J-tangent affine hyperspheres with the involutive contact distribution

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History

In the paper **„Real hypersurfaces in complex centro-affine spaces"** V. Cruceanu considered hypersurfaces with a centro-affine J-tangent vector field.

History

In the paper **„Real hypersurfaces in complex centro-affine spaces"** V. Cruceanu considered hypersurfaces with a centro-affine J-tangent vector field.
He gave a local representation of centro-affine hypersurfaces with a J-tangent centro-affine vector field. More precisely he proved

Theorem (V.Cruceanu, 1988)

Let $f\colon M \to \mathbb{R}^{2n+2}$ be a centro-affine hypersurface with a J-tangent centro-affine vector field. Then there exist an open subset $U \subset \mathbb{R}^{2n}$, an interval $I \subset \mathbb{R}$ and an immersion $g\colon U \to \mathbb{R}^{2n+2}$ such that f can be locally expressed in the form

$$f(x_1, \ldots, x_{2n}, y) = Jg(x_1, \ldots, x_{2n}) \cos y + g(x_1, \ldots, x_{2n}) \sin y.$$

for all $(x_1, \ldots, x_{2n}, y) \in U \times I$.

History

In the paper **„J-tangent affine hyperspheres”** I gave a local characterization of 3-dimensional J-tangent affine hyperspheres with the involutive contact distribution.

In this presentation I would like to give the generalization of the above results. That is, I would like to show the local characterization of J-tangent affine hyperspheres of an arbitrary dimension with the involutive contact distribution.

Affine immersions

Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable n-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then for any transversal vector field C we have

$$\mathrm{D}_X f_* Y = f_*(\nabla_X Y) + h(X, Y)C \qquad \text{(Gauss' formula)}$$

and

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C, \qquad \text{(Weingarten's formula)}$$

where X, Y are vector fields tangent to M. Here

- ∇ — torsion free connection called *the induced connection*,
- h — tensor of type (0,2) called *the second fundamental form*,
- S — tensor of type (1,1) called *the shape operator*,
- τ — 1-form called *the transversal connection form*.

Blaschke hypersurface

We say that f is nondegenerate if the second fundamental form h is nondegenerate.

For a nondegenerate (orientable) hypersurface there exists a (global) transversal vector field C satisfying the conditions:

$$\nabla\theta = 0, \qquad \theta = \omega_h,$$

where ω_h is a volume element determined by h

$$\omega_h(X_1, \ldots, X_n) = \sqrt{|det[h(X_i, X_j)]_{i,j=1\ldots n}|}$$

and θ is an induced volume element on M

$$\theta(X_1, \ldots, X_n) = det[f_*X_1, \ldots, f_*X_n, C].$$

A transversal vector field satisfying these conditions is called *the affine normal field* or *the Blaschke normal field*. It is unique up to sign. A hypersurface with the transversal Blaschke normal field is called *the Blaschke hypersurface*.

Affine hyperspheres

Definition

A Blaschke hypersurface is called *an affine hypersphere* if $S = \lambda I$, where $\lambda = const$.
If $\lambda = 0$, f is called *an improper affine hypersphere*, if $\lambda \neq 0$, f is called *a proper affine hypersphere.*

J-tangent transversal vector field

From now on we are interested in $(2n+1)$-dimensional hypersurfaces $f\colon M \mapsto \mathbb{R}^{2n+2}$. We assume that $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$ is equipped with the standard flat kählerian structure $(\langle\cdot,\cdot\rangle, D, J)$. J is the standard complex structure, that is

$$J(x_1,\ldots,x_{n+1},y_1,\ldots,y_{n+1}) = (-y_1,\ldots,-y_{n+1},x_1,\ldots,x_{n+1}).$$

Definition

A transversal vector field C will be called *J-tangent*, if $JC \in f_*(TM)$.

The biggest J invariant distribution on M we denote by $\mathcal{D}$. It is $2n$-dimensional, smooth distribution on M.

Complex affine hypersurfaces

Let $g\colon M \to \mathbb{R}^{2n+2}$ ($\dim M = 2n$) be a *complex hypersurface* of the complex affine space $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$, that is for each point p of M we have

$$J(T_pM) = T_pM$$

.

Let $\zeta\colon M \to T\mathbb{R}^{2n+2}$ be a local transversal vector field on M. Then $\zeta(p), J\zeta(p)$ and T_pM together span $T_p\mathbb{R}^{2n+2}$.

