Structured Relativistic Continuum. Spherically Symmetric Solutions

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MOTIVATION

Gravitation = dynamical space-time geometry (including perhaps some internal space)

In specially-relativistic physics space-time and elementary particles (fundamental quantum fields) are ruled by some space-time groups and their coverings:

$SO(1,3)^{\uparrow}$, restricted Lorentz — $SL(2,\mathbb{C})$

$O(1,3)^{\uparrow}$, orthochronous Lorentz — Pin

$e^{\mathbb{R}}\, SO(1,3)^{\uparrow}$ Weyl (linear-conformal) — $GL(2,\mathbb{C})^{+}$, Perhaps $GL(2,\mathbb{C})$ or $U(1)\times GL(2,\mathbb{C})^{+}$

$\mathcal{P} = SO(1,3)^{\uparrow} \times_s \mathbb{R}^4$

$SL(2,\mathbb{C}) \times_s \mathbb{R}^4$

etc., i.e., including reflections and dilatations

$CO(1,3)$ Minkowski-conformal

$\begin{cases} SU(2,2) \\ U(2,2) \end{cases}$

$GL(4,\mathbb{R})$

$\widetilde{GL(4,\mathbb{R})}$

(nonlinear !)

$Af(4,\mathbb{R}) = GL(4,\mathbb{R}) \times_s \mathbb{R}^4$

$\widetilde{GL(4,\mathbb{R})} \times_s \mathbb{R}^4$

$GL(4,\mathbb{C})$?

$Af(4,\mathbb{C}) = GL(4,\mathbb{C}) \times \mathbb{C}^4$?

All these groups are „external" symmetries acting on the argument of wave amplitudes and simultaneously – on the internal indices of these amplitudes through appropriate representations of the covering groups.

In „gravitational" physics – space-time M becomes an amorphous manifold; there are no longer „rigid" space-time groups. But at the same time, matter, which in special relativity was ruled by these groups is a source of gravitational field.

The only possibility: these groups should become internal symmetries, and just the gauge groups responsible for the gravitational field and for the matter producing it.

What a group should be used? This is not apriori clear, because everything that survives the transition from Minkowskian space-time to a manifold is the dimension four and the normal-hyperbolic signature.

A kind of „experimental physics" - trial and error method - - to check without any prejudices everything apriori possible.

The general pattern of „internalization" to be followed: the pointwise Lorentz symmetry becomes a rotation of local reference frames (tetrads) even in the curvilinear formulation of Minkowskian theory.

GENERALLY-RELATIVISTIC DIRAC THEORY AND GAUGE-POINCARE GRAVITY

M – space-time manifold, structure-less, $\dim M = 4$

Dynamical variables:

$\sim\ \Psi : M \to \mathbb{C}^4$, $\Psi^r(x^\mu)$ – Dirac field $\Big\}$ matter

$\sim\ M \ni x \mapsto e_x \in L(T_xM, \mathbb{R}^4)$, $e^A{}_\mu$ – cotetrad

$\sim\ M \ni x \mapsto \Gamma_x \in L(T_xM, SO(1,3)')$, $\Gamma^A{}_{B\mu}$,

an abstract connection on M, ruled by $SO(1,3)^\uparrow$ $\Bigg\}$ gravity, dynamical geometry

Target-space-geometries:

~ $\mathbb{R}^4$: $[\eta_{AB}] = \mathrm{Diag}(1,-1,-1,-1)$ - Minkowskian structure

~ $\mathbb{C}^4$: $[G_{\bar{r}s}] = \mathrm{Diag}(1,1,-1,-1)$ - neutral Hermitian form

$U(2,2) \subset GL(4,\mathbb{C})$, $U(2,2)' \subset L(4,\mathbb{C})$ - its symmetries

~ $L(4,\mathbb{C})$: $\gamma : \mathbb{R}^4 \hookrightarrow \underbrace{iU(2,2)'}_{H(4,G)}$ - Clifford injection

$$\gamma_A := \gamma \cdot \epsilon_A \,, \qquad \gamma^A\gamma^B + \gamma^B\gamma^A = 2\eta^{AB} I_4$$

$V := \gamma(\mathbb{R}^4) \subset i\,U(2,2)'$ - linear shell of γ^A-s

~ $H(4)$ - the space of Hermitian forms on $\mathbb{C}^4$:

Injection: $\Gamma : \mathbb{R}^4 \hookrightarrow H(4)$, $\Gamma_A := \Gamma \cdot \epsilon_A$,

where: $\Gamma^A{}_{\bar{r}s} := G_{\bar{r}z}\,\gamma^{Az}{}_s$

$\widetilde{V} := \Gamma(\mathbb{R}^4)$ - linear shell of Γ^A-s.

