

On the Axially Symmetric Willmore Surfaces

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- The so-called **Willmore functional** assigns to each surface in the three dimensional Euclidean space its total squared mean curvature. The surfaces providing local extrema to this functional are referred to as the **Willmore surfaces**. The mean and Gaussian curvatures of these surfaces obey the corresponding Euler-Lagrange equation, which is usually called the **Willmore equation**.
- The present work is concerned with a particular class of axially symmetric solutions to the Willmore equation. Not long ago, it was established that there is a **special class of axially symmetric Willmore surfaces**, regarded in Monge representation, whose profile curve height functions satisfy a one-parameter family of second-order nonlinear ordinary differential equations.
- In this work we give explicit expressions for the foregoing profile curves in terms of Jacobi elliptic functions and integrals and show that the corresponding Willmore surfaces are **nodoid-like**.
- In this talk, we discuss five different types of differential equations, which describe axially symmetric Willmore surfaces. All of them are determined as the Euler-Lagrange equations of the Willmore energy expressed in terms of different independent and dependent variables. Several types of exact analytic solutions will be presented.

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Willmore Surfaces

Willmore Functional

- The so-called **Willmore functional** (energy)

$$\mathcal{W} = \int_{\mathcal{S}} H^2 dA \quad (1)$$

assigns to each surface \mathcal{S} in the three-dimensional Euclidean space \mathbb{R}^3 its total squared mean curvature \mathcal{W} . Here, H is the local mean curvature the surface \mathcal{S} , dA is the area element on the surface \mathcal{S} .

- This functional has drawn much attention after [Willmore, 1965] where T. J. Willmore proposed to study the surfaces providing extremum to the functional (1), which are usually referred to as the **Willmore surfaces**.
- This interest is related to the so-called **Willmore conjecture** [Willmore, 1965] concerning the global problem of minimizing of (1) among the class of immersed tori: *the integral of the square of the mean curvature of a torus immersed in \mathbb{R}^3 is at least $2\pi^2$* , which have been proved recently [Marques & Neves, 2012].
- The Willmore surfaces are of great importance for the conformal geometry because of the invariance of Willmore functional (energy) under the 10-parameter group of special conformal transformations in \mathbb{R}^3 .

Willmore Surfaces

Willmore Equation

- The Euler-Lagrange equation associated with the Willmore functional, which is further referred to as the **Willmore equation**, has the form

$$\Delta H + 2(H^2 - K)H = 0 \quad (2)$$

Δ is the Laplace-Beltrami operator on \mathcal{S} and K is its Gaussian curvature.

- According to [Thomsen, 1923], Schadow was the first who derived Eq. (2) in 1922 as the Euler-Lagrange equation for the variational problem

$$\int_{\mathcal{S}} (1/R_1 - 1/R_2)^2 dA \quad (3)$$

where $1/R_1$ and $1/R_2$ are the two principal curvatures of the surface \mathcal{S} , which was studied by Thomsen in connection with the conformal geometry.

- Actually, according to [Nitsche: 1989, 1993] the history of this variational problem can be traced about two centuries back to the memoir by Siméon Denis [Poisson, 1812] and that by Marie-Sophie [Germain, 1821] where the functional (1) was proposed as the **bending energy of elastic shells**.

Willmore Surfaces

Related Functionals and Equations

- In **2D string theory and 2D gravity** based on the [Polyakov, 1981] integral over surfaces, the functional (1) is known as the **Polyakov's extrinsic action**.
- In **mathematical biology** it appears in the [Helfrich,1973] **model** as one of the terms that contribute to the **energy of cell membranes**

$$\mathcal{F}_b = \int_S \left[\frac{1}{2} k_c (2H + c_0)^2 + k_G K \right] dA + \lambda \int_S dA + p \int dV. \quad (4)$$

Here k_c and k_G are real constants representing the bending and Gaussian rigidity of the membrane, c_0 is the spontaneous curvature, λ is the tensile stress, p is the pressure, V is the enclosed volume. The corresponding Euler-Lagrange equation derived by [Ou-Yang and Helfrich, 1989] reads

$$\Delta H + (2H + c_0) (H^2 - (c_0/2)H - K) - (\lambda/k_c)H = -p/(2k_c). \quad (5)$$

- The Helfrich functional (4) and the corresponding Euler-Lagrange equation (5) play an important role in the **continuum theory of carbon nano structures**, see [Ou-Yang et al.: 1997, 2002, 2008], [Mladenov et al., 2013].

Symmetry Groups of the Willmore Equation

Willmore Equation in Monge Representations

- Let (x^1, x^2, x^3) be a fixed right-handed rectangular Cartesian coordinate system in the 3-dimensional Euclidean space \mathbb{R}^3 in which a surface \mathcal{S} is immersed, and let this surface be given in Monge representations, i.e. by the equation

$$\mathcal{S} : x^3 = w(x^1, x^2), \quad (x^1, x^2) \in \Omega \subset \mathbb{R}^2 \quad (6)$$

where $w : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a single-valued and smooth function possessing as many derivatives as may be required on the domain Ω . Let us take x^1, x^2 to serve as Gaussian coordinates on the surface \mathcal{S} .

- Then the components of the first $g_{\alpha\beta}$, second $b_{\alpha\beta}$ fundamental tensor, and the alternating tensor $\varepsilon^{\alpha\beta}$ of \mathcal{S} are given by the expressions

$$g_{\alpha\beta} = \delta_{\alpha\beta} + w_\alpha w_\beta, \quad b_{\alpha\beta} = g^{-1/2} w_{\alpha\beta}, \quad \varepsilon^{\alpha\beta} = g^{-1/2} e^{\alpha\beta} \quad (7)$$

$$g = \det(g_{\alpha\beta}) = 1 + (w_1)^2 + (w_2)^2 \quad (8)$$

$\delta_{\alpha\beta}$ is the Kronecker delta symbol, $e^{\alpha\beta}$ is the alternating symbol and $w_{\alpha_1 \dots \alpha_k}$ ($k = 1, 2, \dots$) denote the k -th order partial derivatives of the function w with respect to the variables x^1 and x^2 .