Gauss and Weingarten formulas for g

For any transversal vector field ζ we have

$$\mathrm{D}_X\, g_* Y = g_*(\widetilde{\nabla}_X Y) + h_1(X,Y)\zeta + h_2(X,Y)J\zeta \qquad \text{(Gauss' formula)}$$

and

$$\mathrm{D}_X\, \zeta = -g_*(\widetilde{S}X) + \tau_1(X)\zeta + \tau_2(X)J\zeta \qquad \text{(Weingarten's formula)}$$

where

- $\widetilde{\nabla}$ — a torsion free affine connection on M,
- h_1 and h_2 — symmetric bilinear forms on M,
- $\widetilde{S}$ — a tensor of type (1,1) on M,
- τ_1 and τ_2 — 1-forms on M.

Relation between h_1 and h_2

Lemma (F. Dillen, L. Vrancken, L. Verstraelen, 1988)

$$h_1(X, JY) = h_1(JX, Y) = -h_2(X, Y),$$
$$h_2(X, JY) = h_2(JX, Y) = h_1(X, Y).$$

On the manifold M we define the volume form θ_ζ by

$$\theta_\zeta(X_1, \dots, X_{2n}) = \det(g_* X_1, \dots, g_* X_{2n}, \zeta, J\zeta)$$

for tangent vectors X_i (i=1,...,2n).
Then, consider the function H_ζ on M defined by

$$H_\zeta = \det[h_1(X_i, X_j)]_{i,j=1\dots 2n}$$

where $X_1, \dots, X_{2n}$ is a local frame in TM such that $\theta_\zeta(X_1, \dots, X_{2n}) = 1$.
We say that a hypersurface is *nondegenerate* if h_1 (and in consequence h_2) is nondegenerate.
When g is nondegenerate, there exist transversal vector fields ζ satisfying the following two conditions:

$$|H_\zeta| = 1,$$

$$\tau_1 = 0.$$

Such vector fields are called *affine normal vector fields*.

Complex affine hyperspheres

Definition (F. Dillen, L. Vrancken, L. Verstraelen)

A nondegenerate complex affine hypersurface is said to be a *proper complex affine hypersphere* if there exists an affine normal vector field ζ such that $\widetilde{S} = \alpha I$, where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\tau_2 = 0$.

If there exists an affine normal vector field ζ such that $\widetilde{S} = 0$ and $\tau_2 = 0$ the hypersurface is called an *improper affine hypersphere*.

Induced almost contact structure

Definition

Let $f\colon M \to \mathbb{R}^{2n+2}$ ($\dim M = 2n+1$) be a hypersurface with a J-tangent transversal vector field C. Then we define a vector field ξ, a 1-form η and a tensor field φ of type (1,1) as follows:

$$\xi := JC,$$
$$\eta|_{\mathcal{D}} = 0 \text{ and } \eta(\xi) = 1,$$
$$\varphi|_{\mathcal{D}} = J|_{\mathcal{D}} \text{ and } \varphi(\xi) = 0.$$

A structure (φ, ξ, η) is called *an induced almost contact structure on M*.

Characterization of hypersurfaces with centro-affine J-tangent transversal vector field and with the involutive contact distribution

Theorem 1 (Z. Szancer, 2012)

Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a centro-affine J-tangent vector field. The distribution $\mathcal{D}$ is involutive if and only if for every $x \in M$ there exists a complex immersion $g\colon V \to \mathbb{R}^{2n+2}$ defined on an open subset $V \subset \mathbb{R}^{2n}$ such that f can be expressed in the neighborhood of x in the form

$$f(x_1, \ldots, x_{2n}, y) = Jg(x_1, \ldots, x_{2n}) \cos y + g(x_1, \ldots, x_{2n}) \sin y.$$

J-tangent affine hyperspheres

Definition (Z. Szancer)

An affine hypersphere whose Blaschke normal field is J-tangent we call a J-tangent affine hypersphere.

Theorem 2 (Z. Szancer, 2014)

There are no improper J-tangent affine hyperspheres.

