A New Characterization of Euler Elastica

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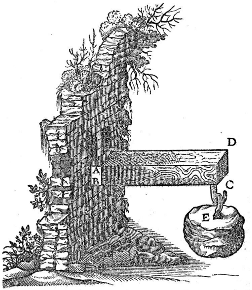
Outline

Elastic Behavior of Roads and Beams

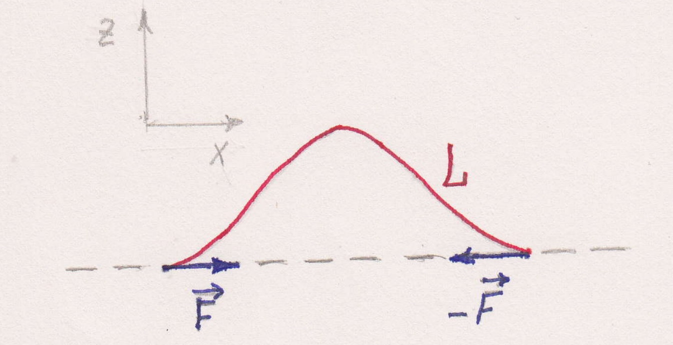
- Galileo Galilei (around 1638) asked the question about the force required to break a beam set into a wall.
- James Bernoulli (1687-1692) raised the question of the shape of a beam.
- James Bernoulli (1694) published the first solution of the rectangular elastica.
- Daniel Bernoulli (1742) proposed variational techniques for the problem of the elastica.
- Leonhard Euler (1743) solved the general problem and classified the elastica shapes.



(From the *Discorsi*, Leiden 1638.)

Figure 6.1: Galileo's 1638 formulation of the elastica.

Bending of an Elastic Rod with Fixed Length L
(elastica with tension)



Physics of the Elasticity of Rods I

Hook's Law (1660): When a thin elastic rod is bent every element of it undergoes a small strain ϵ that is in the same proportions with the stress σ in the rod

$$\sigma = E\epsilon$$

where E is the Young's modulus.

Physics of the Elasticity of Rods II

Daniel Bernoulli's Suggestion (1742): The work done in bending of an elastic rod (the energy of bending U) is proportional to the square of the curvature κ

$$U = IE\int_0^L \kappa^2(s)ds$$

where E is the Young's modulus and I is the moment of inertia of the cross-section about the neutral axis.

Mathematical Problem

Given a thin elastic rod what will be the shape of the rod when it is held by forces applied at its ends only?

Calculus of Variations Problem

Find the shape of the curve that minimizes the functional

$$J = \int_0^L \kappa^2(s)ds$$

subject to the constraint of a fixed length of the curve

$$L = const.$$

Fictitious Dynamical System
(Djondjorov, Hadzhilazova, Mladenov and Vassilev, 2008)

The nonlinearly coupled ordinary differential equations $(\lambda > 0)$

$$\ddot{x} - \lambda z\dot{z} = 0$$
$$\ddot{z} + \lambda z\dot{x} = 0$$

is equivalent to the equation

$$\ddot{\kappa}(s) + \frac{1}{2}\kappa^3(s) + \sigma\kappa(s) = 0$$

which is the intrinsic equation of the elastica with tension $(\mu \in \mathbb{R})$

$$\sigma = \lambda\mu$$

Fictitious Dynamical System
(Djondjorov, Hadzhilazova, Mladenov and Vassilev, 2008)

By integrating once the fictitious dynamical system takes the form

$$\dot{x} = \frac{\lambda z^2}{2} + \mu$$
$$\dot{z}^2 = -\frac{\lambda^2 z^4}{4} - \lambda\mu z^2 - \mu^2 + 1$$

where $\lambda > 0$, $\mu < 1$. In obtaining the above equations it was assumed that the particle trajectory is traced with unit speed

$$\dot{x}^2 + \dot{z}^2 = 1.$$

Euler Elastica for $\mu \in (-1, 1)$
(Djondjorov, Hadzhilazova, Mladenov and Vassilev, 2008)

$$x(s) = \frac{2}{\sqrt{\lambda}} E(\mathrm{am}(\sqrt{\lambda} s), k) - s, \qquad z(s) = a\,\mathrm{cn}(\sqrt{\lambda} s, k)$$

where

$$a = \sqrt{\frac{2(1-\mu)}{\lambda}}, \qquad k = \sqrt{\frac{1-\mu}{2}}$$

$E(u, k)$ incomplete elliptic integral of second order
$\mathrm{am}(u, k)$ Jacobian amplitude function
$\mathrm{cn}(u, k)$ Jacobian elliptic cosine function