Symmetry Groups of the Willmore Equation

Willmore Equation in Monge Representations

- The mean H and Gaussian K curvatures of the surface \mathcal{S} are given as follows

$$H = \frac{1}{2}g^{\alpha\beta}b_{\alpha\beta} = \frac{1}{2}g^{-3/2} (\delta^{\alpha\beta}w_{\alpha\beta} + e^{\alpha\mu}e^{\beta\nu}w_{\alpha\beta}w_{\mu\nu}) \quad (9)$$

$$K = \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}b_{\alpha\beta}b_{\mu\nu} = \frac{1}{2}g^{-2}e^{\alpha\mu}e^{\beta\nu}w_{\alpha\beta}w_{\mu\nu} \quad (10)$$

where

$$g^{\alpha\beta} = g^{-1} (\delta^{\alpha\beta} + e^{\alpha\mu}e^{\beta\nu}w_{\mu\nu}) \quad (11)$$

are the contravariant components of the first fundamental tensor.

- The Willmore functional (1) reads

$$\mathcal{W} = \int \int_{\Omega} L dx^1 dx^2, \quad L = \frac{1}{4}g^{-5/2} (\delta^{\alpha\beta}w_{\alpha\beta} + e^{\alpha\mu}e^{\beta\nu}w_{\alpha\beta}w_{\mu\nu})^2 \quad (12)$$

- Here and in what follows, Greek indices have the range 1, 2, and the usual summation convention over a repeated index is employed.

Symmetry Groups of the Willmore Equation

Willmore Equation in Monge Representations

- The application of the Euler operator

$$E = \frac{\partial}{\partial w} - D_\mu \frac{\partial}{\partial w_\mu} + D_\mu D_\nu \frac{\partial}{\partial w_{\mu\nu}} - \dots \quad (13)$$

$$D_\alpha = \frac{\partial}{\partial x^\alpha} + w_\alpha \frac{\partial}{\partial w} + w_{\alpha\mu} \frac{\partial}{\partial w_\mu} + w_{\alpha\mu\nu} \frac{\partial}{\partial w_{\mu\nu}} + w_{\alpha\mu\nu\sigma} \frac{\partial}{\partial w_{\mu\nu\sigma}} + \dots$$

on the Lagrangian density L of the Willmore functional leads, after taking into account

$$\Delta = g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} + g^{-1/2} \frac{\partial}{\partial x^\alpha} \left(g^{1/2} g^{\alpha\beta} \right) \frac{\partial}{\partial x^\beta}$$

to the Willmore equation (2), which takes the form

$$\mathcal{E} \equiv (1/2)g^{-1/2}g^{\alpha\beta}g^{\mu\nu}w_{\alpha\beta\mu\nu} + \Phi(x_1, x_2, w, w_1, \dots, w_{222}) = 0 \quad (14)$$

where $\Phi(x_1, x_2, w, w_1, \dots, w_{222})$ is a differential function of the independent and dependent variables and the derivatives of the dependent variable up to third order.

Symmetry Groups of the Willmore Equation

The Group of Special Conformal Transformations in \mathbb{R}^3

translations

$$\mathbf{v}_1 = \frac{\partial}{\partial x^1}, \quad \mathbf{v}_2 = \frac{\partial}{\partial x^2}, \quad \mathbf{v}_3 = \frac{\partial}{\partial w}$$

rotations

$$\mathbf{v}_4 = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}, \quad \mathbf{v}_5 = x^1 \frac{\partial}{\partial w} - w \frac{\partial}{\partial x^1}, \quad \mathbf{v}_6 = x^2 \frac{\partial}{\partial w} - w \frac{\partial}{\partial x^2}$$

dilatation

$$\mathbf{v}_7 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + w \frac{\partial}{\partial w}$$

inversions

$$\mathbf{v}_8 = [(x^1)^2 - (x^2)^2 - w^2] \frac{\partial}{\partial x^1} + 2x^1 x^2 \frac{\partial}{\partial x^2} + 2x^1 w \frac{\partial}{\partial w}$$

$$\mathbf{v}_9 = 2x^2 x^1 \frac{\partial}{\partial x^1} + [(x^2)^2 - (x^1)^2 - w^2] \frac{\partial}{\partial x^2} + 2x^2 w \frac{\partial}{\partial w}$$

$$\mathbf{v}_{10} = 2x^1 w \frac{\partial}{\partial x^1} + 2x^2 w \frac{\partial}{\partial x^2} + [w^2 - (x^2)^2 - (x^1)^2] \frac{\partial}{\partial w}$$

Symmetry Groups of the Willmore Equation

The Group of Special Conformal Transformations in \mathbb{R}^3

• The following Propositions [Vassilev & Mladenov, 2004] clarify the invariance properties of the Willmore equation relative to one-parameter Lie groups of point transformations of \mathbb{R}^3 . The coordinates (x^1, x^2, w) on \mathbb{R}^3 represent the involved independent and dependent variables x^1, x^2 and w , respectively. The results are obtained using Lie infinitesimal technique.

Proposition 1. The 10-parameter Lie group G_{SCT} of special conformal transformations in \mathbb{R}^3 (whose basic generators are \mathbf{v}_j , $j = 1 \dots 10$) is the largest group of point (geometric) transformations of the involved independent and dependent variables that a generic equation of form (14) could admit.

