Kepler Problem and Formally Real Jordan Algebras IV

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In the last lecture we arrived at the following procedure for producing many more integrable models of Kepler type:

A simple Euclidean Jordan algebra V

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the conformal Lie algebra $\mathfrak{co}$

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the associative algebra $\mathcal{TKK}$

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Universal Kepler Problem

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A generalized Kepler problem via a suitable realization of $\mathfrak{co}$

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Recall that the conformal algebra of V has a Poisson realization on TV in which X_u and Y_v can be realized as real-valued function

$$\mathcal{X}_u = \langle x \mid \{\pi u \pi\}\rangle \quad \text{and} \quad \mathcal{Y}_v = \langle x \mid v\rangle$$

respectively on TV. Since $H = \frac{1}{2}\frac{X_e}{Y_e} - \frac{1}{Y_e}$, H can be realized as

$$\mathcal{H} = \frac{1}{2}\frac{\langle x \mid \pi^2\rangle}{r} - \frac{1}{r}$$

where $r = \langle x \mid e\rangle = \frac{1}{\mathrm{rank}\, V}\mathrm{tr}\, x$.

However,

$\mathcal{H}$ is NOT even a real-valued function on TV!

To make sense of $\mathcal{H}$, we need to work on a small subspace of TV.

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Kepler cones

Theorem (G. W. Meng, 2011)

Let k be a positive integer which is at most $\mathrm{rank}\, V$, and C_k be the set of rank-k semi-positive elements of V. Then C_k is a manifold. Moreover, for any $x \in C_k$, 1) $T_xC_k = \{x\} \times \mathrm{Im}L_x$, 2) The map

$$\langle \pi | \ \rangle \mapsto \frac{\mathcal{X}_e}{\mathcal{Y}_e} = \frac{\langle x|\pi^2\rangle}{\langle x|e\rangle} = \frac{\langle \pi \mid x\pi\rangle}{r}$$

is a positive-definite quadratic form on $T_x^*C_k$.

These quadratic forms in the theorem define a Riemannian metric on C_k (called the **Kepler metric**), and shall be denoted by $(\ ,\ )_K$.

Claim: *The dynamic model with configuration space C_k, Lagrangian $L = \frac{1}{2}(\dot{x}, \dot{x})_K^2 + \frac{1}{r}$ or Hamiltonian*

$$\mathcal{H} = \frac{1}{2}||p||_K^2 - \frac{1}{r}$$

is a super integrable model of Kepler type.

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Kepler problem and future light-cone

The purpose here is to verify this claim: If $V = \Gamma(3) := \mathbb{R} \oplus \mathbb{R}^3$ and $k = 1$, then the dynamical model mentioned in the last slide is exactly the Kepler problem.

In terms of the standard basis vectors $\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3$, the Jordan multiplication can be determined by the following rules: $\vec{e}_0$ is the identity element, and

$$\vec{e}_i \vec{e}_j = \delta_{ij} \vec{e}_0$$

for $i, j > 0$. The trace $\mathrm{tr} : V \to \mathbb{R}$ is given by the following rules:

$$\mathrm{tr}\, \vec{e}_0 = 2, \quad \mathrm{tr}\, \vec{e}_i = 0.$$

So the inner product on V is the one such that the standard basis is an orthonormal basis. Since V has rank two, the determinant of $x = x^\mu \vec{e}_\mu$ is

$$\det x = \frac{1}{2}((\mathrm{tr}\, x)^2 - \mathrm{tr}\, x^2) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

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Therefore, the Kepler cone

$$C_1 = \{x \in V \mid \det x = 0, \mathrm{tr}\, x > 0\}$$

is precisely the future light-cone in the Minkowski space. Since points on C_1 can be parametrized by $\mathbf{r} \in \mathbb{R}^{\mathbf{3}}_*$: $x(\mathbf{r}) = (|\mathbf{r}|, \mathbf{r})$ where $|\mathbf{r}|$ is the length of $\mathbf{r}$, we have (write $\mathbf{r} = x^i \vec{e}_i$)

$$\partial_{x^i} = \vec{e}_i + \frac{x^i}{|\mathbf{r}|}\vec{e}_0.$$

The dual tangent frame E^j w.r.t. $\langle \mid \rangle$, obtained by solving Eqs $\langle E^j \mid \partial_{x^i} \rangle = \delta^j_i$, is

$$E^j = \vec{e}_j - \frac{x^j}{2|\mathbf{r}|^2}\mathbf{r} + \frac{x^j}{2|\mathbf{r}|}\vec{e}_0.$$

Write the Kepler metric $\mathrm{d}s^2_K$ as $g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j$.

