Inequalities for Submanifolds of Product Spaces

Cihan ÖZGÜR

Department of Mathematics

Balikesir University, TURKEY

cozgur@balikesir.edu.tr

In ([Chen 1993]), B.-Y. Chen recalled that "The Riemannian invariants are the DNA of the Riemannian manifolds and one of the basic interests of submanifold theory is to establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the Ricci curvature and the scalar curvature. There are many works in the literature related to basic inequalities involving the squared mean curvature and one of the classical curvature invariants namely the scalar curvature, the sectional curvature and the Ricci curvature for different kinds of submanifolds of real space forms, complex space forms and contact space forms, etc.

The first results related to basic inequalities were proved by B.-Y. Chen in ([Chen 1993]), ([Chen 1996]) and ([Chen 1999]). In ([Chen 1999]), B.-Y. Chen extends the notion of Ricci curvature to k-Ricci curvature ($2 \leq k \leq n$) in an n-dimensional Riemannian manifold. In this talk we give some relations between main extrinsic invariants and main intrinsic invariants for submanifolds of the product of two real space forms. For a collection of the results in this direction we refer to B.-Y. Chen's book ([Chen 2011]) and references therein.

Let $M^{n_1}(c_1)$ and $M^{n_2}(c_2)$ be two real space forms of constant curvatures c_1, c_2 with dimensions n_1 and n_2, respectively. Let us consider the product space $(M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$. Assume that (M^m, g) be a Riemannian manifold isometrically immersed into the product space $(M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$. Denote by $\widetilde{\nabla}$ and F the Levi-Civita connection of M^m and the product structure of the product space $(M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$, respectively. The product structure $F : TM^{n_1}(c_1) \times TM^{n_2}(c_2) \longrightarrow TM^{n_1}(c_1) \times TM^{n_2}(c_2)$ is a $(1,1)$-tensor field defined by

$$F(X_1 + X_2) = X_1 - X_2$$

for any vector field $X = X_1 + X_2$, X_1, X_2 denote the parts of X tangent to the first and second factors, respectively.

It is easy to see that F satisfies

$$F^2 = I \text{ (and } F \neq I), \tag{1}$$

$$\widetilde{g}(FX, Y) = \widetilde{g}(X, FY), \tag{2}$$

$$\widetilde{\nabla} F = 0, \tag{3}$$

see ([Yano and Kon 1984]). By an easy calculation, we obtain the curvature tensor of $(M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$ as

$$\begin{aligned} \widetilde{R}(X,Y)Z = \frac{c_1 + c_2}{4}[\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y \\ + \widetilde{g}(FY,Z)FX - \widetilde{g}(FX,Z)FY] \\ + \frac{c_1 - c_2}{4}[\widetilde{g}(Y,Z)FX - \widetilde{g}(X,Z)FY + \widetilde{g}(Y,FZ)X - \widetilde{g}(X,FZ)Y], \end{aligned} \tag{4}$$

([Dillen and Kowalczyk 2012]).

Now let $X \in TM^m$ and $\xi \in T^{\perp}M^m$. The decompositions of FX and $F\xi$ into tangent and normal components can be written as

$$FX = kX + hX \text{ and } F\xi = s\xi + t\xi, \tag{5}$$

where $k : TM^m \longrightarrow TM^m$, $h : TM^m \longrightarrow T^{\perp}M^m$,
$s : T^{\perp}M^m \longrightarrow TM^m$, and $t : T^{\perp}M^m \longrightarrow T^{\perp}M^m$ are $(1,1)$-tensor fields. It is easy to see that k and t are symmetric and satisfy the following properties:

$$k^2X = X - shX, \tag{6}$$

$$t^2\xi = \xi - hs\xi, \tag{7}$$

$$ks\xi + st\xi = 0, \tag{8}$$

$$hkX + thX = 0, \tag{9}$$

$$\widetilde{g}(hX, \xi) = \widetilde{g}(X, s\xi), \tag{10}$$

see ([Yano and Kon 1984]).

Let M^n be an n-dimensional submanifold of an $(n+m)$-dimensional Riemannian manifold $\widetilde{M}^{n+m}$. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad \text{and} \qquad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in TM^n$ and $N \in T^{\perp}M^n$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and normal connections in $\widetilde{M}$, M^n and the normal bundle $T^{\perp}M^n$ of M^n, respectively, and h is the second fundamental form related to the shape operator A_N by $g\left(h\left(X, Y\right), N\right) = g\left(A_N X, Y\right)$. The equation of Gauss is given by

$$\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) \tag{11}$$

$$-g\left(h(X, W), h(Y, Z)\right) + g\left(h(X, Z), h(Y, W)\right)$$

for all $X, Y, Z, W \in TM^n$, where R is the curvature tensor of M^n.