Local characterization of 3-dimensional J-tangent affine hyperspheres with the involutive distribution $\mathcal{D}$

Theorem 3 (Z. Szancer, 2014)

Let $f\colon M \mapsto \mathbb{R}^4$ be a J-tangent affine hypersphere with the involutive distribution $\mathcal{D}$. Then f can be locally expressed in the form:

$$f(x,y,z) = \lambda^{-\frac{5}{8}} \begin{bmatrix} \sin\sqrt{\lambda}x \sinh\sqrt{\lambda}y \\ -\cos\sqrt{\lambda}x \sinh\sqrt{\lambda}y \\ cos\sqrt{\lambda}x \cosh\sqrt{\lambda}y \\ \sin\sqrt{\lambda}x \cosh\sqrt{\lambda}y \end{bmatrix} \cos\lambda z$$

$$+\lambda^{-\frac{5}{8}} \begin{bmatrix} \cos\sqrt{\lambda}x \cosh\sqrt{\lambda}y \\ \sin\sqrt{\lambda}x \cosh\sqrt{\lambda}y \\ -\sin\sqrt{\lambda}x \sinh\sqrt{\lambda}y \\ \cos\sqrt{\lambda}x \sinh\sqrt{\lambda}y \end{bmatrix} \sin\lambda z \in \mathbb{R}^4$$

for some $\lambda > 0$.

Local characterization of J-tangent affine hyperspheres of arbitrary dimension with the involutive distribution $\mathcal{D}$

Theorem 4

Let $f\colon M \to \mathbb{R}^{2n+2}$ be a J-tangent affine hypersphere with the involutive distribution $\mathcal{D}$. Then f can be locally expressed in the form:

$$(*) \quad f(x_1,\ldots,x_n,y_1,\ldots,y_n,z) = Jg(x_1,\ldots,x_n,y_1,\ldots,y_n)\cos z + g(x_1,\ldots,x_n,y_1,\ldots,y_n)\sin z$$

where g is a proper complex affine hypersphere. Conversely if g is a proper complex affine hypersphere then f given by the formula $(*)$ is a J-tangent affine hypersphere with the involutive distribution $\mathcal{D}$.

Proof $(\Rightarrow)$ First note that due to Theorem 2 the immersion f must be a proper affine hypersphere. Let C be J-tangent affine normal field. There exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $C = -\lambda f$. Since C is J-tangent and transversal, the same is $\frac{1}{\lambda}C = -f$. Thus f satisfies assumptions of Theorem 1. That is, there exists a complex immersion g from an open subset $U \subset \mathbb{R}^{2n}$ into $\mathbb{R}^{2n+2}$ and there exists an open interval I such that f can be locally expressed in the form

$$f(x_1, \ldots, x_{2n}, z) = Jg(x_1, \ldots, x_{2n}) \cos z + g(x_1, \ldots, x_{2n}) \sin z,$$

where $(x_1, \ldots, x_{2n}) \in U$ and $z \in I$.

We shall now prove that g is a proper complex affine hypersphere.
Let $\zeta := |\lambda|^{\frac{2n+3}{2n+4}} g$. Assume that there exist functions α^i, γ, δ from U into $\mathbb{R}$ such that

$$\alpha^i g_{x_i} + \gamma g + \delta J g = 0$$

for $i = 1, \ldots, 2n$. Then for any $z \in I$ we have

$$\alpha^i g_{x_i} \sin z + \gamma g \sin z + \delta J g \sin z = 0$$

and

$$\alpha^i g_{x_i} \cos z + \gamma g \cos z + \delta J g \cos z = 0.$$

We obtain

$$\alpha^i f_{x_i} + \gamma f - \delta f_z = 0.$$

But since f is an immersion and $C = -\lambda f$ is a transversal vector field, the above implies

$$\alpha^i = \gamma = \delta = 0.$$

Thus $\{g_{x_i}\}$, g, Jg are linearly independent. In consequence $\zeta = |\lambda|^{\frac{2n+3}{2n+4}} g$ is a transversal vector field to g. From the Weingarten formula for g we have

$$\mathrm{D}_{\partial_{x_i}} \zeta = -g_*(\widetilde{S}\partial_{x_i}) + \tau_1(\partial_{x_i})\zeta + \tau_2(\partial_{x_i})J\zeta.$$

On the other hand we compute

$$\mathrm{D}_{\partial_{x_i}} \zeta = \partial_{x_i}(\zeta) = -|\lambda|^{\frac{2n+3}{2n+4}} g_*(\partial_{x_i}).$$