Claim: $g_{ij} = \delta_{ij}$, i.e., $\mathrm{d}s^2_K = \sum_i (\mathrm{d}x^i)^2$.

Proof. It suffice to prove that $g^{ij} = \delta_{ij}$. To do that, we note that

$$xE^j = (\cancel{x^j\vec{e}_0} - \cancel{\frac{x^j}{2}\vec{e}_0} + \cancel{\frac{x^j}{2|\mathbf{r}|}}\mathbf{r}) + (|\mathbf{r}|\vec{e}_j - \cancel{\frac{x^j}{2|\mathbf{r}|}}\mathbf{r} + \cancel{\frac{x^j}{2}\vec{e}_0}) = |\mathbf{r}|\vec{e}_j + x^j\vec{e}_0.$$

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Therefore, the Kepler cone

$$C_1 = \{x \in V \mid \det x = 0, \operatorname{tr} x > 0\}$$

is precisely the future light-cone in the Minkowski space. Since points on C_1 can be parametrized by $\mathbf{r} \in \mathbb{R}^{\mathbf{3}}_*$: $x(\mathbf{r}) = (|\mathbf{r}|, \mathbf{r})$ where $|\mathbf{r}|$ is the length of $\mathbf{r}$, we have (write $\mathbf{r} = x^i \vec{e}_i$)

$$\partial_{x^i} = \vec{e}_i + \frac{x^i}{|\mathbf{r}|}\vec{e}_0.$$

The dual tangent frame E^j w.r.t. $\langle \mid \rangle$, obtained by solving Eqs $\langle E^j \mid \partial_{x^i}\rangle = \delta^j_i$, is

$$E^j = \vec{e}_j - \frac{x^j}{2|\mathbf{r}|^2}\mathbf{r} + \frac{x^j}{2|\mathbf{r}|}\vec{e}_0.$$

Write the Kepler metric $\mathrm{d}s^2_K$ as $g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j$.
Claim: $g_{ij} = \delta_{ij}$, i.e., $\mathrm{d}s^2_K = \sum_i (\mathrm{d}x^i)^2$.
Proof. It suffice to prove that $g^{ij} = \delta_{ij}$. To do that, we note that

$$xE^j = (\cancel{x^j\vec{e}_0} - \cancel{\frac{x^j}{2}\vec{e}_0} + \cancel{\frac{x^j}{2|\mathbf{r}|}}\mathbf{r}) + (|\mathbf{r}|\vec{e}_j - \cancel{\frac{x^j}{2|\mathbf{r}|}}\mathbf{r} + \cancel{\frac{x^j}{2}\vec{e}_0}) = |\mathbf{r}|\vec{e}_j + x^j\vec{e}_0.$$

So, because $E^i = \vec{e}_i - \frac{x^i}{2r^2}\mathbf{r} + \frac{x^i}{2|\mathbf{r}|}\vec{e}_0$ and $xE^j = |\mathbf{r}|\vec{e}_j + x^j\vec{e}_0$, we have

$$g^{ij} = \frac{\langle E^i \mid xE^j\rangle}{|\mathbf{r}|} = \frac{|\mathbf{r}|\delta_{ij} - \frac{x^i x^j}{2|\mathbf{r}|} + \frac{x^i x^j}{2|\mathbf{r}|}}{|\mathbf{r}|} = \delta_{ij}.$$

Let us write the momentum p as $p_i\, \mathrm{d}x^i$, since $\mathrm{d}s_K^2 = \sum_i (\mathrm{d}x^i)^2$, and $r = \langle x \mid e\rangle = \frac{1}{2}\mathrm{tr}\, x = |\mathbf{r}|$, the hamiltonian of the dynamic model can be written as

$$\mathcal{H} = \frac{1}{2}\sum_i p_i^2 - \frac{1}{|\mathbf{r}|}.$$

That is precisely the hamiltonian of the Kepler problem!

Exercise. Continue the above discussion, please verify that

$$\mathcal{L}_{\vec{e}_1,\vec{e}_2} = L_3, \quad \mathcal{L}_{\vec{e}_2,\vec{e}_3} = L_1, \quad \mathcal{L}_{\vec{e}_3,\vec{e}_1} = L_2, \quad \mathcal{A}_{\vec{e}_i} = A_i, \quad \mathcal{A}_{\vec{e}_0} = 1.$$

Here L_i (resp. A_i) is the i-th component of the angular momentum (resp. Lenz vector) in the Kepler problem.

So, because $E^i = \vec{e}_i - \frac{x^i}{2r^2}\mathbf{r} + \frac{x^i}{2|\mathbf{r}|}\vec{e}_0$ and $xE^j = |\mathbf{r}|\vec{e}_j + x^j\vec{e}_0$, we have

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