Surfaces from Deformation Parameters

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Süleyman Tek
University of the Incarnate Word, San Antonio, TX, USA

Metin Gürses
Bilkent University, Ankara, Turkey

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Surface theory in $\mathbb{R}^3$ plays a crucial role in differential geometry, partial differential equations (PDEs), string theory, general theory of relativity, and biology [Parthasarthy and Viswanathan, 2001] - [Ou-Yang et. al., 1999].

Soliton equations play a crucial role for the construction of surfaces.

The theory of nonlinear soliton equations was developed in 1960s.

For details of integrable equations one may look [Drazin, 1989], [Ablowitz and Segur, 1991], and the references therein.

Lax representation of nonlinear PDEs consists of two linear equations which are called Lax equations

$$\Phi_x = U\,\Phi, \quad \Phi_t = V\,\Phi, \tag{1}$$

and their compatibility condition

$$U_t - V_x + [U, V] = 0, \tag{2}$$

where x and t are independent variables. Here U and V are called Lax pairs.

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Using this relation, soliton surface theory was first developed by Sym [Sym, 1982]-[Sym, 1985]. He obtained the immersion function by using the deformation of Lax equations for integrable equations.

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Soliton surface technique is an effective method to develop surfaces in $\mathbb{R}^3$ and in M_3.

In this method, one mainly uses the deformations of the Lax equations of the integrable equations [Sym, 1982]-[Gürses and Tek, 2014],

- Sine Gordon (SG) equation
- Korteweg de Vries (KdV) equation
- Modified Korteweg de Vries (mKdV) equation
- Nonlinear Schrödinger (NLS) equation

There are many attempts to find new examples of two surfaces.

We follow Fokas and Gel'fand [Fokas and Gelfand, 1996] approach to develop surfaces using integrable equations.

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Deformation matrices A and B

Let $\delta U = A$, $\delta V = B$, where A and B satisfy

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Soliton Surface: Let $\langle , \rangle$ defines an inner product in $\mathfrak{g}$.

First fundamental form

$$(ds_I)^2 \equiv g_{ij}\, dx^i\, dx^j = \langle A, A\rangle\, dx^2 + 2\langle A, B\rangle\, dx\, dt + \langle B, B\rangle\, dt^2,$$

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Second fundamental form

$$(ds_{II})^2 \equiv h_{ij}\, dx^i\, dx^j = \langle A_x + [A, U], C\rangle\, dx^2 + 2\langle A_t + [A, V], C\rangle\, dx\, dt + \langle B_t + [B, V], C\rangle\, dt^2,$$

$$[A, B] = AB - BA, \ ||A|| = \sqrt{|\langle A, A\rangle|}, \ \text{and } C = \frac{[A,B]}{||[A,B]||}.$$

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$$[A, B] = AB - BA,\ ||A|| = \sqrt{|\langle A, A\rangle|},\ \text{and } C = \tfrac{[A,B]}{||[A,B]||}.$$

Gaussian and mean curvatures

$$K = \det(g^{-1}\, h)\ ,\ H = \frac{1}{2}\text{trace}(g^{-1}\, h),\ g = (g_{ij}),\ h = (h_{ij}).$$

Since our aim is finding a class of surfaces which correspond to integrable equations, we need to find A and B that satisfy the following equation

$$A_t - B_x + [A, V] + [U, B] = 0. \tag{6}$$

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But in general, solving that equation is not simple. However there are some deformations which provide A and B directly.

Spectral parameter λ invariance of the equation

$$A = \mu_1 \frac{\partial U}{\partial \lambda}, \; B = \mu_1 \frac{\partial V}{\partial \lambda}, \; F = \mu_1 \, \Phi^{-1} \frac{\partial \Phi}{\partial \lambda}, \tag{7}$$

That kind of deformation was first used by Sym [Sym, 1982]-[Sym, 1985].

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Gauge symmetries of the Lax equation

$$A = M_x + [M, U], \; B = M_t + [M, V], \; F = \Phi^{-1} M \Phi, \tag{8}$$

where M is any traceless 2×2 matrix. [Fokas and Gelfand, 1996], [Fokas et. al., 2000], [Cieslinski, 1997].