Byproducts of dynamical variables:

~ $g_{\mu\nu} := \eta_{AB} e^{A}{}_{\mu} e^{B}{}_{\nu}$ metric field

~ $\Gamma^{\alpha}{}_{\beta\mu} := e^{\alpha}{}_{A} \Gamma^{A}{}_{B\mu} e^{B}{}_{\beta} + e^{\alpha}{}_{A} e^{A}{}_{\beta,\mu}$ Einstein-Cartan affine connection, $\overset{\Gamma}{\nabla} g = 0$

~ $\omega_{\mu} := \frac{1}{8} \Gamma_{KL\mu} \left(\gamma^{K} \gamma^{L} - \gamma^{L} \gamma^{K} \right)$ bispinor connection

~ $\widetilde{\Psi}_{r} := \overline{\Psi}^{\bar{s}} G_{\bar{s}r}$ Dirac conjugation

~ $D_{\mu} \Psi := \partial_{\mu} \Psi + \omega_{\mu} \Psi$ covariant differentiation of bispinors

~ $e^{r}{}_{s\mu} := \gamma_{A}{}^{r}{}_{s} e^{A}{}_{\mu}$ V - valued cotetrad form, i.e., the „world Dirac matrices"

$$M \ni x \mapsto e_{x} \in L(T_{x}M, V) \subset L(T_{x}M, \underbrace{i\,U(2,2)'}_{H(4,G)})$$

~ $e_{\bar{r}s\mu} := G_{\bar{r}z} e^{z}{}_{s\mu}$ $\widetilde{V}$ - valued cotetrad form

Matter Lagrangian for the Dirac field

$$L_m(\Psi; e, \omega) :=$$

$$= \frac{i}{2} g^{\mu\nu} \boxed{e^{r}{}_{s\mu}} \left(\widetilde{\Psi}_r D_\nu \Psi^s - D_\nu \widetilde{\Psi}_r \Psi^s \right) \sqrt{|g|} - m \widetilde{\Psi}_r \Psi^r \sqrt{|g|} =$$

$$= \frac{i}{2} g^{\mu\nu} \boxed{e_{\bar{r} s\mu}} \left(\overline{\Psi}^{\bar{r}} D_\nu \Psi^s - D_\nu \overline{\Psi}^{\bar{r}} \Psi^s \right) \sqrt{|g|} - m\, G_{\bar{r}s} \overline{\Psi}^{\bar{r}} \Psi^s \sqrt{|g|}$$

Gravitational Lagrangians. Poincare-gauge models

$$L^{EC}_{gr}(e,\omega) = \frac{1}{k} g^{\mu\nu} R(\Gamma)^{\alpha}{}_{\mu\alpha\nu} \sqrt{|g|} \qquad \text{- Einstein-Cartan}$$

$$L^{YM}_{gr}(e,\omega) = \frac{1}{l} R^{\alpha}{}_{\beta\mu\nu} R^{\beta}{}_{\alpha\varkappa\lambda} g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} \qquad \text{- Yang-Mills}$$

$$L^{cosm}_{gr}(e,\omega) = \Lambda \sqrt{|g|}$$

$$L^{torsion}_{gr}(e,\omega) = A\, g_{\alpha\beta} g^{\mu\varkappa} g^{\nu\lambda} S^{\alpha}{}_{\mu\nu} S^{\beta}{}_{\varkappa\lambda} \sqrt{|g|} \; +$$

$$+ B g^{\mu\nu} S^{\alpha}{}_{\beta\mu} S^{\beta}{}_{\alpha\nu} \sqrt{|g|} + C g^{\mu\nu} S^{\alpha}{}_{\alpha\mu} S^{\beta}{}_{\beta\nu} \sqrt{|g|}$$

Weitzenböck terms, $S^{\mu}{}_{\nu\lambda} = \Gamma^{\mu}{}_{[\nu\lambda]}$

Total: $L(\Psi, e, \omega) = L_m(\Psi; e, \omega) + L_{gr}(e,\omega)$

Specially-relativistic limit: small excitations of vacuums:

$$\Psi = 0, \quad e^{A}{}_{\mu} = \delta^{A}{}_{\mu}, \quad \Gamma^{A}{}_{B\mu} = 0,$$

ruled by the usual Dirac Lagrangian:

$$L^{SR}_{m} = \frac{i}{2} \gamma^{\mu r}{}_{s} \left(\widetilde{\Psi}_{r} \partial_{\mu} \Psi^{s} - \partial_{\mu} \widetilde{\Psi}_{r} \Psi^{s} \right) - m \widetilde{\Psi}_{r} \Psi^{r}$$

Generally-covariant Dirac equation:

$$\boxed{i e^{\mu}{}_{A} \gamma^{A} \left(D_{\mu} + S^{\nu}{}_{\nu\mu} I_{4} \right) \Psi = m \Psi}$$

i.e.,

$$i e^{\mu r}{}_{s} \left(D_{\mu}{}^{s}{}_{z} + S^{\nu}{}_{\nu\mu} \delta^{s}{}_{z} \right) \Psi^{z} = m \Psi^{r}$$

Field equations for e, Γ - Poincare-gauge gravitation theory

Everything invariant under the local (x-dependent) $SL(2,\mathbb{C})$-group

$$A : M \to SL(2,\mathbb{C}), \quad L(A) : M \to SO(1,3)^{\uparrow}, \quad \text{where}$$

$$U(A)\,\gamma_K\, U(A)^{-1} = \gamma_M\, L(A)^M{}_K, \quad \text{and:}$$

$U : SL(2,\mathbb{C}) \hookrightarrow U(2,2)$ denotes the $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$-representation

$$[\psi^r] \mapsto [U(A)^r{}_s\, \psi^s],$$

$$[e^K{}_\mu] \mapsto [L(A)^K{}_M\, e^M{}_\mu],$$

$$[\Gamma^K{}_{M\mu}] \mapsto \left[L(A)^K{}_N\, \Gamma^N{}_{R\mu}\, L(A)^{-1\,R}{}_M - \frac{\partial L(A)^K{}_N}{\partial x^\mu}\, L(A)^{-1\,N}{}_M\right]$$

$$[\omega^r{}_{s\mu}] \mapsto \left[U(A)^r{}_z\, \omega^z{}_{t\mu}\, U(A)^{-1\,t}{}_s - \frac{\partial U(A)^r{}_z}{\partial x^\mu}\, U(A)^{-1\,z}{}_s\right]$$