Proposition 2. In Monge representation, the Willmore equation admits all the transformations of the group G_{SCT} .

Remark. All vector fields \mathbf{v}_j , $j = 1, \dots, 10$ are variational symmetries of the Willmore equation, i.e., infinitesimal divergence symmetries of the Willmore functional. Hence, Noether's theorem implies the existence of ten linearly independent conservation laws that hold on the smooth solutions of the Willmore equation.

Symmetry Groups of the Willmore Equation

Group-Invariant Solutions

- Once a group G is found to be a symmetry group of a given differential equation, it is possible to look for the so-called group-invariant (G -invariant) solutions of this equation – the solutions, which are invariant under the transformations of the symmetry group G .
- The main advantage that one can gain when looking for this kind of particular solutions of the given differential equation consists in the fact that each group-invariant solution is determined by a reduced equation obtained by a symmetry reduction of the original one and involving less independent variables than the latter.
- Let $G(\mathbf{v})$ be a one parameter group generated by a vector field \mathbf{v} belonging to the Lie algebra L_{SCT} , that is \mathbf{v} is a linear combination of the vector fields \mathbf{v}_j , $j = 1 \dots 10$,

$$\mathbf{v} = \sum_{j=1}^{10} c_j \mathbf{v}_j \quad (15)$$

where c_j , $j = 1 \dots 10$, are real numbers – the components of the vector field \mathbf{v} with respect to the basic vector fields \mathbf{v}_j .

Symmetry Groups of the Willmore Equation

Group-Invariant Solutions

- Then, $G(\mathbf{v})$ is a symmetry group of the Willmore equation and so one can look for the $G(\mathbf{v})$ -invariant solutions of this equation. For that purpose, first one should find a complete set of functionally independent invariants of the group $G(\mathbf{v})$. In the present case this is a set of two functionally independent functions $I_\alpha(x^1, x^2, w)$ such that

$$\mathbf{v}I_\alpha = 0$$

the vector field \mathbf{v} being regarded here as an operator acting on the functions $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$. Then, if the necessary condition for the existence of group invariant solutions is satisfied, which in the present case reads

$$\text{rank} \left(\frac{\partial I_\alpha}{\partial w} \right) = 1 \quad (16)$$

assuming that $\partial I_1 / \partial w \neq 0$, one can seek the $G(\mathbf{v})$ -invariant solutions in the form

$$U = U(s), \quad U = I_1, \quad s = I_2. \quad (17)$$

Axially Symmetric Willmore Surfaces

Axially Symmetric Surfaces

Let the components X, Y, Z of the position vector \mathbf{R} of an axially symmetric surface \mathcal{S} in \mathbb{R}^3 be given with respect to a fixed right-handed rectangular Cartesian coordinate system as

$$\mathbf{R} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} r(t) \cos \theta \\ r(t) \sin \theta \\ z(t) \end{pmatrix} \quad (18)$$

where t is an arbitrary parameter along the contour of the surface \mathcal{S} at fixed $\theta \in [0, 2\pi]$. Then, the first and second fundamental tensors of \mathcal{S} read

$$g_{\alpha\beta} = \begin{pmatrix} r_t^2 + z_t^2 & 0 \\ 0 & r^2 \end{pmatrix}, \quad b_{\alpha\beta} = \frac{1}{\sqrt{r_t^2 + z_t^2}} \begin{pmatrix} r_t z_{tt} - z_t r_{tt} & 0 \\ 0 & r z_t \end{pmatrix} \quad (19)$$

respectively, and

$$g = \det(g_{\alpha\beta}) = r^2 (r_t^2 + z_t^2). \quad (20)$$

Here, the subindexes indicate derivatives with respect to the variable t .

Axially Symmetric Willmore Surfaces

Axially Symmetric Surfaces

Consequently, the mean H and Gaussian K curvatures of the surface \mathcal{S} read

$$H = \frac{1}{2} \frac{r(r_t z_{tt} - z_t r_{tt}) + z_t(r_t^2 + z_t^2)}{r(r_t^2 + z_t^2)^{3/2}} \quad (21)$$

$$K = \frac{z_t(r_t z_{tt} - z_t r_{tt})}{r(r_t^2 + z_t^2)^2} \quad (22)$$

The volume V enclosed by the surface \mathcal{S} can be written as follows

$$V = \frac{1}{3} \int_{\mathcal{S}} \mathbf{n} \cdot \mathbf{R} \, dA \quad (23)$$

where

$$\mathbf{n} = \frac{1}{\sqrt{r_t^2 + z_t^2}} \begin{pmatrix} z_t \cos \theta \\ z_t \sin \theta \\ -r_t \end{pmatrix} \quad (24)$$

is the unit outward normal vector and $dA = 2\pi\sqrt{g}dt = 2\pi r\sqrt{r_t^2 + z_t^2}dt$ is the infinitesimal area element on the surface.

Axially Symmetric Willmore Surfaces

Willmore Energy of Axially Symmetric Surfaces

- The Willmore energy of an axially symmetric surface \mathcal{S} reads

$$\mathcal{W} = \int_{\mathcal{S}} H^2 dA = \int \mathcal{L} dt \quad (25)$$

where

$$\mathcal{L} = 2\pi \left[\frac{r(r_t z_{tt} - z_t r_{tt}) + z_t(r_t^2 + z_t^2)}{2r(r_t^2 + z_t^2)^{3/2}} \right]^2 r \sqrt{r_t^2 + z_t^2} \quad (26)$$

- In mathematical biology the Willmore energy appears in the [Helfrich,1973] model as one of the terms that contribute to the energy of cell membranes

$$\mathcal{F}_b = \int_{\mathcal{S}} \left[\frac{1}{2} k_c (2H + c_0)^2 + k_G K \right] dA + \lambda \int_{\mathcal{S}} dA + p \int dV. \quad (27)$$

Here k_c and k_G are real constants representing the bending and Gaussian rigidity of the membrane, c_0 is the spontaneous curvature, λ is the tensile stress, p is the pressure, V is the enclosed volume.