Symmetries of the (integrable) differential equations

$$A = \delta U,\ B = \delta V,\ F = \Phi^{-1}\delta\Phi, \tag{9}$$

where δ represents the classical Lie symmetries and (if integrable) the generalized symmetries of the nonlinear PDE's
[Fokas and Gelfand, 1996], [Fokas et. al., 2000], [Cieslinski, 1997].

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where δ represents the classical Lie symmetries and (if integrable) the generalized symmetries of the nonlinear PDE's [Fokas and Gelfand, 1996], [Fokas et. al., 2000], [Cieslinski, 1997].

Deformation of parameters of solution of integrable equation

$$A = \mu_2 \left(\partial U / \partial k_i \right), \;\; B = \mu_2 \left(\partial V / \partial k_i \right), \;\; F = \mu_2 \Phi^{-1} \left(\partial \Phi / \partial k_i \right), \tag{10}$$

where $i = 1, 2$ and k_i are parameters of the solution $u(x, t, k_1, k_2)$ of the PDEs, μ_2 is constant. [Gürses and Tek, 2015]

In this section, we obtain the immersions of 2-surfaces in $\mathbb{R}^3$.

For this purpose, we use Lie group SU(2) and its Lie algebra $\mathfrak{su}(2)$ with basis $e_j = -i\,\sigma_j$, $j = 1, 2, 3$, where σ_j denote the usual Pauli sigma matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

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Define an inner product on $\mathfrak{su}(2)$ as

$$\langle X, Y \rangle = -\frac{1}{2}\,\mathrm{trace}(XY), \tag{12}$$

where X, $Y \in \mathfrak{su}(2)$ valued vectors.

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Let $u(x,t)$ satisfy the mKdV equation

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x. \tag{13}$$

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Let $u(x,t)$ satisfy the mKdV equation

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x. \tag{13}$$

Substituting the travelling wave ansatz $u_t - \alpha\, u_x = 0$ in Eq. (13), we get

$$u_{xx} = \alpha u - \frac{u^3}{2}. \tag{14}$$

Lax pairs U and V are given as

$$U = \frac{i}{2}\begin{pmatrix} \lambda & -u \\ -u & -\lambda \end{pmatrix}, \tag{15}$$

$$V = -\frac{i}{2}\begin{pmatrix} \frac{1}{2}u^2-(\alpha+\alpha\lambda+\lambda^2) & (\alpha+\lambda)u-iu_x \\ (\alpha+\lambda)u+iu_x & -\frac{1}{2}u^2+(\alpha+\alpha\lambda+\lambda^2) \end{pmatrix}, \tag{16}$$

and λ is a spectral parameter.

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and λ is a spectral parameter.

Consider the one soliton solution of mKdV equation [Eq. (14)] as

$$u = k_1 \operatorname{sech} \xi_1, \tag{17}$$

where $\alpha = k_1^2/4$, $\xi_1 = k_1(k_1^2 t + 4x)/8 + k_0$, and k_0 and k_1 are arbitrary constants.

First we consider mKdV surfaces arising from deformation of parameter k_0.

Proposition

Let u be a travelling wave solution of mKdV equation given by Eq. (17). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the mKdV equation are given by Eqs. (15) and (16), respectively. $\mathfrak{su}(2)$ valued matrices A and B are

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$$A = -\frac{i\mu}{2}\begin{pmatrix} 0 & \phi_0 \\ \phi_0 & 0 \end{pmatrix}, \tag{18}$$

$$B = -\frac{i\mu}{2}\begin{pmatrix} u\,\phi_0 & (k_1^2/4+\lambda)\phi_0 - i\,(\phi_0)_x \\ (k_1^2/4+\lambda)\phi_0 + i\,(\phi_0)_x & -u\,\phi_0 \end{pmatrix} \tag{19}$$

where $A = \mu\,(\partial U/\partial k_0)$, $B = \mu\,(\partial V/\partial k_0)$, $\phi_0 = \partial\,u/\partial k_0$; k_0 is a parameter of the one soliton solution u, and μ is a constant.

