m-2)$ are some functions, and $\mathbf{O}_l$ a zero matrix of degree $l \in \{1, 2, \cdots, m-3\}$.

Generalized B-scroll (Kim, Kim, Yoon)

Given a initial condition

$$X(0) = \big(A(0)B(0)C_1(0)\cdots C_{m-2}(0)\big)$$

satisfying $X^t(0)EX(0) = T$, there exists a unique solution $X(s)$ to $\dot{X} = XM$. Since T is symmetric and MT is skew-symmetric, $\frac{d}{ds}(X^tEX) = 0$ and so $X^t(s)EX(s) = T$. Using such solution, we define

$$M : x(s,t) = \alpha(s) + tB(s).$$

Then, M is a timelike surface which is called a *generalized B-scroll* in $\mathbb{L}^m$.

In particular, M is called an *extended B-scroll* if $z_{ij} = \omega_i \equiv 0$.

Generalized B-scroll (Kim, Kim, Yoon)

Remark

(1) By choosing appropriate unit spacelike vector fields $C_2(s), \cdots, C_{m-2}(s)$, we may assume $z_{ij} \equiv 0$.

(2) Let G be a Gauss map of a generalized B-scroll.
Then, $\Delta^2 G = 2a^2 \Delta G$, $\Delta G \neq 0$.

Theorem (Kim, Kim, Yoon, *2002*)

Let M be a null scroll in $\mathbb{L}^m$. Then, M has finite type Gauss map iff G is 1-type or null 2-type, that is, M is an extended B-scroll or a generalized B-scroll.

Generalized B-scroll (Kim, Kim, Yoon)

Remark

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BS-ruled submanifold

- Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{L}^m$ and let $A(s), B(s), C_1(s), \cdots, C_{m-2}(s)$ be a null frame along α satisfying

$$\langle A, A\rangle = \langle B, B\rangle = \langle A, C_i\rangle = \langle B, C_i\rangle = 0,$$

$$\langle A, B\rangle = -1, \qquad \langle C_i, C_j\rangle = \delta_{ij}, \qquad \alpha'(s) = A(s)$$

  for $1 \leqslant i, j \leqslant m-2$. Let $X(s)$ be the matrix $(A(s) \quad B(s) \quad C_1(s) \quad \cdots \quad C_{m-2}(s))$ consisting of column vectors of $A(s), B(s), C_1(s), \cdots, C_{m-2}(s)$ with respect to the standard coordinate system in $\mathbb{L}^m$.

-
$$X^t(s)EX(s) = T.$$

BS-ruled submanifold

$$X'(s) = X(s)M(s),$$

where $M(s) =$

$$\begin{pmatrix} 0 & 0 & v_2 & v_3 & \cdots & v_r & 0 & \cdots & 0 \\ 0 & 0 & u_2 & u_3 & \cdots & u_r & u_{r+1} & \cdots & u_{m-1} \\ u_2 & v_2 & 0 & 0 & \cdots & 0 & z_{2,r+1} & \cdots & z_{2,m-1} \\ u_3 & v_3 & 0 & 0 & \cdots & 0 & z_{3,r+1} & \cdots & z_{3,m-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ u_r & v_r & 0 & 0 & \cdots & 0 & z_{r,r+1} & \cdots & z_{r,m-1} \\ u_{r+1} & 0 & -z_{2,r+1} & -z_{3,r+1} & \cdots & -z_{r,r+1} & 0 & \cdots & 0 \\ u_{r+2} & 0 & -z_{2,r+2} & -z_{3,r+2} & \cdots & -z_{r,r+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ u_{m-1} & 0 & -z_{2,m-1} & -z_{3,m-1} & \cdots & -z_{r,m-1} & 0 & \cdots & 0 \end{pmatrix}$$

G-kind ruled submanifold

If one of $v_2, \ldots v_r$ is nonzero constant, we define

Definition

(BS-kind Ruled submanifold)

$$x(s, t, t_2, \cdots, t_r) = \alpha(s) + tB(s) + \sum_{i=2}^{r} t_i C_i,$$

where $2 \leqslant r \leqslant m - 2$.