Summarizing we obtain

$$\widetilde{S} = |\lambda|^{\frac{2n+3}{2n+4}} I, \quad \tau_1 = 0, \quad \tau_2 = 0. \tag{1}$$

Now, to prove that ζ is the affine normal vector field it is enough to show that $|H_\zeta| = 1$. Since g is complex, J is a complex structure on TU thus, we can take a local frame $\partial_{x_1}, \ldots, \partial_{x_{2n}}$ such that

$$\partial_{x_{n+i}} = J\partial_{x_i}$$

for $i = 1 \ldots n$. Let us denote

$$A := \Theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_n}, J\partial_{x_1}, \ldots, J\partial_{x_n}).$$

Then the basis

$$\frac{1}{A}\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, J\partial_{x_1}, \ldots, J\partial_{x_n}$$

is unimodular relative to Θ_ζ. Now (according to the definition of H_ζ) we have

$$H_\zeta = \frac{1}{A^2} \cdot \det h_1,$$

where

$$\det h_1 = \det \begin{bmatrix} h_1(\partial_{x_1}, \partial_{x_1}) & h_1(\partial_{x_1}, \partial_{x_2}) & \cdots & h_1(\partial_{x_1}, \partial_{x_{2n}}) \\ h_1(\partial_{x_2}, \partial_{x_1}) & h_1(\partial_{x_2}, \partial_{x_2}) & \cdots & h_1(\partial_{x_2}, \partial_{x_{2n}}) \\ \vdots & \vdots & \ddots & \vdots \\ h_1(\partial_{x_{2n}}, \partial_{x_1}) & h_1(\partial_{x_{2n}}, \partial_{x_2}) & \cdots & h_1(\partial_{x_{2n}}, \partial_{x_{2n}}) \end{bmatrix}.$$

From the Gauss formula for g we have

$$\begin{aligned} g_{x_ix_j} &= g_*(\widetilde{\nabla}_{\partial_{x_i}}\partial_{x_j}) + h_1(\partial_{x_i},\partial_{x_j})\zeta + h_2(\partial_{x_i},\partial_{x_j})J\zeta \\ &= g_*(\widetilde{\nabla}_{\partial_{x_i}}\partial_{x_j}) - |\lambda|^{\frac{2n+3}{2n+4}}h_1(\partial_{x_i},\partial_{x_j})g - |\lambda|^{\frac{2n+3}{2n+4}}h_2(\partial_{x_i},\partial_{x_j})Jg. \end{aligned} \tag{2}$$

From the Gauss formula for f we have

$$\begin{aligned} f_{x_ix_j} &= Jg_{x_ix_j}\cos z + g_{x_ix_j}\sin z \\ &= f_*(\nabla_{\partial_{x_i}}\partial_{x_j}) - \lambda h(\partial_{x_i},\partial_{x_j})(Jg\cos z + g\sin z). \end{aligned} \tag{3}$$

Applyingg (2) to (3) we obtain

$$\begin{aligned} f_*(\nabla_{\partial_{x_i}}\partial_{x_j}) - \lambda h(\partial_{x_i},\partial_{x_j})(Jg\cos z + g\sin z) \\ = f_*(\widetilde{\nabla}_{\partial_{x_i}}\partial_{x_j}) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i},\partial_{x_j})\cdot f \\ - |\lambda|^{\frac{2n+3}{2n+4}} h_2(\partial_{x_i},\partial_{x_j})\cdot Jf. \end{aligned}$$

Since $f_*(\widetilde{\nabla}_{\partial_{x_i}}\partial_{x_j})$ and Jf are tangent, we immediately get that

$$-\lambda h(\partial_{x_i},\partial_{x_j}) = -|\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i},\partial_{x_j}).$$