Axially Symmetric Willmore Surfaces

Euler-Lagrange Equations

- The application of the Euler operators

$$E_r = \frac{\partial}{\partial r} - D_t \frac{\partial}{\partial r_t} + D_t D_t \frac{\partial}{\partial r_{tt}} - \dots \quad (28)$$

$$E_z = \frac{\partial}{\partial z} - D_t \frac{\partial}{\partial z_t} + D_t D_t \frac{\partial}{\partial z_{tt}} - \dots \quad (29)$$

where

$$D_t = \frac{\partial}{\partial t} + r_t \frac{\partial}{\partial r} + z_t \frac{\partial}{\partial z} + r_{tt} \frac{\partial}{\partial r_t} + z_{tt} \frac{\partial}{\partial z_t} + \dots \quad (30)$$

on the Lagrangian density \mathcal{L} of the Willmore functional \mathcal{W} leads to

$$r_t E_r \mathcal{L} \equiv -z_t E_z \mathcal{L} \quad (31)$$

- Thus, we have a **single equation determining the extremals of \mathcal{W}** , say

$$E_r \mathcal{L} = 0 \quad (32)$$

instead of a system of two Euler-Lagrange equations for two dependent variables.

- **The same holds true for the critical points of the Helfrich energy \mathcal{F}_b .**

Axially Symmetric Willmore Surfaces

Euler-Lagrange Equation assuming $r_t^2 + z_t^2 = 1$

- Assume $r_t^2 + z_t^2 = 1$ meaning that t is the arc length s along the contour. Then, in terms of the tangent angle φ we have

$$r_t = \cos \varphi, \quad z_t = \sin \varphi \quad (33)$$

and can rewrite equation $E_r \mathcal{L}_H = 0$ in the form

$$\begin{aligned} \ddot{\varphi} &= -\frac{2 \cos \varphi}{r} \dot{\varphi} - \frac{1}{2} \dot{\varphi}^3 + \frac{\sin \varphi}{r} \dot{\varphi}^2 + \frac{\sin \varphi}{2r} (\dot{\varphi} - c_0)^2 \\ &+ \left[\frac{1}{2} \left(\frac{\sin \varphi}{2} - c_0 \right)^2 + \frac{1 - 2 \sin^2 \varphi}{r^2} + \frac{\lambda}{k_c} \right] \dot{\varphi} \\ &- \frac{\cos^2 \varphi + 1}{2r^3} \sin \varphi + \frac{\lambda \sin \varphi}{k_c r} + \frac{p}{k_c} \end{aligned} \quad (34)$$

where the dots indicate derivatives with respect to the arc length s .

Axially Symmetric Willmore Surfaces

Euler-Lagrange Equation assuming $z = t, r = u(z)$

- Assume $z = t$ and $r = u(z)$. Then, equation $E_r \mathcal{L}_H = 0$ takes the form

$$\begin{aligned} & 2u^3 (u_z^2 + 1)^2 u_{zzzz} \\ & + 4u^2 u_z (u_z^2 + 1) [(u_z^2 + 1) - 5uu_{zz}] u_{zzz} \\ & + 5u^3 (6u_z^2 - 1) u_{zz}^3 - 3u^2 (u_z^2 + 1) (4u_z^2 - 1) u_{zz}^2 \\ & - u (u_z^2 + 1)^2 \left\{ 4u \left[u (u_z^2 + 1) \left(c_0^2 + \frac{\lambda}{k_c} \right) - 2c_0 \sqrt{u_z^2 + 1} \right] + 2u_z^2 - 1 \right\} u_{zz} \\ & + (u_z^2 + 1)^3 \left[4u^2 (u_z^2 + 1) \left(c_0^2 + \frac{\lambda}{k_c} + \frac{p}{k_c} u \sqrt{u_z^2 + 1} \right) - 2u_z^2 - 1 \right] = 0 \quad (35) \end{aligned}$$

Here, the subindexes indicate derivatives with respect to the variable z .

Axially Symmetric Willmore Surfaces

Euler-Lagrange Equation assuming $r = t, z = w(r)$

- Assume $r = t$ and $z = w(r)$. Then, equation $E_r \mathcal{L}_H = 0$ takes the form

$$\begin{aligned} \mathcal{R} \equiv & (2r^3 + 4r^3 w_r^2 + 2r^3 w_r^4) w_{rrrr} \\ & + (4r^2 + 8r^2 w_r^2 + 4r^2 w_r^4 - 20r^3 w_r w_{rr} - 20r^3 w_r^3 w_{rr}) w_{rrr} \\ & - 5r^2 (3w_r + 3w_r^3 + r w_{rr} - 6r w_r^2 w_{rr}) w_{rr}^2 \\ & + (r w_r^6 - 2r - 3r w_r^2) w_{rr} + 2w_r + 7w_r^3 + 9w_r^5 + 5w_r^7 + w_r^9 = 0 \end{aligned} \quad (36)$$

where

$$w_r = \frac{dw}{dr}, \quad w_{rr} = \frac{d^2w}{dr^2}, \quad w_{rrr} = \frac{d^3w}{dr^3}, \quad w_{rrrr} = \frac{d^4w}{dr^4}.$$

- Simultaneously, the mean and Gaussian curvatures take the forms

$$H = \frac{1}{2r} \frac{r w_{rr} + w_r^3 + w_r}{(1 + w_r^2)^{3/2}}, \quad K = \frac{1}{r} \frac{w_{rr} w_r}{(1 + w_r^2)^2}. \quad (37)$$