Proposition

Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(j,k=1,2)$

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k, \tag{20}$$
$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k, \tag{21}$$

where

$$g_{11} = \frac{1}{4}\mu^2\phi_0^2,\;\; g_{12} = g_{21} = \frac{1}{16}\mu^2\phi_0^2(k_1^2 + 4\,\lambda), \tag{22}$$
$$g_{22} = \frac{1}{64}\mu^2\Big(16\,(\phi_0)_x^2 + \phi_0^2\left[16\,u^2 + (k_1^2 + 4\lambda)^2\right]\Big), \tag{23}$$
$$h_{11} = -16\,\Delta_1\,\lambda\,u\,\phi_0^2, \tag{24}$$
$$h_{12} = 4\,\Delta_1\phi_0\Big(4\,(\phi_0)_x\,u_x + u\,\phi_0\left[2\,u^2 - k_1^2(\lambda+1) - 4\,\lambda^2\right]\Big), \tag{25}$$

Proposition

$$h_{22} = -\Delta_1\Big(u\,\phi_0^2\,(k_1^2+4\lambda)\Big[2u^2+4\lambda^2+k_1^2(\lambda+1)\Big] \tag{26}$$

$$+\,4\phi_0\Big[4u(\phi_0)_{xt}-(\phi_0)_x\Big([k_1^2+4\lambda]u_x+4u_t\Big)\Big] \tag{27}$$

$$+\,4u(\phi_0)_x\Big[(\phi_0)_x(k_1^2+4\lambda)-4(\phi_0)_t\Big]\Big)$$

$$\Delta_1 = \frac{\mu}{32\Big((\phi_0)_x^2+u^2\phi_0^2\Big)^{1/2}} \tag{28}$$

and the corresponding Gaussian and mean curvatures are

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and the corresponding Gaussian and mean curvatures are

$$K = \frac{16\lambda^2}{k_1^2\mu^2}, \quad H = -\frac{4\lambda}{k_1\mu}, \tag{29}$$

where $x^1 = x$, $x^2 = t$.

Another parameter of the one soliton solution of mKdV equation is k_1. Now we give mKdV surfaces arising from k_1 parameter deformation.

Proposition

Let u be the soliton solution of mKdV equation and the Lax pairs U and V are given by Eqs. (15) and (16), respectively. $\mathfrak{su}(2)$ valued matrices A and B are

$$A = -\frac{i\mu}{2}\begin{pmatrix} 0 & \phi_1 \\ \phi_1 & 0 \end{pmatrix}, \tag{30}$$

$$B = -\frac{i\mu}{8}\begin{pmatrix} 4\,u\,\phi_1 - 2k_1(\lambda+1) & \tau - 4i(\phi_1)_x \\ \tau + 4i(\phi_1)_x & -4\,u\,\phi_1 + 2k_1(\lambda+1) \end{pmatrix}, \tag{31}$$

where $A = \mu\,(\partial U/\partial k_1)$, $B = \mu\,(\partial V/\partial k_1)$, $\tau = 2k_1 u + (k_1^2 + 4\lambda)\phi_1$ and $\phi_1 = \partial\,u/\partial k_1$; k_1 is a parameter of the one soliton solution u, and μ is a constant.

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$$g_{22} = \frac{1}{64}\mu^2\Big(4\,[k_1^2 + 4\,\phi_1^2]\,u^2 + 4\,k_1\,(k_1^2 - 4)u\,\phi_1 + 16(\phi_1)_x^2 \tag{35}$$

$$+ (k_1^2 + 4\,\lambda)^2\phi_1^2 + 4\,k_1^2(\lambda + 1)^2\Big), \tag{36}$$

$$h_{11} = \frac{1}{16}\Delta_2\,\mu^3\,\lambda\,\phi_1^2\Big(k_1[\lambda + 1] - 2\,u\,\phi_1\Big), \tag{37}$$