This is $SL(2,\mathbb{C})$-ruled gauge theory :

~ $\omega^r{}_{s\mu}$ - connection form

~ ψ^r - matter field - associate bundle cross-section

Non-typical features of this gauge theory :

~ non-compact group $SL(2,\mathbb{C})$

~ much more strange - dynamical use of the tetrad.
Its meaning :

- reference frame
- object which establishes a bundle monomorphism of an abstract principal $SO(1,3)^{\uparrow}$ bundle over M into the bundle of linear frames FM; more detailly - onto some $SO(1,3)^{\uparrow}$-reduction of FM. This reduction is dynamical, non-fixed (belongs to degrees of freedom).

U(2,2) - GAUGE MODEL WITH THE SECOND-ORDER WAVE EQUATION

Dynamical variables:

~ $\psi : M \to \mathbb{C}^4$, $\psi^r(x^\mu)$ - Dirac field — material sector

~ $g_{\mu\nu}$ - normal-hyperbolic metric on M

~ $M \ni x \mapsto \vartheta_x \in L(T_x M, U(2,2)')$, $\vartheta^r{}_{s\mu}(x)$

U(2,2) - ruled connection, gauge field — geometrodynamical sector

Their concomitants:

~ Covariant derivative of wave functions:

$$\nabla_\mu \Psi = \partial_\mu \Psi + g\vartheta_\mu \Psi + \frac{q-g}{4} \operatorname{Tr}\vartheta_\mu \Psi$$

g, q - coupling constants

~ Curvature two-form:

$$\Phi_{\mu\nu} = \nabla\vartheta_{\mu\nu} = \partial_\mu\vartheta_\nu - \partial_\nu\vartheta_\mu + g\left[\vartheta_\mu, \vartheta_\nu\right]$$

U(2,2)-gauge-invariant matter Lagrangian (Klein-Gordon):

$$\boxed{L_m(\Psi;\vartheta,g) := \frac{b}{2} g^{\mu\nu} \nabla_\mu \widetilde{\Psi} \nabla_\nu \Psi \sqrt{|g|} - \frac{c}{2} \widetilde{\Psi} \Psi \sqrt{|g|}}\,,$$

i.e.,

$$\boxed{L_m(\Psi;\vartheta,g) = \frac{b}{2} g^{\mu\nu} \nabla_\mu \overline{\Psi}^{\bar{r}} \nabla_\nu \Psi^s G_{\bar{r}s} \sqrt{|g|} - \frac{c}{2} G_{\bar{r}s} \overline{\Psi}^{\bar{r}} \Psi^s \sqrt{|g|}}$$

~ <u>Connection dynamics:</u> $L_{YM}(\vartheta, g) :=$

$$= \frac{a}{4}\mathrm{Tr}\left(\phi_{\mu\nu}\,\phi_{\varkappa\lambda}\right) g^{\mu\varkappa} g^{\nu\lambda}\sqrt{|g|} + \frac{a'}{4}\mathrm{Tr}\left(\phi_{\mu\nu}\right)\mathrm{Tr}\left(\phi_{\varkappa\lambda}\right) g^{\mu\varkappa} g^{\nu\lambda}\sqrt{|g|}$$

~ <u>Hilbert-Einstein term:</u> $L_{HE}(g) = -d\,R(g)\sqrt{|g|} + \ell\sqrt{|g|}$

Vanishing values $d = 0$, $\ell = 0$ - allowed (Palatini-like schem

Total: $L(\psi; \vartheta, g) = L_m(\psi; \vartheta, g) + L_{YM}(\vartheta, g) + L_{HE}(g)$.

Field equations:

$g^{\mu\nu} \overset{g}{\nabla}_\mu \overset{g}{\nabla}_\nu \Psi + \frac{c}{b} \Psi = 0$	matter, wave equation
$\chi^{\mu\nu}{}_{;\nu} + g[\vartheta_\nu, \chi^{\mu\nu}] = g J^\mu + \frac{q-g}{4} \operatorname{Tr} J^\mu I_4$ $d\left(R(g)^{\mu\nu} - \frac{1}{2} R(g) g^{\mu\nu}\right) = -\frac{\ell}{2} g^{\mu\nu} + \frac{1}{2} T^{\mu\nu}$	geometrodynamics, gravitation

where:

~ $\chi^{\mu\nu} := \dfrac{\partial L_{YM}}{\partial \vartheta_{\mu,\nu}} = -a \Phi^{\mu\nu}\sqrt{|g|} - a' I_4 \operatorname{Tr} \Phi^{\mu\nu} \sqrt{|g|}$

field momentum

~ $T^{\mu\nu}$ – metrical energy-momentum tensor of (Ψ, ϑ)

~ $J_\mu := \frac{b}{2}\left(\Psi \nabla_\mu \widetilde{\Psi} - \nabla_\mu \Psi \widetilde{\Psi}\right)\sqrt{|g|}$ – $U(2,2)$-current of matte

~ $\overset{g}{\nabla}_\mu$ – total covariant differentiation unifying the internal ϑ-connection (r-indices) and the external Levi-Civita connection $\{g\}$ (μ-indices)

Reduction to subgroups $SL(2,\mathbb{C})$, $GL(2,\mathbb{C})$.