By the Gauss formula for f we also have

$$h(\partial_z, \partial_z) = \frac{1}{\lambda}$$

and

$$h(\partial_z, \partial_{x_i}) = h(\partial_{x_i}, \partial_z) = 0$$

for $i = 1 \dots 2n$. Hence

$$\det h := \det \begin{bmatrix} h(\partial_{x_1}, \partial_{x_1}) & h(\partial_{x_1}, \partial_{x_2}) & \cdots & h(\partial_{x_1}, \partial_{x_{2n}}) & 0 \\ h(\partial_{x_2}, \partial_{x_1}) & h(\partial_{x_2}, \partial_{x_2}) & \cdots & h(\partial_{x_2}, \partial_{x_{2n}}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(\partial_{x_{2n}}, \partial_{x_1}) & h(\partial_{x_{2n}}, \partial_{x_2}) & \cdots & h(\partial_{x_{2n}}, \partial_{x_{2n}}) & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\lambda} \end{bmatrix}$$

$$= \frac{1}{\lambda} \det[h(\partial_{x_i}, \partial_{x_j})] = \frac{1}{\lambda} \cdot (\frac{1}{\lambda} \cdot |\lambda|^{\frac{2n+3}{2n+4}})^{2n} \det h_1$$

$$= \frac{1}{\lambda} \cdot |\lambda|^{-\frac{2n}{2n+4}} \det h_1.$$

Finally we get

$$|\det h_1| = |\lambda|^{\frac{4n+4}{2n+4}}|\det h| = |\lambda|^{\frac{2n+2}{n+2}}|\det h|. \tag{4}$$

Now, since $C = -\lambda f$ is the Blaschke field we have

$$\begin{aligned}\omega_h = \sqrt{|\det h|} &= \Theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z) = \det[f_{x_1}, \dots, f_{x_{2n}}, f_z, C] \\ &= -\lambda \det[Jg_{x_1}\cos z + g_{x_1}\sin z, \dots, Jg_{x_{2n}}\cos z + g_{x_{2n}}\sin z, \\ &\qquad - Jg\sin z + g\cos z, Jg\cos z + g\sin z].\end{aligned}$$

Using the fact that the determinant is $(2n+2)$-linear and antisymmetric and since

$$g_{x_{n+i}} = Jg_{x_i}$$

for $i = 1 \dots n$ we obtain

$$\begin{aligned}\Theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z) &= -\lambda \det[g_{x_1}, \dots, g_{x_n}, Jg_{x_1}, \dots, Jg_{x_n}, g, Jg] \\ &= -\lambda(|\lambda|^{\frac{2n+3}{2n+4}})^{-2} \det[g_*(\partial_{x_1}), \dots, g_*(\partial_{x_{2n}}), \zeta, J\zeta] \\ &= -\lambda \cdot (|\lambda|^{-\frac{2n+3}{n+2}})\Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}}).\end{aligned}$$

Now it follows that

$$\begin{aligned}|\det h| &= [\Theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z)]^2 \\ &= |\lambda|^2 \cdot |\lambda|^{\frac{-4n-6}{n+2}}[\Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}})]^2 \\ &= |\lambda|^{\frac{-2n-2}{n+2}}[\Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}})]^2 \\ &= |\lambda|^{\frac{-2n-2}{n+2}} \cdot A^2.\end{aligned}$$

The above implies (see (4)) that

$$|\det h_1| = A^2.$$

Summarizing we get

$$|H_\zeta| = \frac{1}{A^2}|\det h_1| = 1,$$

that is, ζ is the affine normal field and due to (1) g is an affine hypersphere.

($\Leftarrow$) Let $g\colon U \to \mathbb{R}^{2n+2}$ be a proper complex affine hypersphere. There exists $\alpha \neq 0$ such that $\zeta = -\alpha g$ is the affine normal vector field. Without loss of generality we may assume that $\alpha > 0$. Since g is transversal, Jg is transversal too, thus $\{g_{x_1}, \ldots, g_{x_{2n}}, g, Jg\}$ form the basis of $\mathbb{R}^{2n+2}$. The above implies that

$$f\colon U \times I \ni (x_1, \ldots, x_{2n}, z) \mapsto f(x_1, \ldots, x_{2n}, z) \in \mathbb{R}^{2n+2}$$

given by the formula:

$$f(x_1, \ldots, x_{2n}, z) := Jg(x_1, \ldots, x_{2n}) \cos z + g(x_1, \ldots, x_{2n}) \sin z$$

is an immersion and $C := -\alpha^{\frac{2n+4}{2n+3}} \cdot f$ is a transversal vector field. Of course C is J-tangent because $JC = \alpha^{\frac{2n+4}{2n+3}} f_z$.

