# Gauss map of real hypersurfaces in complex projective space and submanifolds in complex 2-plane Grassmannians

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- Then the Gauss map $\gamma : M \to \widetilde{G}_2(\mathbb{R}^{n+2}) \cong Q^n$ is defined by
- $\gamma(p) = x(p) \wedge N_p$ (B. Palmer, 1997).

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- Also for parallel hypersurface $M_r := \cos rx + \sin rN$ $(r \in \mathbb{R})$ of $M$, the Gauss image is not changed: $\gamma(M) = \gamma(M_r)$.
- We define Gauss map $\gamma : M^{2n-1} \to \mathbb{G}_2(\mathbb{C}^{n+1})$ for real hypersurface $M^{2n-1}$ in $\mathbb{CP}^n$.

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- For $p \in M$, take a point $z_p \in \pi^{-1}(p) \subset \pi^{-1}(M)$ and let $N'_p$ be a holizontal lift of unit normal of $M \subset \mathbb{CP}^n$ at $z_p$.

# Gauss map of real hypersurface in $\mathbb{CP}^n$

- If we put $\gamma(p) = \text{span}_{\mathbb{C}}\{z_p, N'_p\}$,then the map $\gamma : M \to \mathbb{G}_2(\mathbb{C}^{n+1})$ is well-defined.

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- We call $\gamma$ as the Gauss map of real hypersurface $M$ in $\mathbb{CP}^n$.
- Note that for parallel hypersurface $M_r := \pi(\cos r z_p + \sin r N'_p)$ of $M$,image of the Gauss map $\gamma : M^{2n-1} \to \mathbb{CP}^n$ is not changed: $\gamma(M) = \gamma(M_r)$.

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- For a real hypersurface $M^{2n-1}$ in Kähler manifold $(\widetilde{M}^n, J)$ and a unit normal vector $N$,
- a vector $\xi := -JN$ tangent to $M$ is called the structure vector of $M$.
- And when $\xi$ is an eigenvector of the shape operator $A$ of $M$, we call $M$ a Hopf hypersurface in $\widetilde{M}$.

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- Conversely, if a Hopf hypersurface $M$ in $\mathbb{CP}^n(4)$ satisfies $A\xi = \mu\xi$ ($\mu$ is necessarily constant), and for $r \in (0, \pi/2)$ with $\mu = 2\cot 2r$, $r \in (0, \pi/2)$, if rank of the focal map $\phi_r : M \to \mathbb{CP}^n$ is constant, then

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- $\phi_r(M)$ is a complex submanifold of $\mathbb{CP}^n(4)$ and $M$ lies on a tube over $\phi_r(M)$. (Cecil-Ryan, 1982).

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- For example, he showed that compact embedded Hopf hypersurface in $\mathbb{CP}^n$ lies on a tube over an algebraic variety.
- In this talk, we will give a characterization of Hopf hypersurface $M$ in $\mathbb{CP}^n$ by using the Gauss map $\gamma : M \to \mathbb{G}_2(\mathbb{C}^{n+2})$.

# Quaternionic Kähler manifold

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- Here, $\tilde{g}$ is a Riemannian metric of $\widetilde{M}$, $Q$ is a subbundle of End$T\widetilde{M}$ with rank $3$, satisfying:
- For each $p \in \widetilde{M}$, there exists a neighborhood $U \ni p$, such that there exists local frame field $\{\tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}$ of $Q$.

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- $$\tilde{I}_1^2 = \tilde{I}_2^2 = \tilde{I}_3^2 = -1, \quad \tilde{I}_1\tilde{I}_2 = -\tilde{I}_2\tilde{I}_1 = \tilde{I}_3,$$
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- For each $L \in Q_p$, $\tilde{g}$ is invariant, i.e., $\tilde{g}_p(LX, Y) + \tilde{g}_p(X, LY) = 0$ for $X, Y \in T_p\widetilde{M}$, $p \in \widetilde{M}$.

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- Vector bundle $Q$ is parallel with respect to the Levi-Civita connection of $\tilde{g}$ at $\mathrm{End}\, T\widetilde{M}$.

# Almost Hermitian submanifolds in Q.K. manifold

- A submanifold $M^{2m}$ in quaternionic Kähler manifold $\widetilde{M}$ is called almost Hermitian submanifold, if there exists a section $\tilde{I}$ of vector bundle $Q|_M$ over $M$ such that

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- if we write the almost complex structure on $M$ which is induced by $\tilde{I}$ as $I$, then
- with respect to the induced metric, $(M, I)$ is an almost Hermitian manifold.

# Totally complex submanifold of Q.K. manifold

- In particular, when almost Hermitian submanifold $(M, \bar{g}, I)$ is Kähler, we call $M$ a Kähler submanifold of quaternionic Kähler manifold $\widetilde{M}$.

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- Similarly, an almost Hermitian submanifold $(M, \bar{g}, I)$ is called totally complex submanifold if at each point $p \in M$, with respect to $\tilde{L} \in Q_p$ which anti-commute with $\tilde{I}_p$, $\tilde{L}T_pM \perp T_pM$ hold.

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- In quaternionic Kähler manifold, a submanifold is totally complex if and only if it is Kähler (Alekseevsky-Marchiafava, 2001).