The adapted basis of $U(2,2)'$:

$$i\gamma^A, \quad i\,{}^A\gamma, \quad \Sigma_{AB} \ (A<B), \quad i\gamma^5, \quad i I_4,$$

where:

$$\gamma^5 := -\gamma^0\gamma^1\gamma^2\gamma^3, \quad {}^A\gamma := -i\gamma^5\gamma^A, \quad \Sigma^{AB} := \frac{1}{4}\left(\gamma^A\gamma^B - \gamma^B\gamma^A\right)$$

$$\{{}^A\gamma, {}^B\gamma\} = -2\eta^{AB} I_4 \quad \text{(inverted signature)}$$

More convenient: $\tau_A := \frac{1}{2}(\gamma_A + {}_A\gamma)$, $\chi^A := \frac{1}{2}(\gamma^A - {}^A\gamma)$

(twistor translations) (twistor conformal boosts)

$$\vartheta_\mu = \frac{1}{2g}\check{\Gamma}^{AB}{}_\mu \Sigma_{AB} + \frac{1}{4g} Q_\mu \frac{1}{i}\gamma^5 + A_\mu i I_4 + e^A{}_\mu i \tau_A + f_{A\mu} i x^A$$

$$= \frac{1}{2g}\check{\Gamma}^{AB}{}_\mu \Sigma_{AB} + \frac{1}{4g} Q_\mu \frac{1}{i}\gamma^5 + A_\mu i I_4 + E^A{}_\mu i \gamma_A + F^A{}_\mu i {}^A\gamma$$

where: $E^A{}_\mu := \frac{1}{2}(e^A{}_\mu + f^A{}_\mu)$, $F^A{}_\mu := \frac{1}{2}(e^A{}_\mu - f^A{}_\mu)$

$GL(2,\mathbb{C})$ – interpretation:

$e^A{}_\mu$ – co-tetrad, if $\det[e^A{}_\mu] \neq 0$

$f_{A\mu}$ – auxiliary cotetrad

homogeneous transformation rule under $GL(2,\mathbb{C})$

$\check{\Gamma}^A{}_{B\mu}$ – Einstein-Cartan connection in e-representation

$\Gamma^A{}_{B\mu} = \check{\Gamma}^A{}_{B\mu} + \frac{1}{2} Q_\mu \delta^A{}_B$ – Einstein-Cartan-Weyl connection

Q_μ – Weyl covector

inhomogeneous, connection-like transformation rule under $GL(2,\mathbb{C})$

$$\boxed{\begin{aligned}\nabla_\mu \Psi &= D_\mu \Psi + g\, e^{A}{}_{\mu}\, i\tau_A \Psi + g f_{A\mu}\, i\chi^A \Psi = \\ &= D_\mu \Psi + g\, E^{A}{}_{\mu}\, i\gamma_A \Psi + g\, F^{A}{}_{\mu}\, i\, {}^{A}\gamma \Psi\end{aligned}},$$

where D is the $GL(2,\mathbb{C})$-covariant differentiation of spinors,

$$\boxed{\begin{aligned}D_\mu \Psi &:= \partial_\mu \Psi + \frac{1}{2}\check{\Gamma}^{AB}{}_{\mu}\Sigma_{AB}\Psi + \frac{1}{4}Q_\mu \frac{1}{i}\gamma^5 \Psi + qA_\mu i\Psi = \\ &= \partial_\mu \Psi + \frac{1}{2}\Gamma^{AB}{}_{\mu}\left(\Sigma_{AB} + \frac{1}{4}\eta_{AB}\frac{1}{i}\gamma^5\right) + qA_\mu i \Psi\end{aligned}}$$

If $\det\left[e^{A}{}_{\mu}\right] \neq 0$, $\det\left[f_{A\mu}\right] \neq 0$, ϑ gives rise to affine connections,

$$\boxed{\begin{aligned}\Gamma(e)^{\alpha}{}_{\mu\nu} &:= e^{\alpha}{}_{A}\,\Gamma^{A}{}_{B\nu}\,e^{B}{}_{\mu} + e^{\alpha}{}_{A}\,e^{A}{}_{\mu,\nu}\\ \Gamma(f)^{\alpha}{}_{\mu\nu} &:= -f_{A\mu}\,\Gamma^{A}{}_{B\nu}\,f^{\alpha B} + f^{\alpha A}\,f_{A\mu,\nu}\end{aligned}}$$

$S(e)^{\alpha}{}_{\mu\nu} := \Gamma(e)^{\alpha}{}_{[\mu\nu]}$, $S(f)^{\alpha}{}_{\mu\nu} := \Gamma(f)^{\alpha}{}_{[\mu\nu]}$ – torsions

$R(e)^{\alpha}{}_{\varkappa\mu\nu}$, $R(f)^{\alpha}{}_{\varkappa\mu\nu}$ – curvature tensors of $\Gamma(e)^{\alpha}{}_{\mu\nu}$, $\Gamma(f)^{\alpha}{}_{\mu\nu}$

$\Phi = \nabla\vartheta$ is an algebraic function of $e, f, S(e), S(f), R(e), R(f)$.