Proposition

$$h_{12} = h_{21} = \frac{1}{64}\Delta_2\,\mu^3\phi_1^2\Big(8\,(\phi_1)_x u_x + \Big[k_1(\lambda+1) - 2u\phi_1\Big]\Big[2(2\lambda^2 - u^2) + k_1^2(\lambda+1)\Big]\Big), \tag{38}$$

$$h_{22} = \frac{1}{256}\Delta_2\,\mu^3\,\phi_1\Big(8\,(\phi_1)_x\Big\{2\,k_1\,u\,u_x + (k_1^2 + 4\,\lambda)\Big[\phi_1\,u_x - u(\phi_1)_x\Big] + 4(u\phi_1)_t\Big\} + \Big[k_1(\lambda+1) - 2\,u\,\phi_1\Big]\Big\{16\,(\phi_1)_{xt} - 4\,k_1\,u(u^2 + 2\,\lambda) + \phi_1(k_1^2 + 4\,\lambda)\Big(2[u^2 + 2\,\lambda^2] + k_1^2[\lambda+1]\Big)\Big\}\Big). \tag{39}$$

Proposition

The Gaussian and mean curvatures are

$$K = \frac{1}{\mu^2\,\eta_0\,\left(4\,\eta_4^2+\eta_3^2\right)^2}\sum_{l=1}^{7} Q_l\,(\operatorname{sech}\xi_1)^l, \tag{40}$$

$$H = \frac{1}{4\,\mu\,\eta_0\,\left(4\,\eta_4^2+\eta_3^2\right)^{3/2}}\sum_{m=0}^{7} Z_m\,(\operatorname{sech}\xi_1)^m, \tag{41}$$

where $\eta_0,\ldots,\eta_4, Q_1,\ldots,Q_7, Z_1,\ldots,Z_6$ *are functions of* x *and* t.

In this section, we explore the position vector

$$\overrightarrow{y} = \left(y_1(x,t), y_2(x,t), y_3(x,t)\right), \tag{42}$$

of the mKdV surfaces that we obtain using deformation of parameters.

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$$u = k_1 \,\mathrm{sech}\, \xi_1, \tag{43}$$

where $\alpha = k_1^2/4$, $\xi_1 = k_1(k_1^2 t + 4x)/8 + k_0$, and k_0 and k_1 are arbitrary constants.

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where $\alpha = k_1^2/4$, $\xi_1 = k_1(k_1^2 t + 4x)/8 + k_0$, and k_0 and k_1 are arbitrary constants.
We solve the Lax equations $\Phi_x = U\,\Phi$ and $\Phi_t = V\,\Phi$ using Lax pairs U and V, and a solution of the mKdV equation.

The components of the 2×2 matrix Φ are

$$\Phi_{11} = -\frac{\Delta_4}{k_1}\Big[A_1(2\,\lambda i - k_1 \tanh\xi_1)\cdot \exp\big(i(k_1^2+4\lambda^2)t/8\big)\cdot \Xi_1 - i\,k_1^2 B_1 \operatorname{sech}\xi_1 \cdot \exp\big(-i(k_1^2+4\lambda^2)t/8\big)\cdot \Xi_2\Big], \quad (44)$$

$$\Phi_{12} = -\frac{\Delta_4}{k_1}\Big[A_2(2\,\lambda i - k_1 \tanh\xi_1)\cdot \exp\big(i(k_1^2+4\lambda^2)t/8\big)\cdot \Xi_1 - i\,k_1^2 B_2 \operatorname{sech}\xi_1 \cdot \exp\Big(-i(k_1^2+4\lambda^2)t/8\Big)\cdot \Xi_2\Big], \quad (45)$$

$$\Phi_{21} = \Delta_4\Big[i\,A_1 \operatorname{sech}\xi_1 \cdot \exp\big(i(k_1^2+4\lambda^2)t/8\big)\cdot \Xi_1 + B_1(2\,\lambda i + k_1 \tanh\xi_1)\cdot \exp\big(-i(k_1^2+4\lambda^2)t/8\big)\cdot \Xi_2\Big], \quad (46)$$