Axisymmetric Solutions of the Willmore Equation

An Intermediate Integral of the Reduced Equation

- Consider the following normal system of two ordinary differential equations

$$\frac{dw}{dr} = v, \quad \frac{dv}{dr} = \pm \frac{1}{r} (v^2 + 1) \sqrt{v^2 + 2\omega\sqrt{v^2 + 1}} \quad (38)$$

which is equivalent to the single second-order equation

$$\frac{d^2w}{dr^2} = \pm \frac{1}{r} \left[\left(\frac{dw}{dr} \right)^2 + 1 \right] \sqrt{\left(\frac{dw}{dr} \right)^2 + 2\omega\sqrt{\left(\frac{dw}{dr} \right)^2 + 1}}. \quad (39)$$

- Each solution of system (38) or equation (39) is a solution of the reduced Willmore equation $\mathcal{R} = 0$, see [Vassilev & Mladenov, 2004]. In this way, we have determined a special class of axially symmetric Willmore surfaces.
- It is worth nothing that system (38) and equation (39) turn out to be invariant under the translations of the variable w and the scaling transformations

$$w \rightarrow w\eta, \quad r \rightarrow r\eta, \quad \eta \in \mathbb{R}.$$

Axisymmetric Solutions of the Willmore Equation

Analytic Representation of a Class of Axiymmetric Willmore Surfaces

- The substitutions $u = \sqrt{v^2 + 1}$, $\rho = \ln r$ transform system (38) as follows

$$\frac{dw}{d\rho} = e^\rho \sqrt{u^2 - 1} \quad (40)$$

$$\left(\frac{du}{d\rho}\right)^2 = u^2 (u^2 - 1) (u^2 + 2au - 1). \quad (41)$$

- In terms of a new variable t such that

$$\frac{d\rho}{dt} = \frac{1}{u} \quad (42)$$

Eq. (41) may be written in the form

$$\left(\frac{du}{dt}\right)^2 = P(u), \quad P(u) = (u^2 - 1) (u^2 + 2au - 1) \quad (43)$$

and Eq. (40) becomes

$$\frac{dw}{dt} = e^\rho \frac{1}{u} \sqrt{u^2 - 1}. \quad (44)$$

Axisymmetric Solutions of the Willmore Equation

Analytic Representation of a Class of Axiymmetric Willmore Surfaces

Using the standard approach [Whittaker & Watson] [Abramowitz & Stegun]), we can express a class of solutions of Eq. (43) corresponding to the root $u = 1$ of the polynomial $P(u)$ as follows

$$u(t) = \frac{2\sqrt{a^2 + 1} - (\sqrt{a^2 + 1} - a + 1) \operatorname{sn}^2(\lambda t, k)}{2\sqrt{a^2 + 1} - (\sqrt{a^2 + 1} + a + 1) \operatorname{sn}^2(\lambda t, k)} \quad (45)$$

where

$$\lambda = \sqrt[4]{a^2 + 1}, \quad k = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{1}{\sqrt{a^2 + 1}}}$$

Then, using Eq. (45), one can write down the solution $\rho(t)$ of Eq. (42)

$$\rho(t) = (\lambda^2 + a)t - \frac{\lambda^2 + a - 1}{\lambda} \Pi\left(\frac{\lambda^2 - a + 1}{2\lambda^2}, \operatorname{am}(\lambda t, k), k\right). \quad (46)$$

Finally, using that $\rho = \ln r$ and Eq. (40) we arrive at the following analytic representation of the parametric equations for the profile curves of the axially symmetric Willmore surfaces determined by Eq. (39)

$$r(t) = e^{\rho(t)}, \quad w(t) = \int e^{\rho(t)} \frac{1}{u(t)} \sqrt{u(t)^2 - 1} dt + b, \quad b = \text{const.} \quad (47)$$

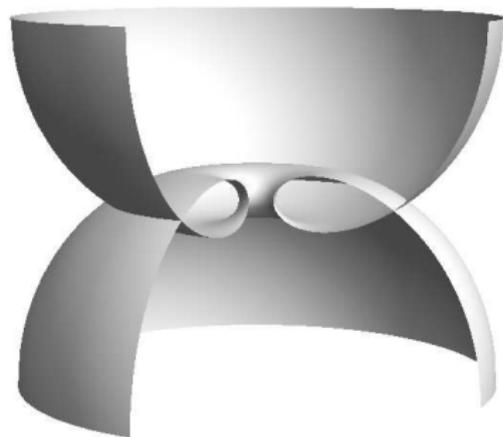
Nodoid-Like Willmore Surfaces

Numerical Calculations

- Unfortunately, the parametric equations (47) are too complicated to be used for displaying the respective surfaces directly. This, however, can be done by solving numerically system (38) taking as initial values at $r = 1$ an arbitrary real number for w (because of the invariance of system (38) under the translation of this variable) and $v = 0$. Indeed, for each surface of the considered class Eqs. (45) and (46) imply $u(0) = 1$ and $\rho(0) = 0$ and hence $v = 0$ at $r = 1$.
- Two Willmore surfaces obtained in this way are depicted in Figure 1. First of them (Figure 1, left) is constructed by joining two profile curves Γ_- and Γ_+ (see Figure 2, left), which are generated by solving numerically system (38), choosing respectively sign "-" and sign "+" of the right-hand side of the second equation in this system, setting $a = 0.2$ and taking $v = 0$ as initial condition at $r = 1$. The second one is constructed by joining another couple of profile curves $\hat{\Gamma}_-$ and $\hat{\Gamma}_+$ (see Figure 2, right) obtained in the same manner, but now $a = 1$. The Gaussian curvatures corresponding to both profile curves Γ_- and Γ_+ are identical, while the respective mean curvatures are symmetric with respect to the r -axis. The same holds true for the Gaussian and mean curvatures of the curves $\hat{\Gamma}_-$ and $\hat{\Gamma}_+$.