$K(e)^{\alpha}{}_{\mu\nu} := \Gamma(e)^{\alpha}{}_{\mu\nu} - {}^{g}\left\{ {\alpha \atop \mu\nu} \right\}$, $K(f)^{\alpha}{}_{\mu\nu} := \Gamma(f)^{\alpha}{}_{\mu\nu} - {}^{g}\left\{ {\alpha \atop \mu\nu} \right\}$

$SL(2,\mathbb{C})$ – invariant tensors, built algebraically of ϑ:

$$\boxed{h(e)_{\mu\nu} := \eta_{AB}\,e^{A}{}_{\mu}\,e^{B}{}_{\nu}\ , \quad h(f)_{\mu\nu} := \eta^{AB}\,f_{A\mu}\,f_{B\nu}}\ ,$$

$t(e,f)_{\mu\nu} := e^{A}{}_{\mu}\,f_{A\nu}$ ($GL(2,\mathbb{C})$ – invariant).

$GL(2,\mathbb{C})$ - adapted representation of Lagrangians:

$$L_m(\psi;\vartheta,g) = \underbrace{bg\frac{i}{2}g^{\mu\nu}E^K{}_\mu\left(D_\nu\tilde{\psi}\gamma_K\psi - \tilde{\psi}\gamma_K D_\nu\psi\right)\sqrt{|g|}}_{\text{Dirac term; signature } (+---)} +$$

$$+\underbrace{bg\frac{i}{2}g^{\mu\nu}F^K{}_\mu\left(D_\nu\tilde{\psi}\,{}_K\gamma\,\psi - \tilde{\psi}\,{}_K\gamma\, D_\nu\psi\right)\sqrt{|g|}}_{\text{Anti-Dirac term; inverted signature } (-+++} +$$

$$+\underbrace{bg\tilde{\psi}W\psi\sqrt{|g|}}_{\text{Algebraic ,,mass'' term}} + \underbrace{\frac{b}{2}g^{\mu\nu}D_\mu\tilde{\psi}D_\nu\psi\sqrt{|g|}}_{\text{Klein-Gordon term}}$$

where:

$$W = \frac{g}{2}g^{\mu\nu}e^K{}_\mu f_{K\nu}I_4 - \frac{c}{2bg}I_4 - \frac{g}{2}ig^{\mu\nu}e^K{}_\mu f_{L\nu}\varepsilon_K{}^L{}_{AB}\Sigma^{AB}$$

$L_{YM}(\vartheta,g)$ = terms quadratic in $R(e)$ + terms linear in $R(e)$ +
+ terms bilinear in $S(e)$, $S(f)$ + terms algebraic in e, f

Substituting Einstein-Cartan Ansatz:

$$f_{K\mu} = \eta_{KM} e^{M}{}_{\mu} \quad , \quad g_{\mu\nu} = h(e)_{\mu\nu} = \eta_{AB} e^{A}{}_{\mu} e^{B}{}_{\nu} \, ,$$

one obtains:

$$L_m(\Psi;\vartheta,g)|_E = \underbrace{bg\frac{i}{2} g^{\mu\nu} e^{K}{}_{\mu} \left(D_\nu \widetilde{\Psi} \gamma_K \Psi - \widetilde{\Psi} \gamma_K D_\nu \Psi \right) \sqrt{|g|}}_{\text{Dirac term}} +$$

$$+ \underbrace{\left(2bg^2 - \frac{c}{2}\right) \widetilde{\Psi} \Psi \sqrt{|g|}}_{\text{algebraic mass term}} + \underbrace{\frac{b}{2} g^{\mu\nu} D_\mu \widetilde{\Psi} D_\nu \Psi \sqrt{|g|}}_{\text{Klein-Gordon term}}$$

$$\boxed{\begin{aligned} L_{YM}(\vartheta,g)|_E = {} & \frac{a}{8g^2} R^{\varkappa}{}_{\lambda\mu\nu} R^{\lambda}{}_{\varkappa}{}^{\mu\nu} \sqrt{|g|} + 2a R^{\mu}{}_{\nu\mu}{}^{\nu} \sqrt{|g|} + \\ & - 4a S^{\varkappa}{}_{\mu\nu} S_{\varkappa}{}^{\mu\nu} \sqrt{|g|} - 48 a g^2 \sqrt{|g|} \end{aligned}}$$

Geometrodynamical sector in the case when e, f are frames:

$$2S(e)^{\mu\nu\varkappa}{}_{;\varkappa} + 2K(e)^{\mu}{}_{\lambda\varkappa}\, S(e)^{\lambda\nu\varkappa} + \widetilde{h(e)}^{\mu\lambda} h(e)_{\varkappa\rho}\, R(e)^{\rho}{}_{\lambda}{}^{\varkappa\nu} +$$

$$-\frac{1}{2}R(e)^{\varkappa}{}_{\varkappa}{}^{\mu\nu} - 2g^2 t^{\varkappa}{}_{\varkappa}\, g^{\mu\nu} + 2g^2 t^{\mu\nu} + 2g^2 h(e)_{\varkappa}{}^{\nu}\, \widetilde{h(e)}^{\lambda\mu} t_{\lambda}{}^{\varkappa} +$$

$$-2g^2 h(e)_{\varkappa}{}^{\varkappa}\, \widetilde{h(e)}^{\mu\lambda} t_{\lambda}{}^{\nu} - 2g^2 t^{\nu\mu} + 2g^2 t^{\mu\nu} = g\frac{b}{2a}\tau^{\mu\nu},$$

$$2S(f)^{\mu\nu\varkappa}{}_{;\varkappa} - 2S(f)^{\lambda\nu\varkappa} K(f)_{\lambda}{}^{\mu}{}_{\varkappa} + \widetilde{h(f)}^{\mu\lambda} h(f)_{\varkappa\rho}\, R(f)^{\rho}{}_{\lambda}{}^{\varkappa\nu} +$$