$$\Phi_{22} = \Delta_4\Big[i\,A_2 \operatorname{sech}\xi_1 \cdot \exp\big(i(k_1^2+4\lambda^2)t/8\big)\cdot \Xi_1 + B_2(2\,\lambda i + k_1 \tanh\xi_1)\cdot \exp\big(-i(k_1^2+4\lambda^2)t/8\big)\cdot \Xi_2\Big], \quad (47)$$

where

$$\Xi_1 = (\tanh \xi_1 + 1)^{i\lambda/2k_1} (\tanh \xi_1 - 1)^{-i\lambda/2k_1}, \tag{48}$$

$$\Xi_2 = (\tanh \xi_1 - 1)^{i\lambda/2k_1} (\tanh \xi_1 + 1)^{-i\lambda/2k_1}, \tag{49}$$

$$\Delta_4 = \sqrt{k_1/(k_1^2 + 4\lambda^2)}. \tag{50}$$

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$$\Delta_4 = \sqrt{k_1/(k_1^2 + 4\lambda^2)}. \tag{50}$$

Here we find the determinant of the matrix Φ as

$$\det(\Phi) = (A_1 B_2 - A_2 B_1) \neq 0. \tag{51}$$

Immersion function of the mKdV surface obtained using k_0 deformation

We find the immersion function F of the mKdV surface obtained using k_0 deformation by using the following equation

$$F = \nu\, \Phi^{-1} \frac{\partial \Phi}{\partial k_0} + \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \tag{52}$$

from which we obtain the position vector.

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from which we obtain the position vector.

Using Φ given in the previous slides and choosing $A_1 = -k_1 \, B_2 \exp(-\lambda \, \pi / k_1)$, $A_2 = k_1 \, B_1 \exp(-\lambda \, \pi / k_1)$, $r_{11} = r_{22} = 0$, $r_{12} = -r_{21}$ to write F in the form $F = -i(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$.

Hence we obtain a family of surfaces parameterized by

$$y_1 = W_6 \cdot \operatorname{sech}^2(\xi_1)\Big[W_3 \cdot \cosh(\xi_1)\cos(\Omega_1) + W_4 \cdot \sinh(\xi_1)\sin(\Omega_1) + 4\lambda\, W_8\big(2W_1\cosh(2\xi_1) + W_7\big)\Big], \quad (53)$$

$$y_2 = \frac{1}{W_5}\operatorname{sech}^2(\xi_1)\Big[W_{10} \cdot \sinh(\xi_1)\cos(\Omega_1) - W_{11} \cdot \cosh(\xi_1)\sin(\Omega_1) + W_9 \cdot \cosh^2(\xi_1)\Big], \quad (54)$$

$$y_3 = W_6 \cdot \operatorname{sech}^2(\xi_1)\Big[W_{13} \cdot \cosh(\xi_1)\cos(\Omega_1) - W_{12} \cdot \sinh(\xi_1)\sin(\Omega_1) + 2\lambda\, W_2\big(2W_1\cosh(2\xi_1) + W_7\big)\Big], \quad (55)$$

where $\Omega_1 = \big(k_1^2(\lambda+1)/4 + \lambda^2\big)t + \lambda\, x + 2\lambda\, k_0/k_1$,
$\xi_1 = k_1(k_1^2 t + 4x)/8 + k_0$ and $W_1, \ldots, W_{13}$ are constants.

Immersion function of the mKdV surface obtained using k_1 deformation

We find the immersion function F of the mKdV surface obtained using k_1 deformation by using the following equation

$$F = \nu \, \Phi^{-1} \frac{\partial \Phi}{\partial k_1} + \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \tag{56}$$

from which we obtain the position vector.