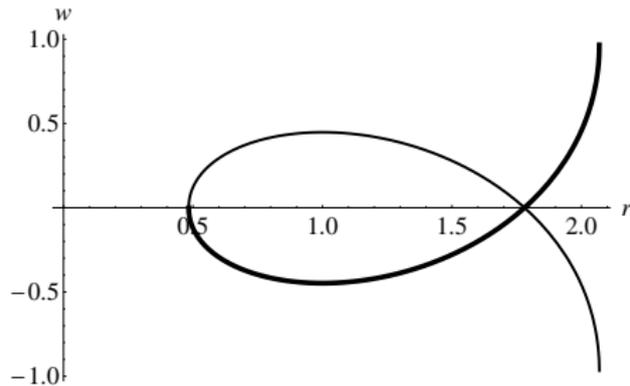
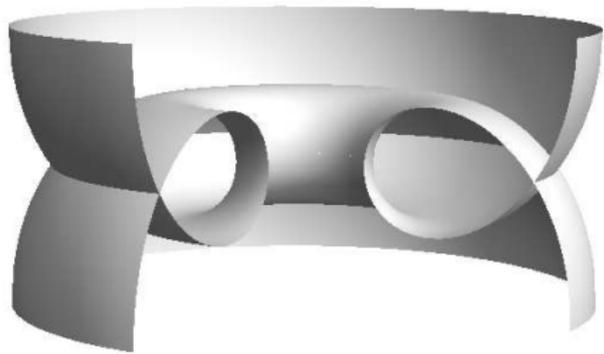
Nodoid-Like Willmore Surfaces

Numerical Calculations

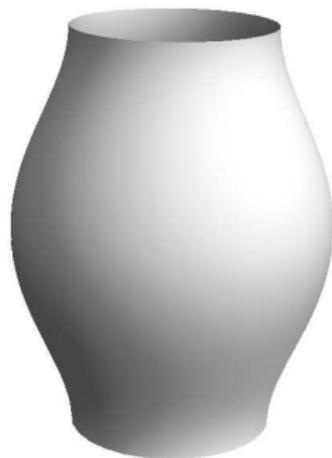


Nodoid-Like Willmore Surfaces

Numerical Calculations



Unduloid-Like Willmore Surfaces



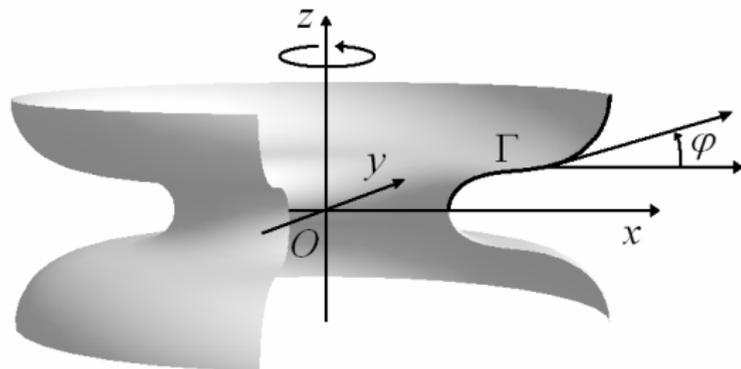
Cup-Like Willmore Surfaces



Axisymmetric Equilibrium Shapes

Sketch of a Surface of Revolution

Suppose that a part of an axisymmetrically deformed SWCNT admits graph parametrization. This means that it may be thought of as a surface of revolution obtained by revolving around the z -axis a plane curve Γ laying in the xOz -plane, which is determined by the graph $(x, z(x))$ of a function $z = z(x)$.



Axisymmetric Equilibrium Shapes

Shape Equation

For each such surface the general shape equation (5) reduces to the following nonlinear third-order ordinary differential equation

$$\begin{aligned} \cos^3 \varphi \frac{d^3 \varphi}{dx^3} &= 4 \sin \varphi \cos^2 \varphi \frac{d^2 \varphi}{dx^2} \frac{d\varphi}{dx} - \cos \varphi \left(\sin^2 \varphi - \frac{1}{2} \cos^2 \varphi \right) \left(\frac{d\varphi}{dx} \right)^3 \\ &+ \frac{7 \sin \varphi \cos^2 \varphi}{2x} \left(\frac{d\varphi}{dx} \right)^2 - \frac{2 \cos^3 \varphi}{x} \frac{d^2 \varphi}{dx^2} \\ &+ \left(\frac{\lambda}{k_c} + \frac{c_0^2}{2} - \frac{2c_0 \sin \varphi}{x} - \frac{\sin^2 \varphi - 2 \cos^2 \varphi}{2x^2} \right) \cos \varphi \frac{d\varphi}{dx} \\ &+ \left(\frac{\lambda}{k_c} + \frac{c_0^2}{2} - \frac{\sin^2 \varphi + 2 \cos^2 \varphi}{2x^2} \right) \frac{\sin \varphi}{x} - \frac{p}{k_c} \end{aligned} \quad (48)$$

(derived in [Hu & Ou-Yang, 1993]) where φ is the angle between the x -axis and the tangent vector to the profile curve Γ , i.e., the tangent (slope) angle, considered as a function of the variable x .