$$-\frac{1}{2}R(f)^{\varkappa}{}_{\varkappa}{}^{\mu\nu} - 2g^2 t^{\varkappa}{}_{\varkappa}\, g^{\mu\nu} + 2g^2 t^{\nu\mu} + 2g^2 h(f)_{\varkappa}{}^{\nu}\, \widetilde{h(f)}^{\nu\lambda} t^{\varkappa}{}_{\lambda} +$$

$$-2g^2 h(f)_{\varkappa}{}^{\varkappa}\, \widetilde{h(f)}^{\mu\lambda} t^{\nu}{}_{\lambda} - 2g^2 t^{\mu\nu} + 2g^2 t^{\nu\mu} = g\frac{b}{2a}\,{}^{\mu\nu}\tau,$$

$$\widetilde{R}(e)^{\varkappa}{}_{\mu}{}^{\nu\varrho}{}_{;\varrho} + K(e)^{\varkappa}{}_{\varrho\lambda}\widetilde{R}(e)^{\varrho}{}_{\nu}{}^{\mu\lambda} - \widetilde{R}(e)^{\varkappa}{}_{\varrho}{}^{\nu\lambda} K(e)^{\varrho}{}_{\nu\lambda} +$$

$$-4g\widetilde{h(e)}^{\varkappa\varrho} h(e)_{\mu\sigma} t_{\varrho\lambda} S(f)^{\lambda\nu\sigma} + 4g t_{\mu\lambda} S(f)^{\lambda\nu\varkappa} - 4g t_{\mu\sigma} S(e)^{\varkappa\nu\sigma} +$$

$$+4g\widetilde{h(e)}^{\varkappa\sigma} h(e)_{\mu\lambda} t_{\sigma\varrho} S(e)^{\lambda\nu\varrho} = -g\frac{b}{2a}\tau^{\varkappa}{}_{\mu}{}^{\nu},$$

where :

$$\widetilde{R}(e)^{\varkappa}{}_{\varrho\mu\nu} := \frac{1}{g}R(e)^{\varkappa}{}_{\varrho\mu\nu} - \frac{1}{4g}\delta^{\varkappa}{}_{\varrho} R(e)^{\lambda}{}_{\lambda\mu\nu} - 2g\delta^{\varkappa}{}_{\mu} t_{\varrho\nu} +$$

$$+2g\delta^{\varkappa}{}_{\nu} t_{\varrho\mu} + 2g h(e)_{\varrho\mu}\widetilde{h(e)}^{\sigma\varkappa} t_{\sigma\nu} - 2g h(e)_{\varrho\nu}\widetilde{h(e)}^{\sigma\varkappa} t_{\sigma\mu},$$

$$\left(\frac{1}{8}R(e)^{\lambda}{}_{\lambda}{}^{\mu\nu} - g^{2}\left(t^{\mu\nu} - t^{\nu\mu}\right)\right)_{;\nu} + 2g^{2}\left(t_{\nu\lambda} - t_{\lambda\nu}\right) S(f)^{\lambda\mu\nu} = -g^{2}\frac{b}{2a}\tau^{\mu}$$

$$\left(1 + \frac{4a'}{a}\right) F^{\mu\nu}{}_{;\nu} = -q\frac{b}{2a}\tau^{\mu\prime}$$

and the source terms are given by the following equations:

$$J(\Psi,\vartheta,g) = {}^{A}J\, i\tau_A + J_A\, i\chi^A + \frac{1}{2}J^{AB}\Sigma_{AB} + J\frac{1}{i}\gamma^5 + J' i I_4 \, ,$$

$${}^{A}J_\mu = \frac{b}{2}{}^{A}\tau_\mu\sqrt{|g|} \, , \qquad J_{A\mu} = \frac{b}{2}\tau_{A\mu}\sqrt{|g|} \, ,$$

$$J^{AB}{}_\mu = \frac{b}{2}\tau^{AB}{}_\mu\sqrt{|g|} \, , \quad J_\mu = \frac{b}{2}\tau_\mu\sqrt{|g|} \, , \quad J'_\mu = \frac{b}{2}\tau'_\mu\sqrt{|g|} \, ,$$

$${}^{A}\tau_\mu = \frac{1}{2i}\left(D_\mu\widetilde{\Psi}\,\chi^A\Psi - \widetilde{\Psi}\chi^A D_\mu\Psi\right) - \frac{g}{2}e^A{}_\mu\widetilde{\Psi}\Psi - \frac{gi}{2}\varepsilon^A{}_{BCD}\, e^B{}_i\,\widetilde{\Psi}\Sigma^{CD}\Psi$$

$$\tau_{A\mu} = \frac{1}{2i}\left(D_\mu\widetilde{\Psi}\tau_A\Psi - \widetilde{\Psi}\tau_A D_\mu\Psi\right) - \frac{g}{2}f_{A\mu}\widetilde{\Psi}\Psi + \frac{gi}{2}\varepsilon_A{}^B{}_{CD}\, f_{B\mu}\widetilde{\Psi}\Sigma^{CD}\Psi ,$$

$$\tau^{AB}{}_\mu = -\left(D_\mu\widetilde{\Psi}\Sigma^{AB}\Psi - \widetilde{\Psi}\Sigma^{AB}D_\mu\Psi\right) + g\,\varepsilon^{AB}{}_{CD}\, f_{C\mu}\widetilde{\Psi}\chi^D\Psi +$$