Immersion function of the mKdV surface obtained using k_1 deformation

We find the immersion function F of the mKdV surface obtained using k_1 deformation by using the following equation

$$F = \nu\, \Phi^{-1} \frac{\partial \Phi}{\partial k_1} + \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \tag{56}$$

from which we obtain the position vector.
Here we use the solution, Φ, of Lax equations and we choose the followings

$$A_1 = -k_1 B_2 \exp(-\lambda\, \pi / k_1),\;\; A_2 = k_1 B_1 \exp(-\lambda \pi / k_1), \tag{57}$$

$$r_{11} = -r_{22} = \frac{\nu\, (\pi\lambda + k_1)\left(B_2^2 - B_1^2\right)}{k_1^2 (B_1^2 + B_2^2)}, \tag{58}$$

$$r_{12} = -\frac{r_{21}\, k_1^2 (B_1^2 + B_2^2) + 2\nu B_1 B_2 (\pi\lambda + k_1)}{k_1^2 (B_1^2 + B_2^2)}, \tag{59}$$

in order to write F in the form $F = -i(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$.

in order to write F in the form $F = -i(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$.

Hence we obtain a family of surfaces parameterized by

$$\begin{aligned} y_1 = W_{14} \cdot \operatorname{sech}^2(\xi_1) \Big[& W_{15} \Big(2\Omega_2 \cdot \sinh(\xi_1) - (16/3)\cosh(\xi_1) \Big) \sin(\Omega_1) \\ & + W_{16} \cdot \Omega_2 \cdot \cosh(\xi_1) \cos(\Omega_1) \\ & + W_8 \Big(2\Omega_3 \cdot \cosh(2\xi_1) + 2k_1^2 \lambda \sinh(2\xi_1) + \Omega_4 \Big) \Big], \end{aligned} \tag{60}$$

$$\begin{aligned} y_2 = W_{14} \cdot \operatorname{sech}^2(\xi_1) \Big[& W_{17} \Big(2\Omega_2 \cdot \sinh(\xi_1) - (16/3)\cosh(\xi_1) \Big) \cos(\Omega_1) \\ & - W_{18} \cdot \Omega_2 \cdot \cosh(\xi_1) \sin(\Omega_1) + W_{19} \Big(\cosh(2\xi_1) + 1 \Big) \Big], \end{aligned} \tag{61}$$

$$\begin{aligned} y_3 = W_{14} \cdot \operatorname{sech}^2(\xi_1) \Big[& W_{20} \Big(2\Omega_2 \cdot \sinh(\xi_1) - (16/3)\cosh(\xi_1) \Big) \sin(\Omega_1) \\ & - W_{21} \cdot \Omega_2 \cdot \cosh(\xi_1) \cos(\Omega_1) \\ & + (W_2/2) \Big(2\Omega_3 \cdot \cosh(2\xi_1) + 2k_1^2 \lambda \cdot \sinh(2\xi_1) + \Omega_4 \Big) \Big], \end{aligned} \tag{62}$$

where
$\Omega_2 = t\,k_1^3 + 4\,x\,k_1/3$, $\Omega_3 = \left(4\,\lambda^2 + k_1^2\right)\left(k_1^3[\lambda+1]t - 4\,\lambda\,k_0\right)$,
$\Omega_4 = tk_1^3\left(4\,\lambda^2[\lambda+1] + k_1^2[7\,\lambda+1]\right)/4 + \lambda\left(k_1^2[2\,x\,k_1 - k_0] - 4\,\lambda^2\,k_0\right)$ and
$W_{14}, \ldots, W_{21}$ are constants.

Graph of Some of the mKdV Surfaces

We obtained the position vector $\overrightarrow{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$, of the mKdV surfaces arising from deformation of parameters.

We plot some of these mKdV surfaces for some special values of the constants.

Graph of Some of the mKdV Surfaces from k_0 deformation

Example: Taking $\lambda = 0.4$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 1.5$, $k_1 = 1.3$ and $r_{21} = 1$ in Eqs. (53)-(55), we get the surface given in Figure 1.

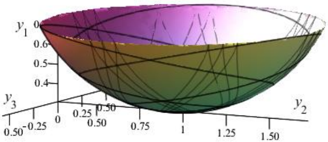


Figure : $(x, t) \in [-15, 15] \times [-15, 15]$

Example: Taking $\lambda = 1.2$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0.5$, $k_1 = 1.4$ and $r_{21} = 1$ in Eqs. (53)-(55), we get the surface given in Figure 2.