Axisymmetric Equilibrium Shapes

Exact Solutions of the Shape Equation

[Naito et al., 1995] discovered that the shape equation (48) has the following class of exact solutions

$$\sin \varphi = ax + b + dx^{-1}, \quad (49)$$

provided that a , b and d are real constants, which meet the conditions

$$\frac{p}{k_c} - 2a^2c_0 - 2a \left(\frac{c_0^2}{2} + \frac{\lambda}{k_c} \right) = 0, \quad (50)$$

$$b \left(2ac_0 + \frac{c_0^2}{2} + \frac{\lambda}{k_c} \right) = 0, \quad (51)$$

$$b(b^2 - 4ad - 4c_0d - 2) = 0, \quad (52)$$

and

$$d(b^2 - 4ad - 2c_0d) = 0. \quad (53)$$

Exact Solutions of the Shape Equation

Six types of solutions of form (49) to Eq. (48) can be distinguished on the ground of conditions (50) – (53) depending on the values of c_0 , λ and p .

Case A. If $c_0 = 0$, $\lambda = 0$, $p = 0$, then the solutions to Eq. (48) of the form (49) are $\sin \varphi = ax$, $\sin \varphi = ax \pm \sqrt{2}$ and $\sin \varphi = dx^{-1}$, the respective surfaces being spheres, Clifford tori and catenoids.

Case B. If $c_0 = 0$, $\lambda \neq 0$, $p = 0$, then the solutions of the considered type reduces to $\sin \varphi = dx^{-1}$ (catenoids).

Case C. If $c_0 = 0$, $\lambda \neq 0$, $p \neq 0$ and $p = 2a\lambda$, then only one branch of the regarded solutions remains, namely $\sin \varphi = ax$ (spheres).

Case D. If $c_0 \neq 0$, $\lambda = 0$, $p = 0$, then one arrives at the whole family of Delaunay surfaces corresponding to the solutions of the form

$$\sin \varphi = -\frac{1}{2}c_0x + \frac{d}{x}. \quad (54)$$

Case E. If $c_0 \neq 0$, $\lambda \neq 0$, $p = 0$ and

$$\frac{\lambda}{k_c} = -\frac{1}{2}c_0(2a + c_0),$$

one gets only solutions of the form $\sin \varphi = ax$ (spheres).

Exact Solutions of the Shape Equation

Case F. If $c_0 \neq 0$, $\lambda \neq 0$, $p \neq 0$, then four different types of solutions of form (49) to Eq. (48) are encountered: (a) $\sin \varphi = ax$ (spheres) if

$$\frac{p}{k_c} = 2a \left(\frac{\lambda}{k_c} + ac_0 + \frac{c_0^2}{2} \right); \quad (55)$$

(b) $\sin \varphi = ax \pm \sqrt{2}$ (Clifford tori) if

$$\frac{p}{k_c} = -2a^2 c_0, \quad \frac{\lambda}{k_c} = -\frac{1}{2} c_0 (4a + c_0); \quad (56)$$

(c) solutions of the form (54) (Delaunay surfaces) if

$$p + c_0 \lambda = 0; \quad (57)$$

(d) solutions of the form

$$\sin \varphi = -\frac{1}{4} c_0 (b^2 + 2) x + b - \frac{1}{c_0 x}, \quad (58)$$

which take place provided that

$$\frac{p}{k_c} = -\frac{1}{8} c_0^3 (b^2 + 2)^2, \quad \frac{\lambda}{k_c} = \frac{1}{2} c_0^2 (b^2 + 1). \quad (59)$$

Parametric Equations of the Unduloid-Like Surfaces

Below, we derive the parametric equations of the surfaces corresponding to the solutions of form (58) to Eq. (48).

First, it is clear that the variable x must be strictly positive or negative, otherwise the right-hand side of Eq. (49) is both undefined and its absolute value is greater than one, which is in contradiction with the sin-function appearing in the left-hand side of this relation.

Next, according to the meaning of the tangent angle

$$\frac{dz}{dx} = \tan \varphi \quad (60)$$

which for the foregoing class of solutions (58) implies

$$\left(\frac{dz}{dx}\right)^2 = \frac{\left[b - \frac{1}{c_0 x} - \frac{1}{4}c_0(b^2 + 2)x\right]^2}{1 - \left[b - \frac{1}{c_0 x} - \frac{1}{4}c_0(b^2 + 2)x\right]^2}. \quad (61)$$

Parametric Equations of the Unduloid-Like Surfaces

In terms of an appropriate new variable t , relation (61) may be written in the form

$$\left(\frac{dx}{dt}\right)^2 = -\frac{1}{u^2} Q_1(x) Q_2(x) \quad (62)$$

$$\left(\frac{dz}{dt}\right)^2 = \frac{1}{4u^2} (Q_1(x) + Q_2(x))^2 \quad (63)$$

where

$$u = -\frac{4}{c_0 (2 + b^2)^{3/4}}$$

$$Q_1(x) = x^2 - \frac{4(b+1)}{c_0(b^2+2)}x + \frac{4}{c_0^2(b^2+2)} \quad (64)$$

$$Q_2(x) = x^2 - \frac{4(b-1)}{c_0(b^2+2)}x + \frac{4}{c_0^2(b^2+2)}. \quad (65)$$

Parametric Equations of the Unduloid-Like Surfaces

It should be noticed that the roots of the polynomial $Q(x) = Q_1(x)Q_2(x)$ read

$$\alpha = \frac{2 \operatorname{sign}(b) h - 1}{c_0 \sqrt{b^2 + 2} h + 1}, \quad \beta = \frac{2 \operatorname{sign}(b) h + 1}{c_0 \sqrt{b^2 + 2} h - 1} \quad (66)$$

$$\gamma = \frac{4b}{c_0 (b^2 + 2)} - \frac{\alpha + \beta}{2} + i \frac{2\sqrt{2|b| + 1}}{c_0 (b^2 + 2)}$$

$$\delta = \frac{4b}{c_0 (b^2 + 2)} - \frac{\alpha + \beta}{2} - i \frac{2\sqrt{2|b| + 1}}{c_0 (\epsilon^2 + 2)}$$

where

$$h = \sqrt{\frac{1 + |b| + \sqrt{2 + b^2}}{1 + |b| - \sqrt{2 + b^2}}}. \quad (67)$$

Hence, Eq. (62) has real-valued solutions if and only if at least two of these roots are real and different. Evidently, the roots γ and δ can not be real, but α and β are real provided that $|b| > 1/2$ as follows by relations (66) and (67).