Since C is equiaffine, it is enough to show that $\omega_h = \Theta$ for some positively oriented (relative to Θ) basis on $U \times I$. Let $\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z$ be a local coordinate system on $U \times I$. Since g is complex, we may assume that $\partial_{x_{n+i}} = J\partial_{x_i}$ for $i = 1 \ldots n$. In a similar way as in the proof of the first implication we compute

$$\Theta(\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z) = \det[f_{x_1}, \ldots, f_{x_{2n}}, f_z, -\alpha^{\frac{2n+4}{2n+3}} f]$$
$$= -\alpha^{-\frac{2n+2}{2n+3}} \Theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}}).$$

Again, in a similar way as in the proof of the first implication, we get

$$\begin{aligned}\det h &= \alpha^{-\frac{2n+4}{2n+3}} \cdot \left(\frac{\alpha}{\alpha^{\frac{2n+4}{2n+3}}}\right)^{2n} \det h_1 \\ &= \alpha^{-\frac{2n+4}{2n+3}} \cdot \alpha^{-\frac{2n}{2n+3}} \det h_1 \\ &= \alpha^{\frac{-4n-4}{2n+3}} \det h_1 .\end{aligned}$$

The above implies

$$\omega_h := \sqrt{|\det h|} = \alpha^{\frac{-2n-2}{2n+3}} \sqrt{|\det h_1|}.$$

It is easy to see that

$$|\det h_1| = |H_\zeta| [\Theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}})]^2,$$

because

$$\frac{1}{\Theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}})} \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{2n}}$$

is a unimodular basis relative to Θ_ζ.

Hence (since $|H_\zeta| = 1$)

$$\omega_h = \alpha^{\frac{-2n-2}{2n+3}} |\Theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}})|.$$

Finally we get $\omega_h = |\Theta(\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z)|$. The proof is completed.

Remark

If in the above theorem f is an affine hypersphere with $S = \lambda I$ and g is a complex affine hypersphere with $\widetilde{S} = \alpha I$ than we have the following relation $|\lambda| = |\alpha|^{\frac{2n+4}{2n+3}}$.

Complex affine circles in $\mathbb{C}^2$

Definition

Complex 1-dimensional affine hyperspheres in $\mathbb{C}^2$ will be called *complex affine circles*.

We have the following classification of the complex affine circles

Theorem 5 (F. Dillen, L. Vrancken, L. Verstraelen, 1988)

A complex affine curve in $\mathbb{C}^2$ is a complex affine circle if and only if it is a quadratic complex curve, respectively of parabolic or hyperbolic type according to the circle being improper or proper.

As a consequence of Theorem 5 and the main theorem we get an alternative proof of Theorem 3.

Proof.

From the main theorem f can be locally expressed in the form:

$$f(x, y, z) = Jg(x, y)\cos z + g(x, y)\sin z,$$

where g is a complex affine hypersphere. Since g is a 1-dimensional complex affine hypersphere, thus it is a complex affine circle. By Theorem 5, g is a quadratic complex curve. Moreover, since g is a proper hypersphere it must be of hyperbolic type, that is,

$$z_1 z_2 = \alpha,$$

where $\alpha > 0$.

Equivalently, using the following complex equiaffine transformation

$$\begin{bmatrix} \frac{i}{2} & \frac{1}{2} \\ -i & 1 \end{bmatrix}$$

g can be locally expressed in a parametric form as follows

$$g(u) = \sqrt{2\alpha} \begin{bmatrix} \cos u \\ \sin u \end{bmatrix}.$$

Moving to real numbers $(u = x + iy, x, y \in \mathbb{R})$ we have

$$g(x,y) = \sqrt{2\alpha} \begin{bmatrix} Re \cos u \\ Re \sin u \\ Im \cos u \\ Im \sin u \end{bmatrix} = \sqrt{2\alpha} \begin{bmatrix} \cosh x \cosh y \\ \sin x \cosh y \\ -\sin x \sinh y \\ \cos x \sinh y \end{bmatrix}$$

and consequently

$$f(x,y,z) = \sqrt{2\alpha} \begin{bmatrix} \sin x \sinh y \\ -\cos x \sinh y \\ cos x \cosh y \\ \sin x \cosh y \end{bmatrix} \cos z$$

$$+\sqrt{2\alpha} \begin{bmatrix} \cos x \cosh y \\ \sin x \cosh y \\ -\sin x \sinh y \\ \cos x \sinh y \end{bmatrix} \sin z$$

Taking into account that $\widetilde{S} = \frac{1}{(2\alpha)^{\frac{2}{3}}} I$ for g we easily get that $\lambda = (2\alpha)^{-\frac{4}{5}}$. Now, replacing x with $\sqrt{\lambda}x$, y with $\sqrt{\lambda}y$ and z with λz we obtain f in the required form.