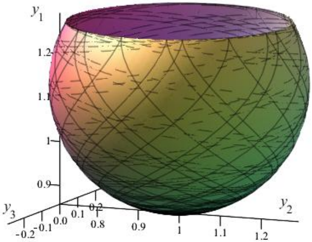
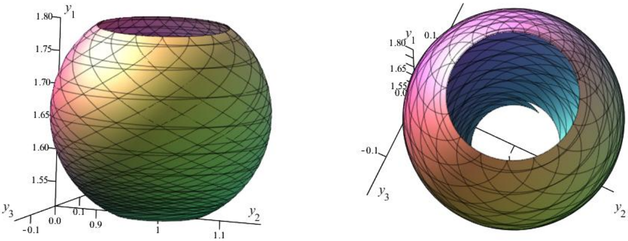


Figure : $(x, t) \in [-5, 5] \times [-5, 5]$

Example: Taking $\lambda = 0.6$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_2 = 0.2$, $k_3 = 0.4$ and $r_{21} = 1$ in Eqs. (53)-(55), we get the surface given in Figure 3.



$a)$ $b)$

Figure : (a), (b) $(x, t) \in [-10, 10] \times [-10, 10]$

Taking $\lambda = 2.7$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0.3$, $k_1 = 1.5$ and $r_{21} = 1$ in Eqs. (53)-(55), we get the surface given in Figure 4.

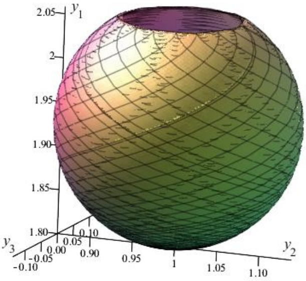


Figure : $(x, t) \in [-5, 5] \times [-5, 5]$

Graph of Some of the mKdV Surfaces from k_1 deformation

Example: Taking $\lambda = 0.15$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0.1$, $k_1 = -0.5$ and $r_{21} = 1$ in Eqs. (60)-(62), we get the surface given in Figure 5.

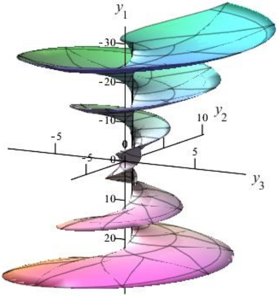


Figure : $(x, t) \in [-200, 200] \times [-200, 200]$

Example:

Taking $\lambda = 0.03$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = -0.1$ and $r_{21} = 1$ in Eqs. (60)-(62), we get the surface given in Figure 6.

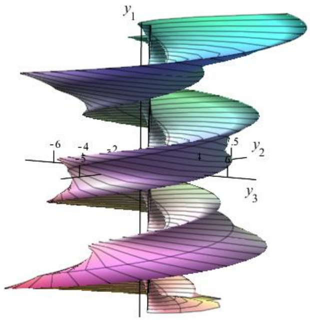


Figure : $(x, t) \in [-3000, 3000] \times [-3000, 3000]$

Example:

Taking $\lambda = -0.2$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = 0.7$ and $r_{21} = 1$ in Eqs. (60)-(62), we get the surface given in Figure 7.

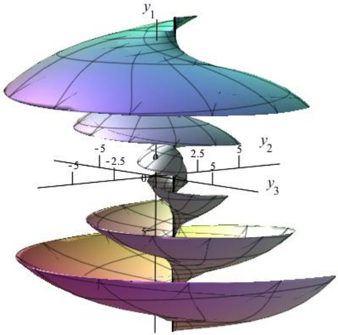


Figure : $(x, t) \in [-100, 100] \times [-100, 100]$

Example:

Taking $\lambda = -1.3$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = 4$ and $r_{21} = 1$ in Eqs. (60)-(62), we get the surface given in Figure 8.