$$- g\,\varepsilon^{AB}{}_C{}^D\, e^C{}_\mu\,\widetilde{\Psi}\tau_D\Psi \, .$$

$$\tau_\mu = \frac{1}{4i}\left(D_\mu \widetilde{\psi}\gamma^5\psi - \widetilde{\psi}\gamma^5 D_\mu \psi\right),$$

$$\tau_\mu{}' = \frac{i}{4}\left(\widetilde{\psi} D_\mu \psi - D_\mu \widetilde{\psi}\psi\right) - \frac{g}{2} e^A{}_\mu \widetilde{\psi}\tau_A \psi - \frac{g}{2} f_{A\mu}\widetilde{\psi}\chi^A\psi,$$

$$\tau^\mu{}_\nu := e^\mu{}_A{}^A\tau_\nu, \quad {}^\mu{}_\nu\tau := f^{\mu A}\tau_{A\nu}, \quad \tau^\varkappa{}_{\mu\nu} := e^\varkappa{}_A e^B{}_\mu \tau^A{}_{B\nu}$$

Geometrodynamical sector-correspondence with standard theories.

Substituting to field equations the empty-space condition ~~and~~ th
Einsteinian Ansatz:

$$\Psi=0,\quad f_{A\mu}=\eta_{AB}e^{B}{}_{\mu};\quad g_{\mu\nu}=h(e)_{\mu\nu},\quad S(e)^{\lambda}{}_{\mu\nu}=S(f)^{\lambda}{}_{\mu\nu}=0,\quad Q_{\mu}=0,\ A'_{\mu}:$$

one reduces the Yang-Mills equations to:

$$\boxed{R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}=-12g^{2}g^{\mu\nu}}$$

The weakened Einsteinian Ansatz:

$$\Psi=0,\quad f_{A\mu}=k\eta_{AB}e^{B}{}_{\mu},\quad g_{\mu\nu}=p\,h(e)_{\mu\nu},\quad S(e)^{\lambda}{}_{\mu\nu}=S(f)^{\lambda}{}_{\mu\nu}=0,\quad Q_{\mu}=0,\ A_{\mu}$$

gives:

$$\boxed{R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}=-\frac{12g^{2}k}{p}g^{\mu\nu}}$$

Compatibility condition with: $\frac{\delta}{\delta g_{\mu\nu}} \int L = 0$ reads:

$$lp = 24 g^2 dk \ ; \quad \left(lp - 24 g^2 dk\right) h_{\mu\nu} = {}^{\prime}T_{\mu\nu}$$

„Cosmological" term controlled by the SU(2,2)-coupling constant g!!!

The pure gauge vacuum solutions, $\boxed{\Psi = 0, \quad \Phi = \nabla\vartheta = 0,}$

become then constant-curvature-spaces,

$$\boxed{R_{\alpha\beta\mu\nu} = \frac{4g^2k}{p}\left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}\right)}$$

Conformally flat; compatible with the idea of conformal symmetry underlying this model.

Wave equation in $SL(2,\mathbb{C})$-representation becomes:

$$\boxed{i\gamma^A \pounds_{E_A}\Psi + i\,{}^A\gamma\,\pounds_{F_A}\Psi - W\Psi + \frac{1}{2g}g^{\mu\nu}\overset{g}{D}_\mu\overset{g}{D}_\nu\Psi = 0}$$

where $\overset{g}{D}$ unifies the Levi-Civita differentiation with the ω-differentiation of bispinors and $\Gamma^A{}_B$-differentiation of objects with capital Lorentz indices A.

$$\left.\begin{aligned} \pounds_{E_A}\Psi &:= E^\mu{}_A D_\mu\Psi + \frac{1}{2}\left(\overset{g}{D}_\mu E^\mu{}_A\right)\Psi, \\ \pounds_{F_A}\Psi &:= F^\mu{}_A D_\mu\Psi + \frac{1}{2}\left(\overset{g}{D}_\mu F^\mu{}_A\right)\Psi \end{aligned}\right\} \text{,, covariant Lie derivatives}$$

Correspondence with standard theory becomes readible under th substitution of the Einstein-Cartan Ansatz:

$$\boxed{f_{A\mu} = \eta_{AB}\, e^{B}{}_{\mu} \quad , \quad g_{\mu\nu} = h(e)_{\mu\nu} = \eta_{AB}\, e^{A}{}_{\mu}\, e^{B}{}_{\nu}} \quad ,$$

$$\underbrace{e^{\mu}{}_{A}\, i\gamma^{A}\left(D_{\mu} + S^{\nu}{}_{\nu\mu} I_{4}\right)\Psi}_{\text{Dirac term}} - \underbrace{\frac{4bg^{2}-c}{2bg}\Psi}_{\text{algebraic term}} + \underbrace{\frac{1}{2g} g^{\mu\nu} \overset{g}{D}_{\mu} \overset{g}{D}_{\nu}}_{\text{d'Alembert operator}} \Psi = 0$$

Specially-relativistic limit: $e^{\mu}{}_{A} = \delta^{\mu}{}_{A}$, $\Gamma^{A}{}_{B\mu} = 0$, $g_{\mu\nu} = \eta_{\mu\nu}$.

$$\underbrace{i\gamma^{\mu}\partial_{\mu}\Psi}_{\text{Dirac operator}} - \underbrace{\frac{4bg^{2}-c}{2bg}\Psi}_{\text{algebraic term}} + \underbrace{\frac{1}{2g}\partial^{\mu}\partial_{\mu}}_{\text{d'Alembert operator}}\Psi = 0$$

Dirac-Klein-Gordon equation. May it be useful?