Euler Elastica for $\mu = -1$
(Djondjorov, Hadzhilazova, Mladenov and Vassilev, 2008)

$$x(s) = \frac{2\tanh(\sqrt{\lambda}s)}{\sqrt{\lambda}} - s, \qquad z(s) = \frac{2\,\mathrm{sech}(\sqrt{\lambda}s)}{\sqrt{\lambda}}$$

Euler Elastica for $\mu < -1$
(Djondjorov, Hadzhilazova, Mladenov and Vassilev, 2008)

$$x(s) = aE(\mathrm{am}(\sqrt{\frac{\lambda(1-\mu)}{2}}s, k), k) + \mu s$$
$$z(s) = a\,\mathrm{dn}(\sqrt{\frac{\lambda(1-\mu)}{2}}s, k)$$

where

$$a = \sqrt{\frac{2(1-\mu)}{\lambda}}, \qquad k = \sqrt{\frac{2}{1-\mu}}$$

$E(u, k)$ incomplete elliptic integral of second order
$\mathrm{am}(u, k)$ Jacobian amplitude function
$\mathrm{dn}(u, k)$ Jacobian elliptic cosine function

Euler Elastica via the Weierstrassian Functions
(a new characterization)

$$x(s) = \frac{2}{\lambda}\left[2\zeta(s) + \frac{12\wp'(s)}{12\wp(s) - 2\lambda\mu + 3\lambda}\right] + \frac{2\mu}{3}s$$

$$z(s) = \frac{2(1-\mu)}{\lambda} \cdot \frac{12\wp(s) - 2\lambda\mu - 3\lambda}{12\wp(s) - 2\lambda\mu + 3\lambda}$$

where $\wp(s)$, $\wp'(s)$ and $\zeta(s)$ are the Weierstrassian functions

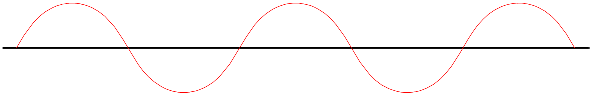
$$\wp(s) \equiv \wp(s; g_2, g_3), \quad \wp'(s) \equiv \wp'(s; g_2, g_3), \quad \zeta(s) \equiv \zeta(s; g_2, g_3)$$

with the invariants

$$g_2 = \frac{\lambda^2(4\mu^2 - 3)}{12}, \qquad g_3 = \frac{\lambda^3\mu(9 - 8\mu^2)}{216}$$

Case I

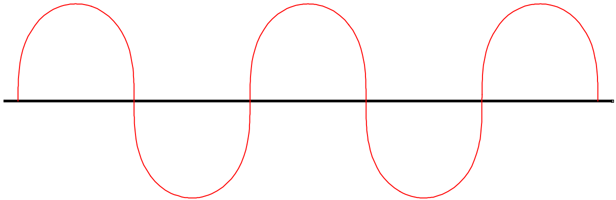
Euler Elastica for $\lambda = 4$, $\mu = 0.5$



Case II

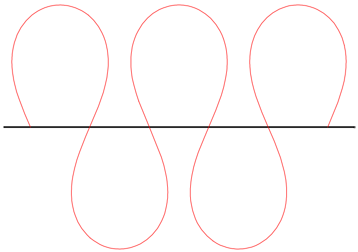
Euler Elastica for $\lambda = 4$, $\mu = 0$

(rectangular elastica)



Case III

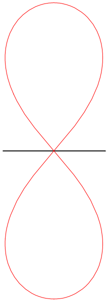
Euler Elastica for $\lambda = 4$, $\mu = -0.4$