The mean curvature vector H is given by $H = \frac{1}{n}\,\mathrm{trace}(h)$. The submanifold M is *totally geodesic* in $\widetilde{M}^{n+m}$ if $h = 0$, and *minimal* if $H = 0$ ([Chen 1973]).
The Gauss equation for the submanifold M^n of a product space $(M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$ is

$$R(X, Y, Z, W) = \frac{c_1 + c_2}{4}[g(Y, Z)X - g(X, Z)Y$$
$$+g(FY, Z)FX - g(FX, Z)FY]$$
$$+\frac{c_1 - c_2}{4}[g(Y, Z)FX - g(X, Z)FY + g(Y, FZ)X - g(X, FZ)Y]$$
$$+g\left(h(X, W), h(Y, Z)\right) - g\ \left(h(X, Z), h(Y, W)\right). \qquad (12)$$

Let $\pi \subset T_x M^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n. For any orthonormal basis $\{e_1, ..., e_m\}$ of the tangent space $T_x M^n$, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

We recall the following algebraic Lemma:

Lemma 1 (Chen 1993)

Let $a_1, a_2, ..., a_n, b$ *be* $(n+1)$ $(n \geq 2)$ *real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$*, with equality holding if and only if* $a_1 + a_2 = a_3 = ... = a_n$.

Let M^n be an n-dimensional Riemannian manifold, L a k-plane section of T_xM^n, $x \in M^n$, and X a unit vector in L.
We choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$.
Ones define ([Chen 1999]) the *Ricci curvature* (or *k-Ricci curvature*) of L at X by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k on M^n is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad x \in M^n,$$

where L runs over all k-plane sections in T_xM^n and X runs over all unit vectors in L.

Recall that the *Chen first invariant* is given by

$$\delta_{M^n}(x) = \tau(x) - \inf\{K(\pi) \mid \pi \subset T_x M^n, x \in M^n, \dim \pi = 2\},$$

(see for example ([Chen 1993]), where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section, $\pi \subset T_x M^n, x \in M^n$ and τ is the scalar curvature at x.

For submanifolds of a product space $(M^{n_1}(c_1) \times M^{n_2}(c_2), \widetilde{g})$ we establish the following optimal inequality, which will call *Chen first inequality*.

Firstly we define

$$\Omega = g(fe_{2,} e_2) g(fe_{1,} e_1) - g^2(fe_{1,} e_2).$$

Theorem 4.1

Let $M^n, n \geq 3,$ be an n-dimensional submanifold of an (n_1+n_2)-dimensional product space $M^{n_1}(c_1) \times M^{n_2}(c_2)$. Then we have:

$$\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 \tag{13}$$

$$+\frac{c_1+c_2}{4}\left[-1-\Omega+\frac{\lambda}{2}\right]+\frac{c_1-c_2}{4}\left[(n-1)trf-(trf)_{|\pi}\right],$$

where

$$\lambda = n^2 - 2n + (trf)^2 + tr(sh)$$

and π is a 2-plane section of $T_xM^n, x \in M^n$.

The equality case of inequality (13) holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_x M^n$ and an orthonormal basis $\{e_{n+1}, ..., e_{n_1+n_2}\}$ of $T_x^{\perp} M^n$ such that the shape operators of M^n in $M^{n_1}(c_1) \times M^{n_2}(c_2)$ at x have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_{e_{n+i}} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 2 \leq i \leq n_1 + n_2 - n,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n + 1 \leq r \leq n_1 + n_2$.

Hence we can state the following corollaries:

Corollary 4.2

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $S^p(1) \times S^{m-p}(1)$. Then we have:

$$\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}\left[-1 - \Omega + \frac{\lambda}{2}\right]. \qquad (14)$$

Corollary 4.3

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $S^p(1) \times H^{m-p}(-1)$. Then we have:

$$\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}\left[(n-1)trf - (trf)_{|\pi}\right]. \quad (15)$$

Corollary 4.4

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $H^p(-1) \times H^{m-p}(-1)$. Then we have:

$$\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}\left[-1 - \Omega + \frac{\lambda}{2}\right]. \tag{16}$$

We first state a relationship between the sectional curvature of a submanifold M^n of an $(n_1 + n_2)$-dimensional product space $M^{n_1}(c_1) \times M^{n_2}(c_2)$ and the associated squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the Ricci curvature and k-Ricci curvature of M^n (intrinsic invariant) and the squared mean curvature $\|H\|^2$ (extrinsic invariant), as another answer of the basic problem in submanifold theory which we have mentioned in the introduction.