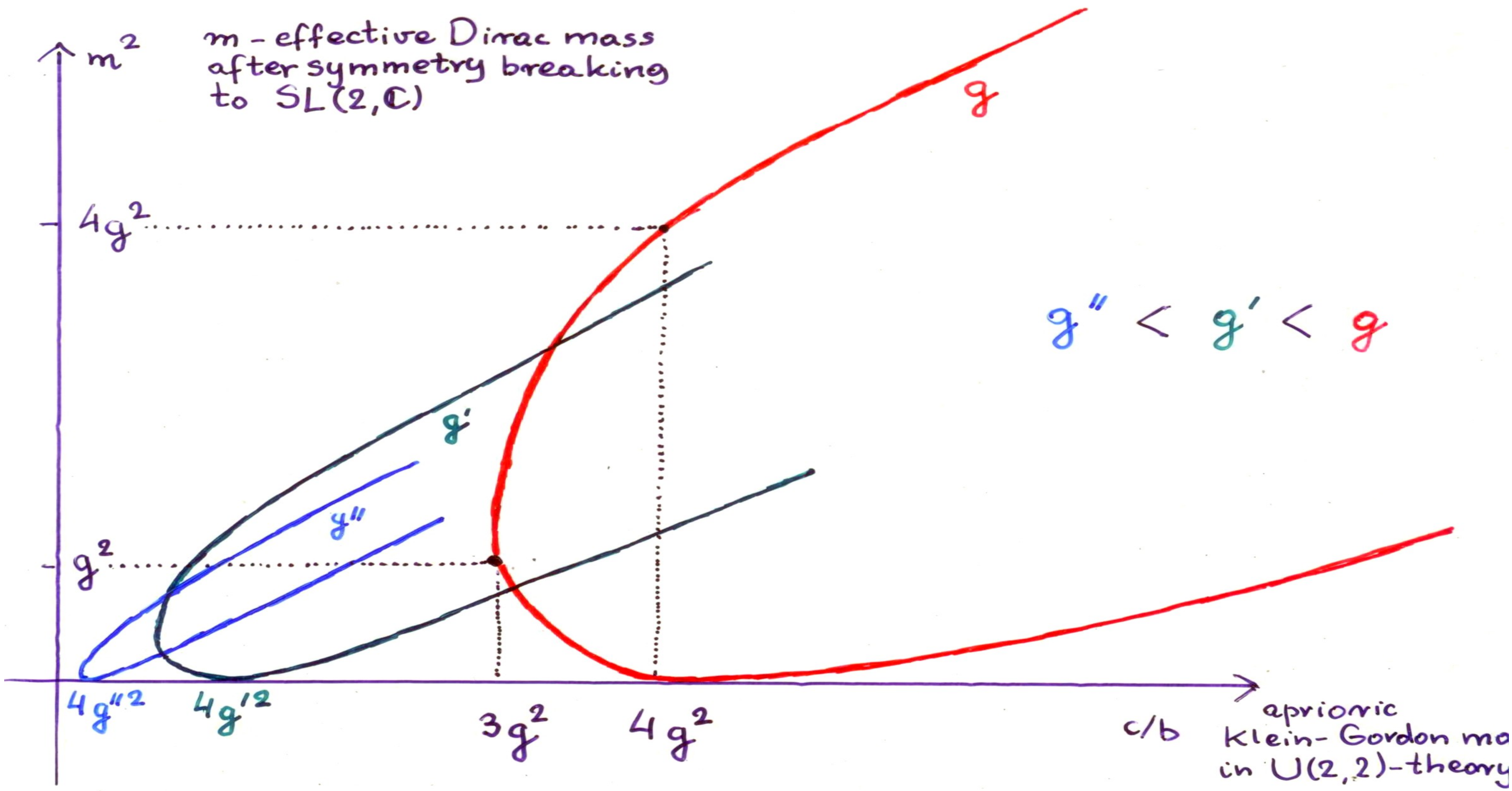
Is it possible to apply the DKG equation as a field model describing fundamental fermions? The crucial point: d'Alembert operator status. Is it admissible at all?

General solution of specially-relativistic D-K-G equation is a superposition of Dirac waves with two masses:

$$\boxed{m_{\pm}^2 = \frac{c}{b} - 2g^2\left(1 \pm \sqrt{\frac{c}{bg^2} - 3}\right)}$$

There is a physical range of approximately Dirac behaviour.

For $\frac{c}{b} = 3g^2$ – exactly Dirac – no mass splitting.



Diagrams of m^2 vrs. c/b for various values of g.

~ Threshold of the effective Dirac behaviour: $c/b = 3g^2$.
There is only one effective mass $m = |g|$.

~ Below the threshold: tachyonic and decay phenomena, no Dirac behaviour

~ Above the threshold: effective Dirac behaviour with two masses possible

~ $c/b = 4g^2$ - one partner is massless, $m_- = 0$, $m_+ = 2|g|$

The mass doubling and experiment:

~ if the energetic gap $m_+ - m_-$ is very small (small $|g|$), perhaps it is below the present accuracy of our experiment

~ if the energetic gap $m_+ - m_-$ is very large, perhaps it is too difficult to excite the higher mass state m_+

~ perhaps the doubling of mass states just underlies the mysterious kinship of fundamental fermions in weak interactions. According to the standard model, they participate pairwise in these interactions.