The Kepler and harmonic oscillator problems on families of coadjoint orbits

(Work in progress, joint with J. Mostovoy)

Varna, Bulgaria

June 10th, 2015

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$$H_k = \frac{1}{2}p^2 - \frac{\alpha}{x} \qquad H_o = \frac{1}{2}\left(p^2 + \omega^2 x^2\right) \qquad (p = \|\mathbf{p}\|, \quad x = \|\mathbf{x}\|)$$

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(example: $M = \mathbb{R}^n \oplus \mathbb{R}^n$, $G = \mathrm{SO}(n)$, $\ell_{ij} = x_i p_j - x_j p_i$, $H = \ell^2$)

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Starting point: $\mathfrak{h}_n \rtimes \mathfrak{o}(n)$ ($\mathbb{R}^n \oplus \mathbb{R}^n$ is coadjoint orbit)

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Commutation relations of $\mathfrak{h}_n \rtimes \mathfrak{o}(n)$:

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Deformation to the *n*-sphere (and hyperbolic *n*-space):

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Can we understand *all* possible deformations?

Infinitesimal problem: $H^2(\mathfrak{h}_n \rtimes \mathfrak{o}(n), \mathfrak{h}_n \rtimes \mathfrak{o}(n))$

Lemma: The space of global deformations is 3-dimensional:

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We are interested in coadjoint orbits of dimension $2n$ (minimal).

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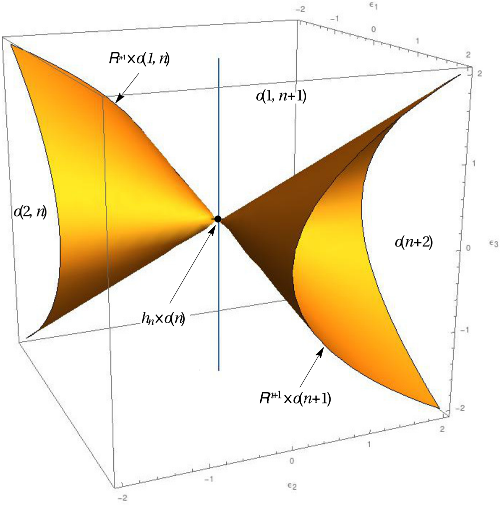
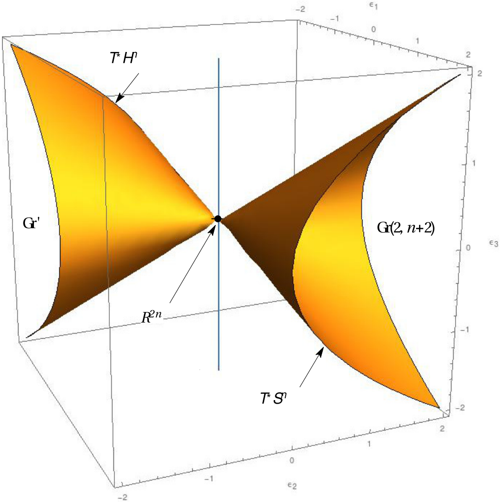
Quadratic Casimir invariant + relations $I\ell_{ij} = x_i p_j - x_j p_i$

Space of global deformations of $\mathfrak{h}_n \rtimes \mathfrak{o}(n)$:



Coadjoint orbits generically invariant under $\mathrm{SO}(2) \times \mathrm{SO}(n)$:



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Hermitian symmetric space $\Rightarrow$ natural invariant complex polarization

More suitable: real **singular** polarization.

Position polarization: $q_i = x_i/I \qquad (\{q_i, q_j\} = 0)$

(correspond to gnomonic coordinates on the sphere)

Darboux coordinates:

$$\pi_i = p_i(1 + \epsilon_2 q^2), \qquad \theta_i = \frac{q_i}{1 + \epsilon_2 q^2}$$

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More complicated normalization of μ_{in+1}, yet possible

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$= H_0 + V(q, p)$ (similar to Higgs solution)

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Thank you!