Definition

(G-kind Ruled submanifold)

$$x(s, t, t_2, \cdots, t_r) = \alpha(s) + tB(s) + \sum_{i=2}^{r} t_i C_i,$$

where $2 \leqslant r \leqslant m - 2$.

Pointwise 1-type Gauss map

Definition

$M \subset \mathbb{L}^m$ is said to have *pointwise 1-type Gauss map* if $\Delta G = f(G + C)$ for some smooth function f and a constant vector C. In particular, if $C = 0$, then it is said to be of pointwise 1-type Gauss map of the first kind. Otherwise, it is said to be of the second kind.

Pointwise 1-type Gauss map

Theorem (Choi-Yoon-Kim), 2009

Let M be a ruled surface in Minkowski 3-space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map. Then, M is an open part of a Euclidean plane, a Minkowski plane, a hyperbolic cylinder, a Lorentz circular cylinder, a circular cylinder of index 1, a cylinder of an infinite type, a helicoid of the first kind, a helicoid of the second kind, a helicoid of the third kind, the conjugate of Enneper's surface of the second kind, a rotational ruled surface of type I or type II, a transcendental ruled surface, or a B-scroll.

Pointwise 1-type Gauss map

Theorem (Kim-Kim), J.Geom.Phy. 2012

Suppose that M is a minimal ruled submanifold in $\mathbb{L}^m$ with Lorentzian rulings. Then, there exists an orthonormal basis $\{E_1, \cdots, E_m\}$ in $\mathbb{L}^m$ such that M is part of one of the following submanifolds (up to cylinders, built on those submanifolds):
(1) an $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$.
(2) $X(s, t_1, t_2, \cdots, t_n) = \sum_{i=1}^{n} t_i e_i(s) + bsE_{2n+1}$, where $e_1(s) = \sinh a_1 s E_1 + \cosh a_1 s E_2$ and for $i \geqslant 2$, $e_i(s) = \sin a_i s E_{2i-1} + \cos a_i s E_{2i}$. Here E_1 is timelike and $E_2, \cdots, E_{2n+1}$ are spacelike.

Pointwise 1-type Gauss map

Theorem (Kim-Kim-Jung), 2013

Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold with non-degenerate rulings in the Minkowski m-space $\mathbb{L}^m$. Then, M has finite type Gauss map G if and only if either M is part of an $(r+1)$-dimensional plane or the Gauss map G is of finite rank k for some k $(1 \leqslant k \leqslant r)$.

Theorem (Kim-Kim-Jung), 2013

Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. M has finite type Gauss map if and only if M is an open portion of a generalized BS-kind ruled submanifold or a G-kind ruled submanifold.

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Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold with non-degenerate rulings in the Minkowski m-space $\mathbb{L}^m$. Then, M has finite type Gauss map G if and only if either M is part of an $(r+1)$-dimensional plane or the Gauss map G is of finite rank k for some k $(1 \leqslant k \leqslant r)$.

Theorem (Kim-Kim-Jung), 2013

Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. M has finite type Gauss map if and only if M is an open portion of a generalized BS-kind ruled submanifold or a G-kind ruled submanifold.

Pointwise 1-type Gauss map

Definition

An $(r+1)$-dimensional cylindrical ruled submanifold M is called a generalized circular cylinder $\Sigma_a \times \mathbb{E}^{r-1}$ if the base curve α is a circle and the generators of rulings are orthogonal to the plane containing the circle α, where Σ_a is a circular cylinder $S^1(a) \times \mathbb{R}$ in $\mathbb{E}^3$.