Some examples

Example 1 Let $\tilde{g}\colon \mathbb{C}^n \to \mathbb{C}^{n+1}$ be given by $\tilde{g}(z_1,\ldots,z_n) =$

$$= \begin{bmatrix} \tilde{g}_1(z_1,\ldots,z_n) \\ \tilde{g}_2(z_1,\ldots,z_n) \\ \tilde{g}_3(z_1,\ldots,z_n) \\ \vdots \\ \tilde{g}_{n-1}(z_1,\ldots,z_n) \\ \tilde{g}_n(z_1,\ldots,z_n) \\ \tilde{g}_{n+1}(z_1,\ldots,z_n) \end{bmatrix} = \begin{bmatrix} \cos z_1 \\ \sin z_1 \cdot \cos z_2 \\ \sin z_1 \cdot \sin z_2 \cdot \cos z_3 \\ \vdots \\ \sin z_1 \cdot \sin z_2 \cdot \ldots \cdot \sin z_{n-2} \cdot \cos z_{n-1} \\ \sin z_1 \cdot \sin z_2 \cdot \ldots \cdot \sin z_{n-1} \cdot \cos z_n \\ \sin z_1 \cdot \sin z_2 \cdot \ldots \cdot \sin z_{n-1} \cdot \sin z_n \end{bmatrix}.$$

It is a complex affine hypersphere.

Let $z_k = x_k + iy_k$ for $k = 1, \ldots, n$. Then

$$g\colon \mathbb{R}^{2n} \ni (x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \mapsto g(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n+2}$$

given by the formula

$$g(x_1, y_1, \ldots, x_n, y_n) = \begin{bmatrix} \operatorname{Re} \tilde{g}_1(z_1, \ldots, z_n) \\ \operatorname{Re} \tilde{g}_2(z_1, \ldots, z_n) \\ \vdots \\ \operatorname{Re} \tilde{g}_n(z_1, \ldots, z_n) \\ \operatorname{Re} \tilde{g}_{n+1}(z_1, \ldots, z_n) \\ \operatorname{Im} \tilde{g}_1(z_1, \ldots, z_n) \\ \operatorname{Im} \tilde{g}_2(z_1, \ldots, z_n) \\ \vdots \\ \operatorname{Im} \tilde{g}_n(z_1, \ldots, z_n) \\ \operatorname{Im} \tilde{g}_{n+1}(z_1, \ldots, z_n) \end{bmatrix}.$$

is a complex affine hypersphere.

Now, by the main theorem

$$f(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, z) := Jg(x_1, y_1, \ldots, x_n, y_n)\cos z \\ + g(x_1, y_1, \ldots, x_n, y_n)\sin z$$

is a J-tangent affine hypersphere with involutive contact distribution.

Example 2 Let us consider complex affine hypersphere given by the formula

$$z_1 \cdot z_2 \cdot \ldots \cdot z_n \cdot z_{n+1} = 1$$

(when $n > 1$ this hypersphere is not affinely equivalent with the hypersphere from Example 1). Rewriting the above equality in a parametric form we get

$$\tilde{g}(z_1, \ldots, z_n) = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ 1/(z_1 \cdot z_2 \cdot \ldots \cdot z_n) \end{bmatrix}.$$

Moving to real numbers we have

$$g(x_1, y_1, \ldots, x_n, y_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \operatorname{Re} 1/(z_1 \cdot z_2 \cdot \ldots \cdot z_n) \\ y_1 \\ y_2 \\ \vdots \\ y_n \\ \operatorname{Im} 1/(z_1 \cdot z_2 \cdot \ldots \cdot z_n) \end{bmatrix}$$

and by the main theorem

$$f(x_1, y_1, x_2, y_2, \dots, x_n, y_n, z) := Jg(x_1, y_1, \dots, x_n, y_n)\cos z \\ + g(x_1, y_1, \dots, x_n, y_n)\sin z$$

is a J-tangent affine hypersphere with the involutive contact distribution.

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Thank you!