Case IV

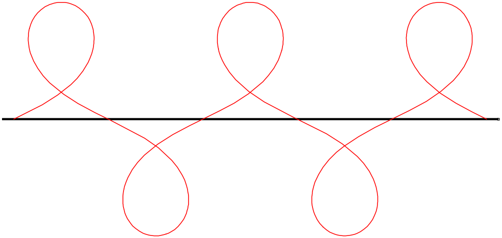
Euler Elastica for $\lambda = 4$, $\mu = -0.652231$

(figure eight)



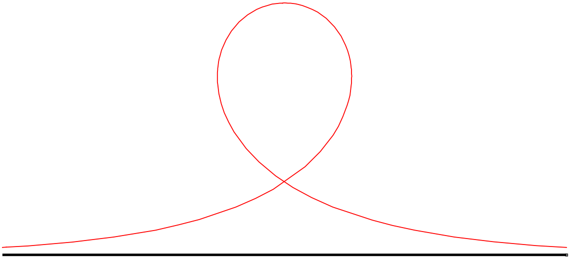
Case V

Euler Elastica for $\lambda = 4,\ \mu = -0.9$



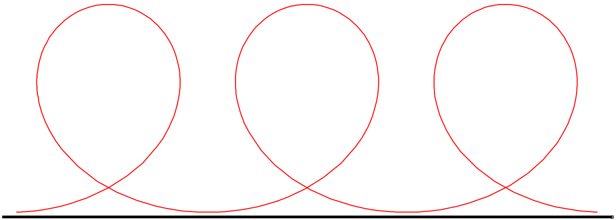
Case VI

Euler Elastica for $\lambda = 4$, $\mu = -1$

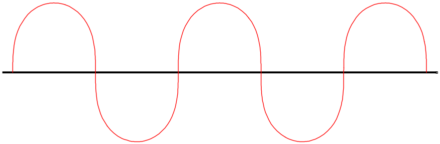


Case VII

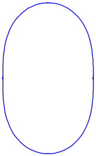
Euler Elastica for $\lambda = 4$, $\mu = -1.2$



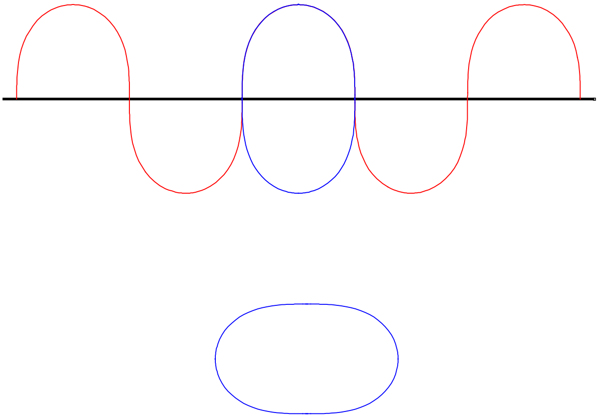
Rectangular Elastica ($\mu = 0$)



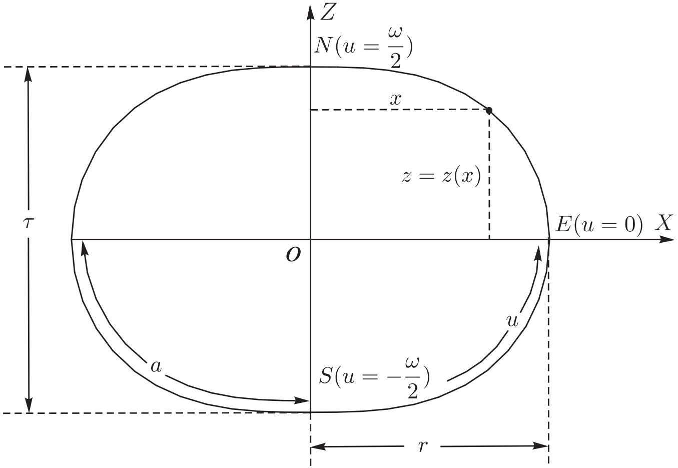
The Meridional Profile of the Mylar Balloon ($r = \sqrt{2/\lambda}$)