Pointwise 1-type Gauss map

Definition

Suppose $\beta = \beta(s)$ is a circle on the unit sphere centered at the origin. Let $\mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_r$ be orthonormal constant vectors satisfying $\langle \beta'(s), \mathbf{a}_i \rangle = \langle \beta(s), \mathbf{a}_i \rangle = 0$ for all $i = 2, 3, \ldots, r$ and s. A ruled submanifold M parametrized by

$$x(s, t_1, t_2, \ldots, t_r) = t_1 \beta(s) + \sum_{i=2}^{r} t_i \mathbf{a}_i + D \qquad (1)$$

is called a generalized right cone $C_a \times \mathbb{E}^{r-1}$, where C_a is a right cone in $\mathbb{E}^3$, D a constant vector and $t_i \in I_i$ for some open intervals I_i and $i = 2, 3, \ldots, r$.

Pointwise 1-type Gauss map

Theorem (Kim-Turgay, (2013))

Let M be a helicoidal surface in $\mathbb{E}^3$. Then, M has $\square$-pointwise 1-type Gauss map of the second kind if and only if M is an open part of a plane, a right circular cylinder, a right circular cone or a surface which is locally congruent to the rotational surface satisfying certain differential equations.

Theorem (Kim-Kim-Jung-Yoon), 2014

The only ruled submanifold M of Euclidean space $\mathbb{E}^m$ with pointwise 1-type Gauss map of the first kind is an open part of a generalized circular cylinder or a generalized helicoid.

Pointwise 1-type Gauss map

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References I

- J. M. Barbosa, M. Dajczer and I. P. Jorge, *Minimal ruled submanifolds in spaces of constant curvature*, Indiana Univ. Math. J. 33 (1984), 531-547.
- B.-Y. Chen and P. Piccini, *Submanifolds with finite type Gauss map*, Bull. Austral. Math. Soc. 35(1987), 161-186.
- B.-Y. Chen, D.-S. Kim and Y. H. Kim, *New charactyerization of W-curves*, Publ. Math. Debrecen 69/4 (2006), 457.472
- M. Choi, Y. H. Kim and D. W. Yoon, *Helicoidal surfaces with pointwise 1-type Gauss map*, J. Korean Math. Soc. 46 (2009), 215-223.
- M. Choi, Y. H. Kim and D. W. Yoon, *Ruled surfaces with pointwise 1-type Gauss map in a Minkowski 3-space*, Taiwanese J. Math. 14 (2010), 1297-1308.

References II

- F. Dillen, *Ruled submanifolds of finite type*, Proc. Amer. Math. Soc. 114 (1992), 795-798.
- U-H. Ki, D.-S. Kim, Y. H. Kim and Y.-M. Roh, *Surfaces of revolution with pointwise 1-type Gauss map in Minkowski 3-space*, Taiwanese J. Math. Soc. 13 (2009), 317-338.
- D.-S. Kim and Y. H. Kim, $\mathbb{B}$*-scrolls with non-diagonalizable shape operators*, Rocky Mount. J. Math. 33(2003), 175-190.
- D.-S. Kim, Y. H. Kim and D. W. Yoon, *Finite type ruled surfaces in Lorentz-Minkowski space*, Taiwanese J. Math. 11 (2007), 1-13.
- D.-S. Kim, Y. H. Kim, S. M. Jung and D. W. Yoon, *Gauss maps of ruled submanifolds and applications* I, submitted for publication.

References III

- D.-S. Kim, Y. H. Kim, S. M. Jung and D. W. Yoon, *Gauss maps of ruled submanifolds and applications* II, submitted for publication.
- D.-S. Kim, Y. H. Kim and D. W. Yoon, *Characterization of generalized* B*-scrolls and cylinders over finite type curves*, Indian J. Pure Appl. Math. 34 (2003), 1523-1532.
- Y. H. Kim and D. W. Yoon, *Ruled surfaces with pointwise 1-type Gauss map*, J. Geom. Phys. 34 (2000), 191-205.
- Y. H. Kim and D. W. Yoon, *On the Gauss map of ruled surfaces in Minkowski space*, Rocky Mountain J. Math. 35 (2005), 1555-1581.

References IV

Y. H. Kim and N¿ C. Turgay, *Classifications of helicoidal surfaces with* L_1*-pointwise 1-type Gauss map*, Bull. Korean Math. Soc., 50 (2013), 1345-1356.

Thank You!