Kepler Problem and Formally Real Jordan Algebras V

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God always geometrizes — Plato

We have learned that there are Kepler-type integrable models associated with a simple Euclidean Jordan algebra V, one for each Kepler cone of V.

We have also learned that the Kepler problem is one of these integrable models.

Since the Kepler problem has magnetized versions, one naturally wonders whether these Kepler-type integrable models also have magnetized versions.

The simple answer is "Yes".

To know more, we must start with the introduction of Sternberg phase space [S. Sternberg. Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field. *Proc Nat. Acad. Sci.* **74** (1977), 5253-5254].

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A technical setup

- G — a compact connected Lie group
- $\mathfrak{g}$, $\mathfrak{g}^*$ —- the Lie algebra of G and its dual
- $P \to X$ — a principal G-bundle over X
- Θ — a fixed principal connection form, i.e., Θ be a $\mathfrak{g}$-valued differential one-form on P which satisfies the following two conditions:

$$1)\ {R_{a^{-1}}}^* \Theta = \mathrm{Ad}_a \Theta \text{ for any } a \in G, \qquad 2)\ \Theta(X_\xi) = \xi \text{ for any } \xi \in \mathfrak{g}.$$

Here, $R_{a^{-1}}$ denotes the right action of a^{-1} on P, Ad_a denotes the adjoint action of a on $\mathfrak{g}$, and vector field X_ξ denotes the infinitesimal right action of ξ on P.

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F	a Hamiltonian G-space
Ω	the symplectic form on F
$\Phi : F \to \mathfrak{g}^*$	the G-equivariant moment map
$\mathcal{F} \to X$	the associated fiber bundle with fiber F
$\mathcal{F}^\sharp$	the pullback of diagram $T^*X \to X \leftarrow \mathcal{F}$

I.e., square

$$\begin{array}{ccc} \mathcal{F}^\sharp & \to & \mathcal{F} \\ \downarrow & & \downarrow \\ T^*X & \to & X \end{array}$$

is a pullback diagram in the category of smooth manifolds and smooth maps.

For notational sanity here, we shall use the same notation for both a differential form (or a map) and its pullback under a fiber bundle projection map.

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Sternberg Phase Space

Theorem (Sternberg, 1977)

• *There is a closed real differential two-form* Ω_Θ *on* $\mathcal{F}$ *which is of the form* $\Omega - \mathrm{d}\langle A, \Phi\rangle$ *under a local trivialization of* $P \to X$ *in which the connection form* Θ *is represented by the* $\mathfrak{g}$*-valued differential one-form* A *on* X.

• *The differential two-form* $\omega_\Theta := \omega_X + \Omega_\Theta$ *is a symplectic form on* $\mathcal{F}^\sharp$, *where* ω_X *is the canonical symplectic form on* T^*X, *pulled back under* $\mathcal{F}^\sharp \to T^*X$, *and* Ω_Θ *is the pullback of* Ω_Θ *under* $\mathcal{F}^\sharp \to \mathcal{F}$.

- Ω_Θ is the right substitute for Ω when we go from a product bundle with the product connection to a generic bundle.
- If $G = \mathrm{U}(1)$, then $(\mathcal{F}^\sharp, \omega_\Theta) = (T^*X, \omega_X - q_e\,\mathrm{d}A)$ where q_e is the electric charge of the particle.
- In the Hamiltonian formalism, as shown by Sternberg and others, the Sternberg phase space $(\mathcal{F}^\sharp, \omega_\Theta)$ is the right substitute for (T^*X, ω_X) when particles move in a background gauge field.

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Examples [G. Meng. J. Math. Phys. **54**, 052902 (2013)]

- $X = \mathbb{R}_*^{2k+1}$ or the Kepler cone C_1 of the Jordan algebra $\Gamma(2k+1)$
- $G = \mathrm{SO}(2k)$
- $P \to X$ is the pullback bundle of $\mathrm{SO}(2k+1) \to \mathrm{S}^{2k}$ under the map

$$\begin{array}{ccc} X & \to & \mathrm{S}^{2k} \\ \mathbf{r} & \mapsto & \frac{\mathbf{r}}{r} \end{array} \quad (1)$$

- Θ is the pullback of the canonical connection

$$\mathrm{Proj}_{\mathfrak{so}(2k)}(g^{-1}\, dg)$$

on $\mathrm{SO}(2k+1) \to \mathrm{S}^{2k}$.

- F is a co-adjoint orbit of G which is either $\{0\}$ or diffeomorphic to $\mathrm{SO}(2k)/\mathrm{U}(k)$.

Fact: The conformal Lie algebra of the Jordan algebra $\Gamma(2k+1)$ has a suitable Poisson realization on $\mathcal{F}^\sharp$, which yields a magnetized Kepler problem.

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Remarks

We have seen that there are Kepler-type classical dynamical models and their magnetized versions associated with each Kepler cone of a simple euclidean Jordan algebras. Here are some further facts:

- The quantum versions of these models are expected to give, among other things, a concrete geometric realizations for all unitary highest weight modules of (the universal cover) of the following real non-compact Lie groups

$$\mathrm{SO}(2, n),\ \mathrm{Sp}(2n, \mathbb{R}),\ \mathrm{SU}(n, n),\ \mathrm{SO}^*(4n),\ E_{7(-25)}.$$

- The n-dimensional isotropic harmonic oscillator is (essentially) the Kepler-type of problem associated with C_1 of $\mathrm{H}_n(\mathbb{R})$, with the Fradkin tensor being the (generalized) Lenz-vector.

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