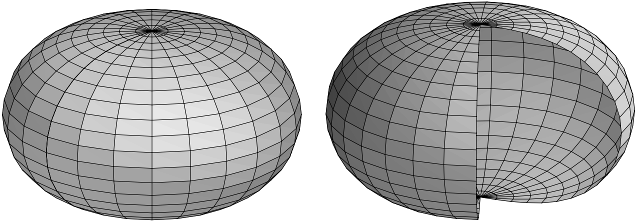
The meridional profile of the Mylar Balloon
is obtained by rotating at $\pi/2$ the Rectangular Elastica



The meridional profile of the Mylar Balloon



Two Views of the Mylar Balloon



The Profile of the Mylar Balloon
(Pulov, Hadzhilazova and Mladenov, 2014)
$(\mu = 0,\ r = \sqrt{2/\lambda})$

$$x(u) = r\frac{2\wp(u) - r^2}{2\wp(u) + r^2}$$

$$z(u) = 2\zeta(u) + \frac{2\wp'(u)}{2\wp(u) + r^2}$$

where $\wp(s)$, $\wp'(s)$ and $\zeta(s)$ are the Weierstrassian functions

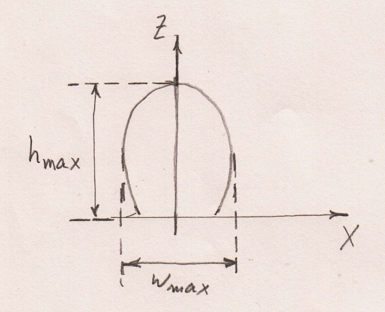
$$\wp(s) \equiv \wp(s; g_2, g_3), \quad \wp'(s) \equiv \wp'(s; g_2, g_3), \quad \zeta(s) \equiv \zeta(s; g_2, g_3)$$

with the invariants

$$g_2 = -r^4, \qquad g_3 = 0$$

Aspect Ratio: $\eta = h_{\max}/w_{\max}$

$$\eta(\mu) = \frac{\sqrt{1-\mu}}{\sqrt{2}\left(2E\left(\arccos\sqrt{-\mu/(1-\mu)},\sqrt{(1-\mu)/2}\right)-F\left(\arccos\sqrt{-\mu/(1-\mu)},\sqrt{(1-\mu)/2}\right)\right)}$$



The aspect ratio of the Mylar Balloon (rectangular elastica) is

$$\eta(0) = \tilde{\omega}/\pi$$

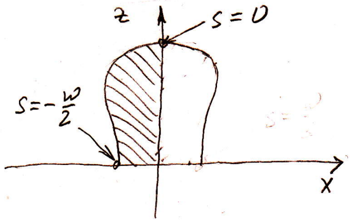
where

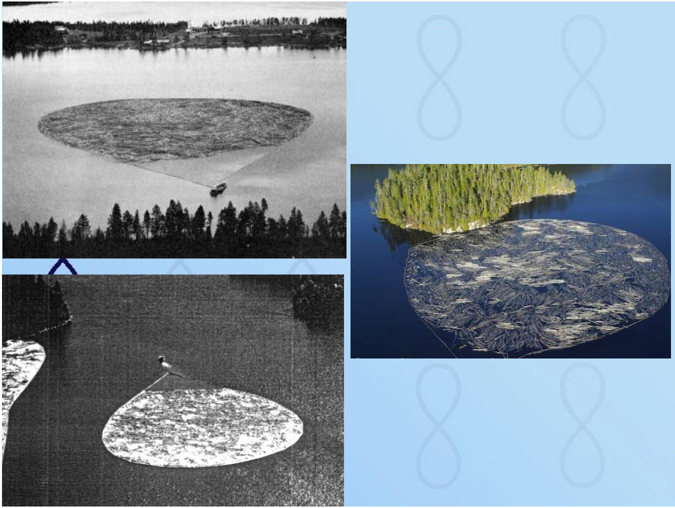
$$\tilde{\omega} \approx 2.6220$$

is the lemniscate constant.

The Total Area A_{tot}
bounded by the elastica and the X axis

$$A_{\mathrm{tot}} = 2\sqrt{1-\mu^2}/\lambda$$





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