Denote by

$$N(x) = \{X \in T_x M^n : h(X, Y) = 0, \ \forall Y \in T_x M^n\}.$$

Theorem 5.1

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(n_1 + n_2)$-dimensional product space $M^{n_1}(c_1) \times M^{n_2}(c_2)$. Then the following statements are true.

i) For each unit vector X in $T_x M^n$ we have

$$Ric(X) \leq \frac{1}{4} n^2 \|H\|^2 + \frac{c_1 + c_2}{4} [n - 2 + tr(f) g(fX, X) + g(shX, X)] + \frac{c_1 - c_2}{4} [trf + (n-2) g(fX, X)]. \tag{17}$$

ii) *If* $H(x) = 0$ then a unit tangent vector X at x satisfies the equality case of (17) if and only if $X \in N(x)$.
iii) The equality case of inequality (17) holds identically for all unit tangent vectors at x if and only if either x is a totally geodesic point, or $n = 2$ and x is a totally umbilical point.

Hence we can state the following corollaries:

Corollary 5.2

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $S^p(1) \times S^{m-p}(1)$. Then for each unit vector X in $T_x M^n$ we have

$$Ric(X) \leq \frac{1}{4} n^2 \|H\|^2 + \frac{c_1 + c_2}{4} [n - 2 + tr(f) g(fX, X) + g(shX, X)]$$

Corollary 5.3

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $S^p(1) \times H^{m-p}(-1)$. Then we have:

$$Ric(X) \leq \frac{1}{4}n^2 \left\|H\right\|^2 + \frac{1}{2}[trf + (n-2)g(fX, X)].$$

Corollary 5.4

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $H^p(-1) \times H^{m-p}(-1)$. Then we have:

$$Ric(X) \leq \frac{1}{4}n^2 \left\| H \right\|^2 - \frac{1}{2}\left[n - 2 + tr(f)g(fX, X) + g(shX, X) \right].$$

Theorem 5.5

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(n_1 + n_2)$-dimensional product space $M^{n_1}(c_1) \times M^{n_2}(c_2)$. Then we have

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c_1 + c_2}{4n(n-1)}\lambda - \frac{c_1 - c_2}{2n} tr(f). \qquad (18)$$

Hence we can state the following corollaries:

Corollary 5.6

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $S^p(1) \times S^{m-p}(1)$. Then we have:

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{\lambda}{2n(n-1)}. \tag{19}$$

Corollary 5.7

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $S^p(1) \times H^{m-p}(-1)$. Then we have:

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{1}{n} tr(f). \tag{20}$$

Corollary 5.8

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $H^p(-1) \times H^{m-p}(-1)$. Then we have:

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} + \frac{\lambda}{2n(n-1)}. \tag{21}$$

Theorem 5.9

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(n_1 + n_2)$-dimensional product space $M^{n_1}(c_1) \times M^{n_2}(c_2)$. Then, for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have

$$\|H\|^2 \geq \Theta_k(p) - \frac{c_1 + c_2}{4n(n-1)}\lambda - \frac{c_1 - c_2}{2n}tr(f). \tag{22}$$

Hence we can state the following corollaries:

Corollary 5.10

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $S^p(1) \times S^{m-p}(1)$. Then we have:

$$\|H\|^2 \geq \Theta_k(p) - \frac{\lambda}{2n(n-1)}. \tag{23}$$

Corollary 5.11

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $S^p(1) \times H^{m-p}(-1)$. Then we have:

$$\|H\|^2 \geq \Theta_k(p) - \frac{1}{n} tr(f). \tag{24}$$

Corollary 5.12

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an m-dimensional product space $H^p(-1) \times H^{m-p}(-1)$. Then we have:

$$\|H\|^2 \geq \Theta_k(p) + \frac{\lambda}{2n(n-1)}. \tag{25}$$

Theorem 6.1

Let M^n, $n \geq 3$, be an n-dimensional submanifold of an $(n_1 + n_2)$-dimensional product space $M^{n_1}(c_1) \times M^{n_2}(c_2)$. Then

$$\tau(x) \leq \frac{1}{2} n^2 \|H\|^2 + \frac{c_1 + c_2}{8} \lambda + \frac{c_1 - c_2}{4} (n-1) tr(f).$$

The equality is satisfied if and only if x is a totally umbilical point.

References

Chen 1973 Chen, Bang-yen. Geometry of submanifolds. Pure and Applied Mathematics, No. 22. Marcel Dekker, Inc., New York, 1973.

Chen 1993 Chen, Bang-Yen. Some pinching and classification theorems for minimal submanifolds. Arch. Math. (Basel) 60 (1993), no. 6, 568-578.

Chen 1996 Chen, Bang-Yen. Mean curvature and shape operator of isometric immersions in real-space-forms. Glasgow Math. J. 38 (1996), no. 1, 87–97.

Chen 1999 B.-Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions, Glasgow Math. J. 41 (1999), 33-41.

Chen 2011 Chen, Bang-Yen. Pseudo-Riemannian geometry, δ-invariants and applications. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
Dillen and Kowalczyk 2012 Dillen, Franki; Kowalczyk, Daniel. Constant angle surfaces in product spaces. J. Geom. Phys. 62 (2012), no. 6, 1414–1432.
Yano-Kon 1984 Yano, Kentaro and Kon, Masahiro. Structures on manifolds. Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, 1984.