Parametric Equations of the Unduloid-Like Surfaces

Now, using the standard procedure for handling elliptic integrals (see [?, 22.7]), one can express the solution $x(t)$ of equation (62) in the form

$$x(t) = \frac{2 \operatorname{sign}(b)}{c_0 \sqrt{b^2 + 2}} \left(1 - \frac{2h}{h + \operatorname{cn}(t, k)} \right) \quad (68)$$

where

$$k = \sqrt{\frac{1}{2} - \frac{3}{4\sqrt{2 + b^2}}}.$$

Consequently, using expressions (64) and (65), one can write down the solution $z(t)$ of equation (63) in the form

$$z(t) = \frac{1}{u} \int \left[x^2(t) - \frac{4bx(t)}{c_0(b^2 + 2)} + \frac{4}{c_0^2(b^2 + 2)} \right] dt. \quad (69)$$

Parametric Equations of the Unduloid-Like Surfaces

Finally, performing the integration in the right-hand-side of Eq. (69), one obtains

$$z(t) = u \left[\mathbf{E}(\operatorname{am}(t, k), k) - \frac{\operatorname{sn}(t, k) \operatorname{dn}(t, k)}{h + \operatorname{cn}(t, k)} - \frac{t}{2} \right]. \quad (70)$$

Thus, for each couple of values of the parameters c_0 and b , (68) and (70) are the sought parametric equations of the contour of an axially symmetric unduloid-like surface corresponding to the respective solution of the membrane shape equation (48) of form (58).

Examples

References I

-  Abramowitz M. and Stegun I., *Handbook of Mathematical Functions*, Dover, New York, 1972.
-  Germain S.,
Recherches sur la théorie des surfaces élastiques, Paris, 1821.
-  Helfrich W.
Elastic properties of lipid bilayers: theory and possible experiments. *Z. Naturforsch C*, 28:693–703, 1973.
-  Marques F. C., Neves A.
Min-Max Theory and the Willmore Conjecture arXiv:1202.6036v1 [math.DG]
(Submitted on 27 Feb 2012)
-  Mladenov, I. M., P. A. Djondjorov, M. Ts. Hadzhilazova, V. M. Vassilev,
Equilibrium Configurations of Lipid Bilayer Membranes and Carbon Nanostructures, *Commun. Theor. Phys.* **59** (2013) 213–228.

References II



Naito H., Okuda M. and Ou-Yang Z.,
New Solutions to the Helfrich Variational Problem for the Shapes of Lipid Bilayer Vesicles: Beyond Delaunay's Surfaces, Phys. Rev. Lett. **74** (1995) 4345–4348.



Ou-Yang Z. and Helfrich W.,
Bending Energy of Vesicle Membranes: General Expressions for the First, Second, and Third Variation of the Shape Energy and Applications to Spheres and Cylinders, Phys. Rev. A **39** (1989) 5280–5288.



Nitsche J.,
Lectures on Minimal Surfaces, Vol. 1. Introduction, Fundamentals, Geometry and Basic Boundary Value Problems, Cambridge University Press, Cambridge, 1989.



Nitsche J.,
Boundary Value Problems for Variational Integrals Involving Surface Curvatures, Q. Appl. Math. **60** (1993) 363–387.

References III



Ou-Yang Z.-C. and W. Helfrich,

Bending energy of vesicle membranes: general expressions for the first, second, and third variation of the shape energy and applications to spheres and cylinders, *Phys. Rev. A* **39** (1989) 5280–5288.



Ou-Yang Z.-C., Z. B. Su, and C. L. Wang.

Coil formation in multishell carbon nanotubes: competition between curvature elasticity and interlayer adhesion, *Phys. Rev. Lett.* **78** (1997) 4055–4058.



Poisson S. D.,

Mémoire sur les surfaces élastiques, Mem. Cl. Sci. Math. Phys., Inst. de France, (1812) 167–225.



Polyakov A.,

The Membrane Properties of Condensing Strings, *Phys. Lett.* **B103** (1981), 207–210.



Thomsen G.,

Grundlagen der konformen Flächentheorie, Abh. Math. Sem. Univ. Hamburg **3** (1923) 31–56.

References IV



Tu Z.-C. and Z.-C. Ou-Yang,
Single-walled and multiwalled carbon nanotubes viewed as elastic tubes with the effective Young's moduli dependent on layer number. *Phys. Rev. B*, **65** (2002) 233407.



Tu Z.-C. and Z.-C. Ou-Yang,
Elastic Theory of Low-Dimensional Continua and its Applications in Bio- and Nano-Structures, *J. Comput. Theoret. Nanoscience*, **5** (2008) 422–448.



Vassilev V. and I. Mladenov,
Geometric symmetry groups, conservation laws and group-invariant solutions of the Willmore equation. In: *Geometry, Integrability and Quantization V*, pp 246–265, SOFTEX, Sofia, 2004.



Willmore T.,
Note on Embedded Surfaces, An. Ştiinţ. Univ. “Al. I. Cuza” Iaşi Secţ. Ia Mat. **11** (1965) 493–496.



Whittaker E. and Watson G.,
A Course of Modern Analysis, Cambridge University Press, Cambridge, 1922.