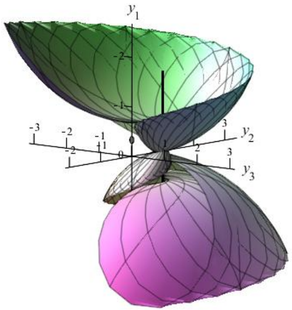


Figure : $(x, t) \in [-5, 5] \times [-5, 5]$

Example:

Taking $\lambda = 0.4$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0.6$, $k_1 = 0.7$ and $r_{21} = -2$ in Eqs. (60)-(62), we get the surface given in Figure 9.

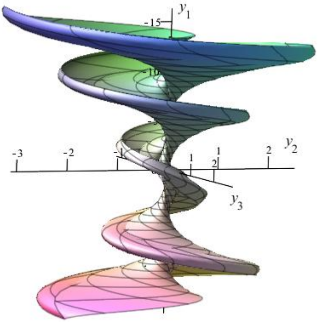


Figure : $(x, t) \in [-50, 50] \times [-50, 50]$

Example:

Taking $\lambda = 0$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = 0.7$ and $r_{21} = 1$ in Eqs. (60)-(62), we get the surface given in Figure 10.

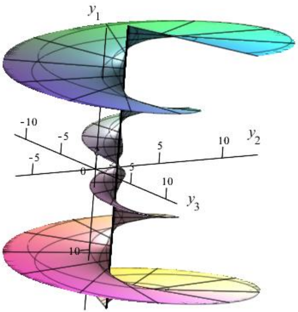


Figure : $(x, t) \in [-100, 100] \times [-100, 100]$

Example:

Taking $\lambda = -0.8$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = -0.2$ and $r_{21} = 1$ in Eqs. (60)-(62), we get the surface given in Figure 11.

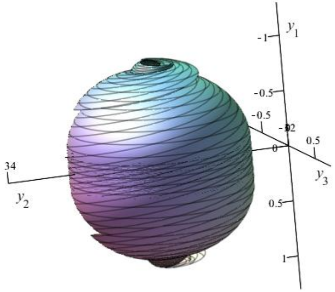


Figure : $(x, t) \in [-20, 20] \times [-20, 20]$

Example:

Taking $\lambda = -0.8$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 5$, $k_1 = -0.2$ and $r_{21} = 1$ in Eqs. (60)-(62), we get the surface given in Figure 12.

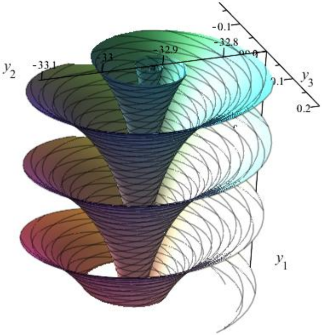


Figure : $(x, t) \in [-20, 20] \times [-20, 20]$

Example:

Taking $\lambda = -0.1$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = -4$, $k_1 = -0.2$ and $r_{21} = 1$ in Eqs. (60)-(62), we get the surface given in Figure 13.

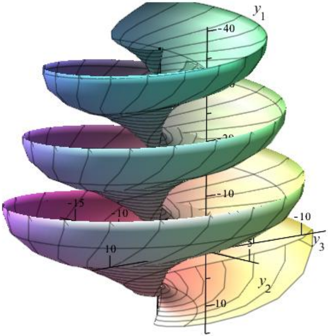


Figure : $(x, t) \in [-500, 500] \times [-500, 500]$

Example:

Taking $\lambda = 0.4$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = 0.2$ and $r_{21} = 1$ in Eqs. (60)-(62), we get the surface given in Figure 14.

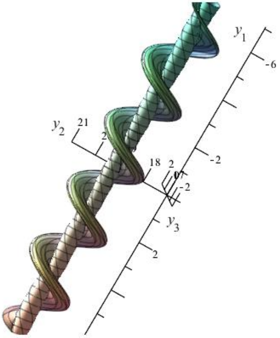


Figure : $(x, t) \in [-80, 80] \times [-80, 80]$

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• We also give the graph of interesting mKdV surfaces arise from parametric deformations.

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