# Theorem: Gauss map of real hypersurface in $\mathbb{CP}^n$

- Theorem (K., Diff. Geom. Appl. 2014) Let $M^{2n-1}$ be a real hypersurface in complex projective space $\mathbb{CP}^n$, and let $\gamma : M \to \mathbb{G}_2(\mathbb{C}^{n+1})$ be the Gauss map.

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- If $M$ is not Hopf, then the Gauss map $\gamma$ is an immersion.
- If $M$ is a Hopf hypersurface, then the image $\gamma(M)$ is a half-dimensional totally complex submanifold of $\mathbb{G}_2(\mathbb{C}^{n+1})$.

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- If $M$ is a Hopf hypersurface, then the image $\gamma(M)$ is a half-dimensional totally complex submanifold of $\mathbb{G}_2(\mathbb{C}^{n+1})$.
- And a Hopf hypersurface $M$ in $\mathbb{CP}^n$ is a total space of a circle bundle over a Kähler manifold such that the fibration is nothing but the Gauss map $\gamma : M \to \gamma(M)$.

# Converse construction

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- then we have a submanifold $\tilde{I}(\Sigma)$ of the twistor space $\mathcal{Z} = \{\tilde{I} \in Q \mid \tilde{I}^2 = -1\}$ of $\mathbb{G}_2(\mathbb{C}^{n+1})$(natural lift).

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- Since $\Sigma$ is a totally complex submanifold of $\mathbb{G}_2(\mathbb{C}^{n+1})$, $\tilde{I}(\Sigma)$ is a Legendrian submanifold of the twistor space $\mathcal{Z}$ with respect to a complex contact structure (Alekseevsky-Marchiafava, 2004).

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- With respect to the following diagram:
- $$\begin{array}{ccccc} \tilde{I}^*E & \xrightarrow[\eta]{} & E & \xrightarrow[\psi]{} & \mathbb{CP}^n \\ \downarrow & & \downarrow & & \\ \Sigma^{n-1} & \xrightarrow[\tilde{I}]{} & \mathcal{Z} \cong L(\mathbb{CP}^n) & & \end{array},$$

# Converse construction

- The map $\Phi := \psi \circ \eta : \tilde{I}^* E \to \mathbb{CP}^n$ gives Hopf hypersurface with $A\xi = 0$ (on open subset of regular points of $M = \tilde{I}^* E$), and

# Converse construction

- The map $\Phi := \psi \circ \eta : \tilde{I}^* E \to \mathbb{CP}^n$ gives Hopf hypersurface with $A\xi = 0$ (on open subset of regular points of $M = \tilde{I}^* E$), and
- its parallel hypersurface $\phi_r(\tilde{I}^* E)$ gives Hopf hypersurface with $A\xi = 2 \tan 2r \xi$ (on open subset of regular points of $M = \tilde{I}^* E$).

# Remarks

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- For real hypersurfaces in complex hyperbolic space $\mathbb{CH}^n$, we define Gauss map $\gamma : M \to \mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$, and
- we obtain similar results for Hopf hypersurfaces in $\mathbb{CH}^n$ by using para-quaternionic Kähler structure (J.T. Cho and M.K., Topol. Appl. 2015).

# Split-quaternions

- $\widetilde{\mathbb{H}} = C(2,0) = C(1,1)$, Split-quaternions (or coquaternions, para-quaternions):
  $q = q_0 + iq_1 + jq_2 + kq_3$, $i^2 = -1$, $j^2 = k^2 = 1$,
  $ij = -ji = -k$, $jk = -kj = i$, $ki = -ik = -j$,
  $|q|^2 = q_0^2 + q_1^2 - q_2^2 - q_3^2$, $\exists$ zero divisors,

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- http://en.wikipedia.org/wiki/Split-quaternion
- Introduced by James Cockle in 1849.

# Para-quaternionic structure

- $\{I_1, I_2, I_3\}$, $I_1^2 = -1$, $I_2^2 = I_3^2 = 1$, $I_1 I_2 = -I_2 I_1 = -I_3$, $I_2 I_3 = -I_3 I_2 = I_1$, $I_3 I_1 = -I_1 I_3 = -I_2$ gives para-quaternionic structure,

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- $\tilde{V} = \{aI_1 + bI_2 + cI_3 \mid a, b, c \in \mathbb{R}\} \cong \mathfrak{su}(1,1) \cong \mathbb{R}^3_1$, and
- $Q_+ = \{I \in \tilde{V} \mid I^2 = 1\} \cong S^2_1$: de-Sitter space, $Q_- = \{I \in \tilde{V} \mid I^2 = -1\} \cong H^2$: hyperbolic space, $Q_0 = \{I \in \tilde{V} \mid I^2 = 0,\ I \neq 0\} \cong$ lightcone.

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- such that $dg_x(T_xM)$ is invariant under $\tilde{I}_{\tilde{1}}$ and $\tilde{I}_{\tilde{2}}dg_x(T_xM), \tilde{I}_3dg_x(T_xM)$ are orthogonal to $dg_x(T_xM)$.

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- satisfies $p + q \leq n - 1$.